REPRESENTATIONS OF SEMIGROUPS OF PARTIAL ISOMETRIES

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Abstract

We present a new proof that the irreducible representations of the von Neumann algebra generated by a strongly continuous semigroup of partial isometries of index 1 are unique up to equivalence, as well as a proof that when such an algebra is a factor, its representations are completely reducible. As an application, we show that the irreducible representations of a strongly continuous semigroup of isometries \{U(\alpha), \alpha \geq 0\} such that \(U(\alpha)x \xrightarrow{\alpha \to \infty} 0\) are equivalent.

1. Introduction

Partial isometries naturally arise in physics when a (one-dimensional) quantum system is constrained to the segment \(I = [0, 1]\). There, the translation operators \(U(\alpha), \alpha \geq 0\), defined in \(\mathcal{H} = L^2[0, 1]\) by

\[
U(\alpha)f(x) = g(x) = \begin{cases} 
0 & \text{if } 0 \leq x < \alpha, \\
(f(x - \alpha) & \text{if } \alpha \leq x \leq 1,
\end{cases}
\]

(1.1)

are isometries for \(f(x) \in \mathcal{H}' = L^2[0, 1-\alpha]\), whereas \(\mathcal{H}'' = \mathcal{H} \ominus \mathcal{H}' = L^2[1-\alpha, 1]\) is the ker of \(U(\alpha)\). We call \(S(1)\) the strongly continuous semigroup \(\{U(\alpha)\}\). Then \(S(\mu)\) is defined in a similar way. More generally, a nilpotent semigroup of index \(\mu\) of partial isometries is a semigroup \(U(\mu)\) of operators \(U(\alpha)\) such that

\[
\|U(\alpha)\| = 1, \quad 0 \leq \alpha < \mu, \\
U(\mu) = 0, \\
U(0) = I.
\]

(1.2) (1.3) (1.4)

Here, \(U(\mu)\) is called a truncated shift if it is unitarily equivalent to a direct sum of copies of \(S(\mu)\).

A natural question to ask is whether the irreducible representations of the von Neumann algebra \(\mathcal{A}(U(\mu))\) generated by \(U(\mu)\) are unique up to equivalence, the physical counterpart of the problem being whether the algebra of the translations uniquely determines the dynamics of the system. The question was answered in the affirmative long ago by Wallen [8, 9], who proved that any strongly continuous nilpotent semigroup of partial isometries on a separable Hilbert space \(\mathcal{H}\) obeying equations (1.2) – (1.4) is a direct integral of truncated shifts, and consequently the irreducible representations of \(\mathcal{A}(U(\mu))\) are equivalent to those of \(\mathcal{A}(S(\mu))\).

Wallen’s proof started from the case in which the von Neumann algebra generated by \(U(\mu)\) is a factor, and relied on a deep analysis of power partial isometries [4], although it was later much simplified [3]. In this note we prove the results of Wallen for the factors in a very direct way, thanks to a simple yet powerful theorem (Theorem 1) which follows immediately from the general properties of partial isometries [2]. This theorem is equivalent to [3, Theorem A], and our proof is an alternative proof of it. After introducing the necessary definitions and obtaining some preliminary results in Section 2, in Section 3 we prove the key theorem which drastically simplifies the rest of the paper. In Section 4 we prove that the irreducible representations of
\[ U(\mu) \text{ in a Hilbert space are unique up to equivalence, and that if the algebra generated by } U(\mu) \text{ is a factor then it is completely reducible. In Section 5 we apply our results to a wide class of semigroups of isometries (those that do not contain a unitary component), for which we give a proof, based on the results of Section 4, that the irreducible representations are equivalent.} \]

2. Definitions and preliminary results

A partial isometry in a Hilbert space \( \mathcal{H} \) is a linear operator \( T \) such that
\[ \mathcal{H} = \mathcal{H}' \oplus \mathcal{H}'', \quad \| Tx' \| = \| x' \| \text{ if } x' \in \mathcal{H}', \quad T x'' = 0 \text{ if } x'' \in \mathcal{H}''. \]
If \( T^\dagger \) is the adjoint of \( T \), the operators \( T^\dagger T \) and \( TT^\dagger \) are the projections onto \( \mathcal{H}' \) and \( \mathcal{T} \) respectively [2]. As a consequence, the following equations hold:
\[ T = TT^\dagger T, \quad T^\dagger = T^\dagger TT^\dagger, \quad (2.1) \]
which in turn imply that \( T \) and \( T^\dagger \) are partial isometries.

We assume that a strongly continuous semigroup \( \mathcal{U} = \{ U(\alpha), 0 \leq \alpha \leq 1 \} \) is given in a separable Hilbert space \( \mathcal{H} \):
\[ U(\alpha)U(\beta) = U(\alpha + \beta) \quad (2.2) \]
such that each \( U(\alpha) \) is a partial isometry and equations (1.2) – (1.4) are satisfied with \( \mu = 1 \).

For the sake of completeness we briefly report some well-known facts about semigroups of partial isometries.

**Lemma 1.** \( \{ U^\dagger(\alpha) \} \) is a strongly continuous semigroup.

**Proof.** The semigroup property is seen to hold by taking the adjoint of equation (2.2). Strong continuity at \( \alpha = 0 \) follows from
\[ 0 \leq \| x - U^\dagger(\alpha)x \|^2 \leq 2\| x \|^2 - \langle x, U(\alpha)x \rangle - \langle U(\alpha)x, x \rangle. \]
The semigroup property forces strong continuity for any \( \alpha \).

We define
\[ E(\alpha) \equiv U(\alpha)U^\dagger(\alpha), \quad P(\alpha) \equiv U^\dagger(1 - \alpha)U(1 - \alpha). \quad (2.3) \]

**Lemma 2.** If \( \alpha \leq \beta \), then
\[ E(\alpha)E(\beta) = E(\beta) = E(\beta)E(\alpha) \quad \text{and} \quad P(\alpha)P(\beta) = P(\alpha) = P(\beta)P(\alpha). \]

**Proof.** \( E(\alpha)E(\beta) = [U(\alpha)U^\dagger(\alpha)U(\alpha)]U(\beta - \alpha)U^\dagger(\beta) = E(\beta) \), due to equation (2.1). Taking the adjoint yields \( E(\beta)E(\alpha) = E(\beta) \). The relations for \( P(\alpha) \) are proved in a similar way.

Thus, \( \{ E(\alpha) \} \) and \( \{ P(\alpha) \} \) are two families of strongly continuous commuting projections, due to the strong continuity of \( \mathcal{U} \). It is convenient to extend by continuity the definition of \( E(\alpha) \) and \( P(\alpha) \): \( E(\alpha) = I \) for \( \alpha < 0 \), and \( P(\alpha) = I \) for \( \alpha > 1 \).

We have the following lemma.

**Lemma 3.** \( [E(\alpha), P(\beta)] = 0 \).

**Proof.** The product \( E(\alpha)P(\beta) \) is idempotent and \( \| E(\alpha)P(\beta) \| \leq 1 \). Hence \( E(\alpha)P(\beta) \) is a projection, so that it equals its adjoint.
**LEMMA 4.** Let \( \alpha \leq \beta \). The following relations hold.

If \( \beta \geq 0 \), then \( E(\alpha)U(\beta) = U(\beta) \) and \( P(\alpha)U(\beta) = 0 \).

If \( \alpha \geq 0 \), then \( E(\beta)U(\alpha) = U(\alpha)E(\beta - \alpha) \) and \( P(\beta)U(\alpha) = U(\alpha)P(\beta - \alpha) \).

For \( \alpha \geq 0 \) and any \( \lambda \), then

\[
U(\alpha)E(\lambda) = E(\alpha + \lambda)U(\alpha) \quad \text{and} \quad U(\alpha)P(\lambda) = P(\alpha + \lambda)U(\alpha).
\]

**Proof.** By equation (2.1) and Lemma 3,

\[
E(\alpha)U(\beta) = [U(\alpha)U(\alpha)]U(\beta - \alpha) = U(\alpha)U(\beta - \alpha) = U(\beta);
\]

\[
E(\beta)U(\alpha) = U(\alpha)[U(\beta - \alpha)U(\beta - \alpha)][U(\alpha)U(\alpha)]
\]

\[
= [U(\alpha)U(\alpha)U(\alpha)]U(\beta - \alpha) = U(\alpha)E(\beta - \alpha);
\]

\[
U(\alpha)E(\lambda) = U(\alpha)P(1 - \alpha)E(\lambda) = U(\alpha)E(\alpha)P(1 - \alpha) = E(\alpha + \lambda)U(\alpha).
\]

The relations involving \( P(\alpha) \) are proved in a similar way. \( \square \)

3. A useful theorem

In what follows we will need the operators

\[
Z(\lambda) \equiv \int_{0}^{\lambda} X(\alpha) d\alpha \quad (3.1)
\]

with

\[
X(\alpha) \equiv E(\alpha) + P(\alpha), \quad \alpha \in [0, 1]. \quad (3.2)
\]

\( X(\alpha) \) is a projection, since \( P(\alpha)E(\alpha) = 0 \). The integral in equation (3.1) clearly converges, and hence the operators \( Z(\lambda) \) belong to the von Neumann algebra \( \mathcal{A}(U) \) generated by \( U \).

Moreover, if

\[
Z \equiv \int_{0}^{1} X(\alpha) d\alpha, \quad (3.3)
\]

we have

\[
\|Z(\lambda) - Z\| \overset{\lambda \to 1}{\to} 0 \quad (3.4)
\]

since

\[
\left\| \int_{0}^{\lambda} X(\alpha) d\alpha \right\| \leq \int_{0}^{\lambda} \|X(\alpha)\| d\alpha = 1 - \lambda. \quad (3.5)
\]

The purpose of this section is to prove the following theorem.

**THEOREM 1.** If the von Neumann algebra \( \mathcal{A}(U) \) is a factor, then \( X(\alpha) = I \) for \( \alpha \geq 0 \).

We begin with some lemmas.

**LEMMA 5.** \( Z(\lambda)U(\lambda) = \lambda U(\lambda) \).

**Proof.** By Lemma 4, \( X(\alpha)U(\lambda) = U(\lambda) \) if \( \alpha \leq \lambda \). Integration over \([0, \lambda]\) yields the result. \( \square \)

**LEMMA 6.** \( Z \) commutes with \( U(\alpha), \alpha \in [0, 1] \).

**Proof.** By Lemma 5, we have

\[
\int_{0}^{1} X(\alpha) d\alpha U(\lambda) = \lambda U(\lambda) + \int_{\lambda}^{1} X(\alpha) d\alpha U(\lambda).
\]
On the other hand, by Lemma 4, \( U(\lambda)X(\alpha) = X(\alpha + \lambda)U(\lambda) \). Integration over \([0,1]\) yields
\[
U(\lambda) \int_0^1 X(\alpha) \, d\alpha = \int_0^{1+\lambda} X(\alpha) \, d\alpha U(\lambda) = \int_0^1 X(\alpha) \, d\alpha U(\lambda) + \lambda U(\lambda).
\]

Lemma 7. If \( \mathcal{A}(\mathcal{U}) \) is a factor, then \( Z = \mu I \).

Proof. Being self-adjoint, by Lemma 6 the operator \( Z \) is in the centre of the algebra. If the algebra is a factor, then \( Z = \mu I \).

We are now in the position to prove Theorem 1: \( X(\alpha) = I \) for \( \alpha \geq 0 \).

Proof of Theorem 1. By Lemma 5 and equation (1.2), for \( 0 \leq \lambda < 1 \) we have
\[
\lambda = \|Z(\lambda)U(\lambda)\| < \|Z(\lambda)\| \|U(\lambda)\| = \|Z(\lambda)\|.
\]
By equation (3.5), we have
\[
\|Z\| - \|Z(\lambda)\| < \|Z - Z(\lambda)\| < 1 - \lambda.
\]
As a consequence,
\[
\lim_{\lambda \to 1} \|Z(\lambda)\| = 1.
\]
By Lemma 7, since \( Z \) is positive, we conclude that \( Z = I \). Since the operators \( X(\alpha) \) are strongly continuous projections for any \( \alpha \), we must have \( X(\alpha) = I \) for \( 0 \leq \alpha < 1 \). For \( \alpha \geq 1 \), we have \( X(\alpha) = I \) by the definition of \( E(\alpha) \) and \( P(\alpha) \).

We note that Theorem 1 is a restatement of the crucial [8, Lemma 4], or [3, Theorem A], which says that, if \( \mathcal{A}(\mathcal{U}) \) is a factor, then \( \ker(U(\alpha)) = \text{range}(U(1 - \alpha)) \). We also stress that the hypothesis of Theorem 1 cannot be weakened. Actually, in the next section, in Theorem 2, we will prove the converse of Theorem 1: if \( X(\alpha) = I \), then \( \mathcal{A}(\mathcal{U}) \) is a factor. Therefore, if \( \mathcal{A}(\mathcal{U}) \) is not a factor, then \( X(\alpha) \neq I \).

As an example, let \( \mathcal{H} = L^2(0,1) \oplus L^2(0,1) \equiv \mathcal{H}_1 \oplus \mathcal{H}_2 \), and let \( \hat{U}(\alpha) \) be defined as follows:
\[
\hat{U}(\alpha)\{f(x), g(x)\} \equiv \{f(x-\alpha), g(x-2\alpha)\} = \{U(\alpha)f(x), U(2\alpha)g(x)\}.
\]
The operators \( \hat{U}(\alpha) \) are a semigroup of partial isometries of index 1, and the von Neumann algebra generated by the operators \( \hat{U}(\alpha) \) is not a factor: the operator \( \hat{Z} \equiv \int_0^1 \hat{X}(\alpha) \, d\alpha \), with
\[
\hat{X}(\alpha) \equiv \hat{E}(\alpha) + \hat{P}(\alpha) \equiv \hat{U}(\alpha)\hat{U}^\dagger(\alpha) + \hat{U}^\dagger(1-\alpha)\hat{U}(1-\alpha),
\]
belongs to the centre of the algebra (by Lemma 6) and is not a multiple of the identity operator. In fact, we have
\[
\hat{E}(\alpha)\{f(x), g(x)\} = \begin{cases} \{\chi_{[\alpha,1]}f(x), \chi_{[2\alpha,1]}g(x)\} & \alpha \leq 1/2, \\ \{\chi_{[0,1]}f(x), 0\} & \alpha > 1/2; \end{cases}
\]
\[
\hat{P}(\alpha)\{f(x), g(x)\} = \begin{cases} \{\chi_{[0,\alpha]}f(x), 0\} & \alpha \leq 1/2, \\ \{\chi_{[0,\alpha]}f(x), \chi_{[0,2\alpha-1]}g(x)\} & \alpha > 1/2. \end{cases}
\]
As a consequence, \( \hat{X}(\alpha) \neq I \):
\[
\hat{X}(\alpha)\{f(x), g(x)\} = \begin{cases} \{f(x), \chi_{[2\alpha,1]}g(x)\} & \alpha \leq 1/2, \\ \{f(x), \chi_{[0,2\alpha-1]}g(x)\} & \alpha > 1/2. \end{cases}
\]
It is easy to verify that \( \hat{Z}\{f(x), g(x)\} = \{f(x), \frac{1}{2} g(x)\} \).
For any factor into which \( \mathcal{A}(U) \) decomposes (generally as a direct integral), \( \hat{Z} \) is the corresponding index. For example, if \( \mathcal{H} = L^2(0, 1) \oplus L^2(0, 1) \oplus L^2(0, 1) \) and 
\[
\hat{U}(\alpha)\{f(x), g(x), h(x)\} = \{f(x-\alpha), g(x-\alpha), h(x-2\alpha)\},
\]
then \( \hat{Z}\{f(x), g(x), h(x)\} = \{f(x), g(x), \frac{1}{2}h(x)\} \).

4. **Uniqueness and complete reducibility**

In this section we will prove the main results of the paper:
(a) the equation \( X(\alpha) = I \) implies that \( \mathcal{A}(U) \) is a factor;
(b) the irreducible representations of \( \mathcal{A}(U) \) are unique up to equivalence; and
(c) if \( \mathcal{A}(U) \) is a factor, then it is completely reducible.
These results will be obtained by proving that any algebra \( \mathcal{A}(U) \) with \( X(\alpha) = I \) can be seen as a Weyl algebra \( \mathcal{W}_1 \) pertaining to the segment \([0, 1]\) which is a factor.

A Weyl algebra \( \mathcal{W}_1 \) is generated by a set of operators \( \{\hat{U}(\alpha), \hat{V}(n), \alpha \in \mathbb{R}, n \in \mathbb{Z}\} \) obeying the Weyl form of the commutation relations
\[
\hat{V}(n)\hat{U}(\alpha) = \exp(-2\pi in\alpha)\hat{U}(\alpha)\hat{V}(n) \quad (4.1)
\]
with \( \hat{V}(n) \) unitary operators such that \( \hat{V}(p)\hat{V}(q) = \hat{V}(p+q) \), and \( \{\hat{U}(\alpha)\} \) a strongly continuous group of unitary operators with \( \hat{U}(0) = I \). If \( \mathcal{W}_1 \) is a factor, then \( \hat{U}(1) = \exp(i\phi)I \). Since these algebras turn out to be completely reducible, and their irreducible representations are equivalent (for a given \( \phi \)), points (a), (b) and (c) are a consequence.

**Lemma 8.** The irreducible representations of \( \mathcal{W}_1 \) with a given value of \( \phi \) are unique up to equivalence. Any representation of \( \mathcal{W}_1 \) with \( \hat{U}(1) = \exp(i\phi)I \) is completely reducible, and is a factor.

**Proof.** Let \( \phi = 0 : \hat{U}(1) = I \). Then \( \{\hat{U}(\alpha)\} \) is a periodic group and the operators
\[
Q_n = \int_0^1 \hat{U}(\alpha) \exp(-2\pi in\alpha) \, d\alpha \quad (4.2)
\]
are orthogonal projections such that \( \sum_{n=-\infty}^{\infty} Q_n = I \); see [5]. Equation (4.1) implies that
\[
\hat{V}(n)Q_0 = Q_n \hat{V}(n),
\]
and hence the spaces \( \mathcal{H}_n = Q_n\mathcal{H} \) are isometrically connected. From equation (4.2) we find that \( \hat{U}(\alpha)Q_0 = Q_0 \), which implies that if \( x_0 \in \mathcal{H}_0 \), then \( \hat{U}(\alpha)x_0 = x_0 \). As a consequence of equation (4.1), \( \hat{U}(\alpha)\hat{V}(n)x_0 = \exp(2\pi in\alpha)\hat{V}(n)x_0 \), and the subspace spanned by \( \{\hat{V}(n)x_0, n \in \mathbb{Z}\} \) is invariant under all the operators of \( \mathcal{W}_1 \).

Then, for an irreducible representation \( \mathcal{H}_0 \) must be one-dimensional, and two irreducible representations are equivalent.

If \( \mathcal{H}_0 \) is not one-dimensional, given an orthonormal basis \( \{e_k^0\} \) of \( \mathcal{H}_0 \), the subspaces \( \mathcal{V}_k \) generated by \( \{\hat{V}(n)e_k^0, n \in \mathbb{Z}\} \) are irreducible and the representations in \( \mathcal{V}_k \) are equivalent. As a consequence, if \( T \) belongs to the centre of \( \mathcal{W}_1 \), it must be the same multiple of the identity in any \( \mathcal{V}_k \); that is, \( \mathcal{W}_1 \) is a factor.

If \( \phi \neq 0 \), the operators \( \hat{U}'(\alpha) \equiv \exp(-i\alpha\phi)\hat{U}(\alpha) \) are such that \( \hat{U}'(1) = I \), and the same conclusions follow as well.

The irreducible representations of \( \mathcal{W}_1 \) are unitarily equivalent to the algebra generated by the unitary translations in \( L^2[0, 1] \) with generator \( P_\phi \):
\[
P_\phi f = -if', \quad \mathcal{D}_{P_\phi} = \{f : f \text{ absolutely continuous, } f(0) = \exp(i\phi)f(1), f' \in L^2\},
\]
and multiplications by \(\exp(-2\pi inx)\). Henceforth we will focus our attention on the case \(\phi = 0\),
but any choice of \(\phi\) would work as well.
In the next two lemmas we prove that an algebra \(A(U)\) with \(X(\alpha) = I\) includes a Weyl
algebra with \(\phi = 0\).

**Lemma 9.** Let \(U\) be such that \(X(\alpha) = I\). Then the operators
\[
\tilde{U}(\alpha) \equiv U(\alpha) + U^\dagger(1 - \alpha), \quad \alpha \in [0, 1],
\]
are unitary. If the definition of \(\tilde{U}(\alpha)\) is extended to any \(\alpha \in \mathbb{R}\) by periodicity, that is
\[
\tilde{U}(\alpha + n) \equiv \tilde{U}(\alpha), \quad n \in \mathbb{Z}, \; \alpha \in [0, 1],
\]
a strongly continuous periodic group with \(\tilde{U}(0) = \tilde{U}(1) = I\) is obtained.

**Proof.** We have
\[
\tilde{U}(\alpha)\tilde{U}^\dagger(\alpha) = U(\alpha)U^\dagger(\alpha) + U^\dagger(1 - \alpha)U(1 - \alpha) = E(\alpha) + P(\alpha) = X(\alpha) = I.
\]
Since \(\tilde{U}^\dagger(\alpha) = \tilde{U}(1 - \alpha)\), we also have \(\tilde{U}^\dagger(\alpha)\tilde{U}(\alpha) = I\). Since \(\tilde{U}(1) = I\), by Lemma 1 the
periodic extension of \(\tilde{U}(\alpha)\) to any \(\alpha \in \mathbb{R}\) is strongly continuous.

The group composition law is satisfied. Given \(\alpha, \beta \in [0, 1]\) and \(n, k \in \mathbb{Z}\), then if \(\alpha + \beta \leq 1\), we have
\[
\tilde{U}(\alpha + n)\tilde{U}(\beta + k) = U(\alpha + \beta) + U^\dagger(1 - \alpha)U(\beta) + U(\alpha)U^\dagger(1 - \beta)
\]
\[
= U(\alpha + \beta) + U^\dagger(1 - \alpha - \beta)P(1 - \beta) + E(\alpha)U^\dagger(1 - \alpha - \beta)
\]
\[
= U(\alpha + \beta) + U^\dagger(1 - \alpha - \beta)X(1 - \beta) = \tilde{U}(\alpha + \beta + n + k);
\]
if \(1 \leq \alpha + \beta = 1 + \delta\), we have
\[
\tilde{U}(\alpha + n)\tilde{U}(\beta + k) = U^\dagger(1 - \alpha)U(\beta) + U(\alpha)U^\dagger(1 - \beta) + U^\dagger(1 - \delta)
\]
\[
= P(1 - \beta + \delta)U(\delta) + U(\delta)E(\alpha - \delta) + U^\dagger(1 - \delta)
\]
\[
= X(\alpha)U(\delta) + U^\dagger(1 - \delta) = \tilde{U}(\alpha + \beta + n + k),
\]
by Lemma 4 and the hypothesis that \(X(\alpha) = I\).

**Lemma 10.** Let
\[
\tilde{V}(n) \equiv \int_0^1 \exp(-2\pi in\gamma) dP(\gamma), \quad n \in \mathbb{Z}
\]
with \(\{P(\gamma)\}\) defined by equation (2.3). \(\{\tilde{V}(n)\}\) is a group of unitary operators such that
\(\tilde{V}(n)\tilde{V}(k) = \tilde{V}(n + k)\). If \(U\) is such that \(X(\alpha) = I\), the multiplication law, given in equation
(4.1), with the operators \(\tilde{U}(\alpha)\) defined in Lemma 9, is satisfied.

**Proof.** By Lemma 2, the projections \(P(\alpha), \alpha \in [0, 1]\), are a resolution of the identity. As a
consequence, \(\tilde{V}(n)\) is unitary for any \(n\), and the group composition law holds. For \(\alpha \in [0, 1]\),
by Lemma 4 we have
\[
\tilde{U}(\alpha)\tilde{V}(n) = \int_0^1 \exp(-2\pi in\gamma) dP(\alpha + \gamma) U(\alpha) + \int_{1-\alpha}^1 \exp(-2\pi in\gamma) dP(\alpha + \gamma - 1) U^\dagger(1 - \alpha)
\]
\[
= \exp(2\pi in\alpha) \left[ \int_0^1 \exp(-2\pi in\sigma) dP(\sigma) U(\alpha) + \int_{1-\alpha}^\alpha \exp(-2\pi in\sigma) dP(\sigma) U^\dagger(1 - \alpha) \right].
\]
Both integrals can be extended to the full interval \([0, 1]\), since, again by Lemma 4, \(P(\sigma)U(\alpha)\) and
\(P(\sigma)U^\dagger(1 - \alpha)\) are independent of \(\sigma\) in the intervals \([0, \alpha]\) and \([\alpha, 1]\) respectively. In conclusion,
one obtains $\tilde{U}(\alpha)\tilde{V}(n) = \exp(2\pi i\alpha)\tilde{V}(n)\tilde{U}(\alpha)$. Due to the periodicity of both $\tilde{U}(\alpha)$ and $\exp(2\pi i\alpha)$, the result holds for any $\alpha$. \hfill \Box

Now we prove the converse of Theorem 1, as follows.

THEOREM 2. Let $\mathcal{U}$ be a strongly continuous semigroup of partial isometries of index 1. If $X(\alpha) = I$, $0 \leq \alpha \leq 1$, then $\mathcal{A}(\mathcal{U})$ is a factor.

Proof. By Lemmas 9 and 10, the algebra $\mathcal{A}(\mathcal{U})$ includes an algebra $\mathcal{W}_1$ with $\tilde{U}(1) = I$. Hence, denoting by $Z$ the centre of an algebra, we have $T \in Z_A \Rightarrow T \in Z_{\mathcal{W}}$. Since, by Lemma 8, we know that $\mathcal{W}_1$ is a factor, $T = \lambda I$. \hfill \Box

LEMMA 11. Let $\mathcal{U}$ be such that $X(\alpha) = I$. For $\alpha \in [0, 1]$, $n \in \mathbb{Z}$, we have $E(\alpha)\tilde{U}(\alpha + n) = U(\alpha)$, where $\tilde{U}(\alpha)$ is as defined in Lemma 9.

Proof. Due to the periodicity of $\tilde{U}(\alpha)$ and equation (2.1), we have

$$E(\alpha)\tilde{U}(\alpha + n) = U(\alpha)U^\dagger(\alpha)U(\alpha) = U(\alpha).$$ \hfill \Box

LEMMA 12. Let $\mathcal{A}(\mathcal{U})$ be a factor. Then the algebra $\mathcal{W}_1$ defined in Lemmas 9 and 10 is equal to $\mathcal{A}(\mathcal{U})$.

Proof. We have $\mathcal{W}_1 \subset \mathcal{A}(\mathcal{U})$. If $T \in \mathcal{W}_1$, the commutant of $\mathcal{W}_1$, $T$ commutes with the projections $P(\alpha)$ and $E(\alpha)$ since it commutes with the operators $\tilde{V}(n)$. Then, by Lemma 11, $T \in \mathcal{A}(\mathcal{U})'$; that is, $\mathcal{W}_1 \subset \mathcal{A}(\mathcal{U})'$. Taking the commutant and recalling von Neumann's density theorem ($\mathcal{M}'' = \mathcal{M}$ for von Neumann algebras), we find that $\mathcal{W}_1 \supset \mathcal{A}(\mathcal{U})$, and hence $\mathcal{W}_1 = \mathcal{A}(\mathcal{U})$. \hfill \Box

In the next theorems we prove the uniqueness of the irreducible representations of $\mathcal{A}(\mathcal{U})$ and the complete reducibility of $\mathcal{A}(\mathcal{U})$ when it is a factor. Note that by Theorems 1 and 2 the hypothesis that $X(\alpha) = I$ of Lemmas 9, 10 and 11 is equivalent to the hypothesis that $\mathcal{A}(\mathcal{U})$ is a factor.

THEOREM 3. Let $\mathcal{H}$ be irreducible under $\mathcal{A}(\mathcal{U})$. Then $\mathcal{U}$ is unitarily equivalent to the semigroup $\mathcal{S}(1)$ of truncated shifts defined by equation (1.1).

Proof. By Lemma 12, $\mathcal{H}$ is irreducible with respect to the algebra $\mathcal{W}_1$ defined in Lemmas 9 and 10. The uniqueness of the irreducible representations of $\mathcal{W}_1$ implies that the representation with $\{\tilde{U}(\alpha), \tilde{V}(n)\}$ is equivalent to the standard representation in $L^2[0, 1]$ defined as follows:

$$\tilde{U}(\alpha)f(x) = \begin{cases} f(x - \alpha), & \alpha < x \leq 1, \\ f(x + 1 - \alpha), & 0 \leq x \leq \alpha, \end{cases}$$

$$\tilde{V}(n)f(x) = \exp(-2\pi inx)f(x).$$

If $S : \mathcal{H} \rightarrow L^2[0, 1]$ is the operator such that

$$S\tilde{U}(\alpha)S^{-1} = \tilde{U}(\alpha), \quad S\tilde{V}(n)S^{-1} = \tilde{V}(n),$$

and $\tilde{E}(\alpha) = SE(\alpha)S^{-1}, \tilde{P}(\alpha) = SP(\alpha)S^{-1}$, then by Lemma 11 the operators $U_T(\alpha) \equiv \tilde{E}(\alpha)\tilde{U}(\alpha) = SU(\alpha)S^{-1}$, $\alpha \in [0, 1]$, constitute a semigroup of truncated shifts in $L^2[0, 1]$ of index 1; that is, $\|U_T(\alpha)\| = 1$ if $\alpha < 1$ and $U_T(1) = 0$. The equivalence of $\mathcal{U}$ with $\mathcal{S}(1)$ is proved. \hfill \Box
Theorem 4. If $\mathcal{A}(\mathcal{U})$ is a factor, then it is completely reducible.

Proof. If $\mathcal{A}(\mathcal{U})$ is a factor, then $X(\alpha) = I$, by Theorem 1. By Lemma 12, $\mathcal{A}(\mathcal{U}) = \mathcal{W}_1$, which is completely reducible. \qed

For the sake of completeness we prove that the roles of $\mathcal{A}(\mathcal{U})$ and $\mathcal{W}_1$ can be exchanged, in the sense that if $\mathcal{W}_1$ is a Weyl algebra with $\tilde{U}(1) = I$, then it contains an algebra $\mathcal{A}(\mathcal{U})$ with $X(\alpha) = I$ which, on the grounds of arguments as in Lemma 12, turns out to be equal to $\mathcal{W}_1$.

Statement 1. Let $\mathcal{W}_1$ be a Weyl algebra $\{\tilde{U}(\alpha), \tilde{V}(n)\}$ with $\tilde{U}(1) = I$. Let $d_\alpha(n)$ be the Fourier coefficients of the characteristic function $e_\alpha(x)$ of the interval $[0, \alpha]$ with respect to $\{\exp(-2\pi nix)\}$. The operators

$$P(\alpha) \equiv \sum_{-\infty}^{\infty} d_\alpha(n)\tilde{V}(n), \quad \alpha \in [0, 1],$$

are a spectral resolution of the identity. If $\alpha + \beta \leq 1$, then $\tilde{U}(\alpha)P(\beta) = [P(\alpha + \beta) - P(\alpha)]\tilde{U}(\alpha)$.

Proof. If $\alpha \leq \beta$, from $e_\alpha(x)e_\beta(x) = e_\alpha(x)$ we find that $\sum d_\alpha(n)d_\beta(s - n) = d_\alpha(s)$, whence it follows that $\{P(\alpha)\}$ is a spectral resolution of the identity. Since the Fourier coefficients of $e_{\alpha + \beta}(x) - e_\alpha(x)$ are $d_\beta(n)\exp(2\pi \pi \alpha)$, we find that

$$\tilde{U}(\alpha)P(\beta) = \sum d_\beta(n)\exp(2\pi \pi \alpha)\tilde{V}(n)\tilde{U}(\alpha) = [P(\alpha + \beta) - P(\alpha)]\tilde{U}(\alpha), \quad (4.3)$$

as required. \qed

Statement 2. Let $\mathcal{W}_1$ be a Weyl algebra with $\tilde{U}(1) = I$. Then $\mathcal{W}_1$ contains an algebra $\mathcal{A}(\mathcal{U})$ generated by a semigroup $\mathcal{U}$ of partial isometries of index 1, with $X(\alpha) = I$.

Proof. Given $\mathcal{W}_1$ with operators $\{\tilde{U}(\alpha), \tilde{V}(n)\}$, we define $P(\alpha), \alpha \in [0, 1]$, as in Statement 1, $E(\alpha) \equiv I - P(\alpha)$ and $U(\alpha) \equiv E(\alpha)\tilde{U}(\alpha)$. Using equation (4.1) and Statement 1 one can verify that the operators $U(\alpha)$ are a semigroup of partial isometries of index 1. Moreover, $U(\alpha)U^\dagger(\alpha) = E(\alpha)$, and by equation (4.3) with $\alpha = 1 - \gamma$ and $\beta = \gamma$, we have $U^\dagger(1 - \gamma)U(1 - \gamma) = P(\gamma)$. Hence $X(\alpha) = U(\alpha)U^\dagger(\alpha) + U^\dagger(1 - \alpha)U(1 - \alpha) = I$. \qed

Finally, we have the following proposition.

Proposition. Any Weyl algebra $\mathcal{W}_1$ with $\tilde{U}(1) = I$ is equal to an algebra $\mathcal{A}(\mathcal{U})$ that is a factor.

Proof. Given a Weyl algebra $\mathcal{W}_1$ that is a factor, we construct an algebra $\mathcal{A}(\mathcal{U})$ according to Statements 1 and 2. Since $X(\alpha) = I$, this latter algebra is a factor, by Theorem 2. The argument of Lemma 12 can be applied with the roles of $\mathcal{W}_1$ and $\mathcal{A}(\mathcal{U})$ exchanged. In fact, $\tilde{U}(\alpha)$ is recovered from the operators $U(\alpha)$ and $U^\dagger(\alpha)$ as $\tilde{U}(\alpha) = U(\alpha) + U^\dagger(1 - \alpha)$:

$$E(\alpha)\tilde{U}(\alpha) + \tilde{U}^\dagger(1 - \alpha)E(1 - \alpha) = [I - P(\alpha)]\tilde{U}(\alpha) + \tilde{U}(\alpha)[I - P(1 - \alpha)]$$

$$= 2\tilde{U}(\alpha) - P(\alpha)\tilde{U}(\alpha) - [P(1)\tilde{U}(\alpha) - P(\alpha)\tilde{U}(\alpha)] = \tilde{U}(\alpha)$$

having used equation (4.3), and the operators $\tilde{V}(n)$ can be expressed as in Lemma 10 by means of the projections $P(\alpha)$ (equal to $U^\dagger(1 - \alpha)U(1 - \alpha)$, as proved in Statement 2). Hence $T \in \mathcal{A}(\mathcal{U})^\dagger \Rightarrow T \in \mathcal{W}'_1$. As a consequence, $\mathcal{W}_1$ is equal to $\mathcal{A}(\mathcal{U})$. \qed
Note that, since the operators $\tilde{U}'(\alpha) = \exp(i\alpha \phi)\tilde{U}(\alpha)$ belong to the algebra if the operators $\tilde{U}(\alpha)$ do, the above result holds for any Weyl algebra with $\tilde{U}(1) = \exp(i\phi)I$.

5. Application to semigroups of isometries

In this section we will apply the previous results to prove that the irreducible representations of the algebra $\mathcal{A}_+$ generated by a strongly continuous semigroup of isometries $\mathcal{U}_+ = \{U_+(\alpha), \alpha \geq 0\}$ such that $U^\dagger_+(\alpha)x \xrightarrow{\alpha \to \infty} 0$ are unique up to isomorphism. The most obvious example of $\mathcal{U}_+$ is given in $L^2(0, \infty)$ by the operators

$$U_+(\alpha)f(x) = g(x) = \begin{cases} 0 & \text{if } 0 \leq x < \alpha, \\ f(x - \alpha) & \text{if } \alpha \leq x. \end{cases}$$  

(5.1)

In order to clarify the restriction on $\mathcal{U}_+$, we recall some facts. The multiplication law of $U_+(\alpha_1), U_+(\alpha_2)$ is most easily expressed in terms of the projections

$$E_+^{(\alpha)} = U_+(\alpha)U_+^\dagger(\alpha).$$  

(5.2)

If $\alpha_1 \leq \alpha_2$, the following equations are verified:

$$U_+^\dagger(\alpha_2)U_+(\alpha_1) = U_+^\dagger(\alpha_2 - \alpha_1); \quad U_+^\dagger(\alpha_1)U_+(\alpha_2) = U_+(\alpha_2 - \alpha_1);$$  

(5.3)

$$U_+(\alpha_2)U_+^\dagger(\alpha_1) = U_+(\alpha_2 - \alpha_1)E_+(\alpha_1) = E_+(\alpha_2)U_+(\alpha_2 - \alpha_1);$$  

(5.4)

$$U_+(\alpha_1)U_+^\dagger(\alpha_2) = E_+(\alpha_1)U_+^\dagger(\alpha_2 - \alpha_1) = U_+^\dagger(\alpha_2 - \alpha_1)E_+(\alpha_2).$$  

(5.5)

The operators $E_+^{(\alpha)}$ are a strongly continuous decreasing family of projections such that $E_+^{(\alpha_1)}E_+^{(\alpha_2)} = E_+^{(\alpha_2)}E_+^{(\alpha_1)} = E_+^{(\alpha_2)}$ if $\alpha_1 \leq \alpha_2$. If $\mathcal{K}$ is the Hilbert space where $\mathcal{A}_+$ is represented and $E_+^{(\infty)}$ is the strong limit of $E_+^{(\alpha)}$ for $\alpha \to \infty$, the subspace $\mathcal{K}\equiv E_+(\infty)\mathcal{K}$ is invariant under the operators of $\mathcal{A}_+$. The restriction of $E_+^{(\infty)}$ to $\mathcal{K} \equiv \mathcal{K}\infty$ vanishes, whereas the restriction of $E_+^{(\alpha)}$ to $\mathcal{K}\infty$ is the identity: $E_+^{(\alpha)}E_+(\infty)x = E_+(\infty)x$. The condition that $U_+^\dagger(\alpha)x \xrightarrow{\alpha \to \infty} 0$ is equivalent to the condition

$$E_+^{(\alpha)}x \xrightarrow{\alpha \to \infty} 0, \quad \text{for all } x \in \mathcal{K},$$  

(5.6)

which implies that the projections $\{P_+(\alpha) \equiv I - E_+^{(\alpha)}\}$ are a resolution of the identity.

The complete reducibility of $\mathcal{A}_+$ was first proved in [1], and is warranted by the Wold decomposition theorem, which states that any strongly continuous semigroup of isometries decomposes uniquely into a unitary part and a shift [6]. Since equation (5.6) implies that $\mathcal{U}_+$ does not contain a unitary part, the space $\mathcal{K}$ decomposes into a direct sum of invariant subspaces in which $\mathcal{U}_+$ acts as a semigroup of shift operators. Also, the uniqueness of the irreducible representations of $\mathcal{A}_+$ follows from the Wold decomposition. We will give an alternative proof by showing that $\mathcal{K}$ can be decomposed into a direct sum $\mathcal{K} = \oplus \mathcal{H}_n$, each $\mathcal{H}_n$ being identifiable with a space $\mathcal{H}$ in which $\mathcal{U}_+$ induces an irreducible representation of $\mathcal{A}(\mathcal{U})$ with $X(\alpha) = I$. The uniqueness of the irreducible representations of $\mathcal{A}_+$ will then follow from Theorem 3.

**Lemma 13.** Let $\mathcal{H} \equiv P_+(1)\mathcal{K}$ and $U(\alpha) \equiv P_+(1)U_+^{(\alpha)}$. Then $\{U(\alpha), \alpha \geq 0\}$ is a strongly continuous semigroup of partial isometries of index 1 in $\mathcal{H}$ such that, for $0 \leq \alpha \leq 1$,

$$X(\alpha) \equiv U(\alpha)U^\dagger(\alpha) + U^\dagger(1 - \alpha)U(1 - \alpha) = P_+(1) = I_{\mathcal{H}}.$$

**Proof.** Since, by equation (5.5), $P_+(1)U_+^{(\alpha)} = P_+(1)U_+^{(\alpha)}P_+(1)$, the semigroup composition law holds: $P_+(1)U_+^{(\alpha_1)}P_+(1)U_+^{(\alpha_2)} = P_+(1)U_+^{(\alpha_1 + \alpha_2)}$. If $\alpha_1 + \alpha_2 \geq 1$, then $P_+(1)U_+^{(\alpha_1 + \alpha_2)} = 0$, because of equation (5.3). For $\alpha < 1$, we have $\|P_+(1)U_+^{(\alpha)}x\|^2 = (x, P_+(1 - \alpha)x) = \|x\|^2$ if $x = P_+(1 - \alpha)y$. Strong continuity follows from the strong continuity of the operators $U_+^{(\alpha)}$. 
Finally, \( U(\alpha)U^\dagger(\alpha) = P_1(1)E_2(\alpha) \) and, by equation (5.4), \( U^\dagger(1 - \alpha)U(1 - \alpha) = P_1(\alpha) = P_1(1)P_1(\alpha) \). Hence \( X(\alpha) = P_1(1) \). 

The purport of the following lemma is obvious when viewed in \( L^2(0, \infty) \).

**Lemma 14.** Let \( \mathcal{H}_n \equiv [P_1(n) - P_1(n-1)]K \), \( n \geq 1 \), \( \mathcal{H} \equiv \mathcal{H}_1 \). The spaces \( \mathcal{H}_n \) are isometrically connected with \( \mathcal{H} \). There is an isomorphism \( Z \) between \( \mathcal{K} \) and \( \mathcal{K}' \),

\[
\mathcal{K}' \equiv \left\{ \{x_n\}_n \mathbb{N}, x_n \in \mathcal{H}, \sum_{n=1}^\infty \|x_n\|^2 < \infty \right\},
\]

such that \( U'(\alpha) \equiv ZU_+^{(\alpha)}Z^{-1} \) is defined as follows: if \( \alpha = k + \delta, k \in \mathbb{N}, 0 \leq \delta < 1 \), then \( U'(\alpha) \{x_n\} = \{y_n\}, U'^\dagger(\alpha) \{x_n\} = \{z_n\}, \) with

\[
y_n = \begin{cases} 
0, & n \leq k, \\
U(\delta)x_{n-k} + U^\dagger(1 - \delta)x_{n-k-1}[1 - \delta_nk+1], & n > k;
\end{cases} 
\tag{5.7}
\]

\[
z_n = U^\dagger(\delta)x_{n+k} + U(1 - \delta)x_{n+k+1}. 
\tag{5.8}
\]

**Proof.** By equations (5.3), \( U_+^{(n-1)} : \mathcal{H}_n \rightarrow \mathcal{H} \) is an invertible isometry, the inverse being the operator \( U_+^{(n-1)} \). For all \( x \in \mathcal{K} \),

\[
x = \sum_{k=1}^\infty [P_1(n) - P_1(n-1)]x = \sum_{k=1}^\infty v_k, \quad v_k \in \mathcal{H}_k.
\]

Define \( x_k = U_+^{(n-1)}(k-1)v_k, x_k \in \mathcal{H} \); then the operator \( Z \) such that \( Zx \equiv \{x_k\} \), with inverse \( Z^{-1} \), \( Z^{-1} \{x_n\} \equiv \sum_{n=1}^\infty U_+^{(n-1)}x_n \), establishes an isomorphism between \( \mathcal{K} \) and \( \mathcal{K}' \). The equation \( X(\alpha) = P_1(1) = I_\mathcal{H} \) (Lemma 13) ensures that equations (5.7) and (5.8) hold true.

**Theorem 5.** The irreducible representations of \( \mathcal{A}_+ \) are unique up to equivalence.

**Proof.** With the notation of Lemmas 13 and 14, we first prove that \( \mathcal{H} \) is irreducible under \( \mathcal{A}(U) \). If \( T \) is a bounded operator in \( \mathcal{H} \) which commutes with \( U(\alpha) \) and \( U^\dagger(\alpha) \), let \( T_{K'} \) be defined as follows: \( T_{K'} \{x_n\} \equiv \{Tx_n\} \). By equations (5.7) and (5.8) one can verify that \( T_K \equiv Z^{-1}T_{K'}Z \) commutes with \( U_+^{(\alpha)} \) and \( U_+^{(\alpha)} \). Hence \( T_K = \lambda T_K' \), which implies that \( T_{K'} = \lambda T_{K'}' \), and consequently \( T = \lambda T_K \). So \( \mathcal{H} \) is irreducible under \( \mathcal{A}(U) \).

Given two irreducible representations \( U_1^{(\alpha)} \) and \( U_2^{(\alpha)} \) of \( \mathcal{A}_+ \) in \( K^{(1)} \) and \( K^{(2)} \) respectively, we prove their equivalence by showing that the representations \( U_1^{(\alpha)} \) and \( U_2^{(\alpha)} \) in \( K^{(1)} \) and \( K^{(2)} \) respectively (see Lemma 14) are equivalent. Since the spaces

\[
\mathcal{H}^{(i)} = P_1^{(i)}(1)K^{(i)}
\]

are irreducible under the algebras \( \mathcal{A}(U^{(i)}) \), by Theorem 3 there is an isomorphism \( S_{\mathcal{H}} \) such that \( S_{\mathcal{H}} U_1^{(\alpha)}(\alpha) = U_2^{(\alpha)}(\alpha) S_{\mathcal{H}} \). The isometric operator \( S_{K'} : K^{(1)} \rightarrow K^{(2)} \), defined as

\[
S_{K'} \{x_n^{(1)}\} = \{S_{\mathcal{H}}x_n^{(1)}\},
\]

by Lemma 14 is such that, for \( \alpha = k + \delta \) (where \( \theta(n) = 1 \) if \( n > 0; \theta(n) = 0 \) if \( n \leq 0 \)),

\[
S_K U_1^{(\alpha)}(k + \delta) \{x_n^{(1)}\} = S_{K'} \{[U_1^{(\delta)} x_n^{(1)} + U_1^{(\delta)} (1 - \delta)x_n^{(1)} - \delta_nk+1)] \theta(n - k)\} = \{[(U_2^{(\delta)} x_n^{(1)} + U_2^{(\delta)} (1 - \delta)x_n^{(1)} - \delta_nk+1)] \theta(n - k)\} = U_2^{(\delta)}(k + \delta) S_{K'} \{x_n^{(1)}\}.
\]

Hence \( U_1^{(\alpha)} \) and \( U_2^{(\alpha)} \) are equivalent, and \( U_1^{(\alpha)} \) and \( U_2^{(\alpha)} \) are equivalent too. 

\[\square\]
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References


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