Topological susceptibility and string tension in the lattice $\text{CP}^{N-1}$ models

Massimo Campostrini, Paolo Rossi, and Ettore Vicari

*Dipartimento di Fisica dell’Università and Istituto Nazionale di Fisica Nucleare, I-56126 Pisa, Italy*

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In the lattice $\text{CP}^{N-1}$ models we study the problems related to the measure of observables closely connected to the dynamically generated gauge field, such as the topological susceptibility and the string tension. We perform numerical simulations at $N = 4$ and $N = 10$. In order to test the universality, we adopt two different lattice formulations. Scaling and universality tests lead to the conclusion that at $N = 10$ the geometrical approach gives a good definition of lattice topological susceptibility. On the other hand, $N = 4$ proves not to be large enough to suppress the unphysical configurations, called dislocations, contributing to $\chi^f$ (at least up to $\xi \simeq 30$ in our lattice formulations). We obtain other determinations of $\chi_f$ by the field-theoretical method, which relies on a local definition of the lattice topological charge density, and the cooling method. They give quite consistent results, showing scaling and universality. The large-$N$ expansion predicts an exponential area law behavior for sufficiently large Wilson loops, which implies confinement, due to the dynamical matter fields and absence of the screening phenomenon. We determine the string tension, without finding evidence of screening effects.

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I. INTRODUCTION

Two-dimensional $\text{CP}^{N-1}$ models play an important role as a theoretical laboratory for testing non-perturbative analytical and numerical methods in a confining, asymptotically free quantum field theory. A pleasant feature of these models is the possibility of performing a systematic $1/N$ expansion around the large-$N$ saddle-point solution. Indeed, most properties of $\text{CP}^{N-1}$ models have been obtained in the context of the $1/N$ expansion [1–3].

An alternative and more general nonperturbative approach is the simulation of the theory on the lattice. Recently there has been considerable interest in simulations of lattice $\text{CP}^{N-1}$ models [4–11].

The purpose of the present paper is that of presenting rather complete numerical results concerning the $\text{CP}^3$ and the $\text{CP}^5$ models. We especially analyze the problems related to the measure of observables closely connected to the dynamical gauge field, such as the topological susceptibility and the string tension.

A troublesome point in the lattice simulation technique is the study of the topological properties. Measuring the topological susceptibility $\chi_f$ from discrete configurations proves to be a nontrivial task.

It is well known that, in the two-dimensional (2D) $O(3)$ $\sigma$ model or $\text{CP}^1$ model, geometrical definitions of topological charge are plagued by the presence of dislocations [12, 13], i.e., topological structures of the size of one lattice spacing, whose unphysical contribution to $\chi_f$ does not vanish in the continuum limit. As a consequence, the topological susceptibility derived from these definitions does not show the expected scaling behavior.

The general belief is that for higher values of $N$ the above problems should disappear. But the situation for the $\text{CP}^3$ model appears still problematic. There have been recent attempts [5, 7, 8, 10] to determine $\chi_f$ by using the geometrical method, but the results are not consistent. The authors of Refs. [8] and [10] both claim to observe scaling but they find different values for $\chi_f$ (about a factor 2 of difference). Since they use different lattice formulations, the geometrical measure would violate universality. Furthermore, the authors of Ref. [7], using the same lattice formulation as in Ref. [10], do not even see scaling.

An alternative approach relies on a definition of the topological charge density by a local polynomial in the lattice variables. Local operator definitions are not affected by the dislocation problem but unavoidably lead to mixing with lower and equal dimension operators and to the need of subtracting perturbative tails and performing finite renormalizations [14]. Nevertheless, this method allowed for the determination of the topological susceptibility of the $\text{CP}^3$ or $O(3)$ $\sigma$ model [4, 14].

Another important property of the $\text{CP}^{N-1}$ models is the appearance of a linear confining potential between non-gauge-invariant states. The large-$N$ expansion predicts an exponential area law behavior for sufficiently large Wilson loops. The persistence of the area law at large distance would imply the absence of the screening phenomenon due to the dynamical “matter fields.” The point we wish to clarify is whether at finite $N$ the screening phenomenon is recovered, or the large-$N$ prediction is confirmed.

This paper is organized as follows.

In Sec. II the lattice actions adopted for numerical simulations are presented and the lattice definitions of physical observables are introduced.

In Sec. III we discuss specific features of the simulations of the $\text{CP}^3$ and $\text{CP}^5$ models, and present the corre-
sponding numerical results.

In Sec. IV problems related to the evaluation of the topological susceptibility are carefully analyzed.

In Sec. V we discuss the determination of the string tension from the Wilson loops.

II. LATTICE FORMULATION

We choose to regularize the theory on the lattice by considering the action:

\[ S_g = -N\beta \sum_{n,\mu} \left( \bar{\xi}_n \xi_n \lambda_{n,\mu} + \bar{\xi}_n \xi_{n+\mu} \tilde{\lambda}_{n,\mu} - 2 \right), \tag{1} \]

Tests of rotation invariance and stability of dimensionless ratios of physical quantities showed that the above actions lead to scaling for rather small correlation lengths [4]. A comparison of measurements performed using these two actions will provide a check of universality, implying that the two actions are different regularizations of a unique quantum field theory.

Since the two actions are linear with respect to each lattice variable, it is easy to construct efficient local algorithms based on overrelaxation procedures. In our simulations we employed algorithms consisting in efficient mixtures of over-heat-bath [16] and microcanonical [17] algorithms. The detailed description of this simulation algorithm with a discussion of its dynamical features is contained in Ref. [4].

An important class of observables can be constructed by considering the local gauge-invariant composite operator

\[ P_{ij}(x) = \bar{\xi}_i(x) \xi_j(x) \tag{4} \]

and its group-invariant correlation function

\[ G_P(x) = \langle \text{Tr} P(x) P(0) \rangle_{\text{conn}}. \tag{5} \]

The standard correlation length \( \xi_w \) is extracted from the long-distance behavior of the zero space momentum correlation function ("wall-wall" correlation). The expected large-distance behavior, including periodic boundary condition effects, is

\[ G_w(x) \sim \frac{A_w}{2} \left[ \exp \left( -\frac{x}{\xi_w} \right) + \exp \left( -\frac{L-x}{\xi_w} \right) \right] \]

\[ \text{for } \frac{L}{2} > x \gg \xi_w. \tag{6} \]

Moreover we measured the "diagonal wall-wall" correlation length \( \xi_s \), obtained by summing on the correlations between points located on two distinct parallel lines oriented at 45° with respect to the coordinate axes, whose large distance behavior should be

\[ G_s(x) \sim \frac{A_s}{2} \left[ \exp \left( -\frac{x}{\xi_s} \right) + \exp \left( -\frac{L-x}{\xi_s} \right) \right] \]

\[ \text{for } \frac{L}{2} > x \gg \xi_s. \tag{7} \]

where \( \xi_n \) is an \( N \)-component complex scalar field, constrained by the condition

\[ \bar{\xi}_n \xi_n = 1 \tag{2a} \]

and \( \lambda_{n,\mu} \) is a U(1) gauge field satisfying

\[ \bar{\lambda}_{n,\mu} \lambda_{n,\mu} = 1. \tag{2b} \]

We also considered its tree Symanzik improved counterpart [15]

\[ G_d(x) \approx \frac{A_d}{2} \left[ \exp \left( -\frac{x}{\xi_d} \right) + \exp \left( -\frac{L/\sqrt{2} - x}{\xi_d} \right) \right] \]

\[ \text{for } \frac{L}{2\sqrt{2}} > x \gg \xi_d. \tag{8} \]

In practice, \( \xi_w, \xi_d, A_w, \) and \( A_d \) will be obtained by fitting the data for \( G_w \) and \( G_d \) by the functions (6) and (7), using all the values of \( x \) larger than a value \( x_{\text{min}} \) to be determined. The comparison between \( \xi_w \) and \( \xi_d \) provides a test of rotation invariance. Indeed in the scaling region, rotation invariance implies \( \xi_w = \xi_d \) and \( A_w = A_d \). Both \( \xi_w \) and \( \xi_d \) should reproduce in the continuum limit the inverse mass gap.

An alternative definition of the correlation length \( \xi_G \) comes from considering the second moment of the correlation function \( G_P \). In the small momentum regime we expect the behavior

\[ \bar{G}_P(k) \approx \frac{Z_P}{\xi_G^2 + k^2}, \tag{9} \]

where \( \bar{G}_P(k) \) is the Fourier transform of \( G_P(x) \). The zero component of \( \bar{G}_P(k) \) is by definition the magnetic susceptibility \( \chi_m \). On the lattice we can use the two lowest components of \( \bar{G}_P(k) \) to obtain the following definition of \( \xi_G \):

\[ \frac{\xi_G^2}{\xi_w^2} = \frac{1}{4\sin^2 \pi/L} \left[ \frac{\bar{G}_P(0,0)}{\bar{G}_P(0,1)} - 1 \right]. \tag{10} \]

In the scaling region the ratio \( \xi_G/\xi_w \) must be a constant, scale-independent number. The large-\( N \) expansion predicts [18]

\[ \frac{\xi_G}{\xi_w} \to \sqrt[3]{\frac{2}{3}} \tag{11} \]

when \( N \to \infty \), while for \( N = 2 \) the ratio is equal to 1 within 1% [4].

The quantity \( Z_P = \chi_m \xi_G^2 \) is related to the renormal-
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I. INTRODUCTION

The topological susceptibility and string tension are important quantities in the study of deconfinement in QCD. They are defined by the expectation values of the composite operators $P$ and $\Delta g$, respectively.

The behavior of these quantities as the lattice size $N$ becomes large is of great interest. The finite-size scaling (FSS) of the susceptibility and tension can be used to study the deconfinement transition and to test theoretical predictions.

II. THEORETICAL FORMULATIONS

The susceptibilities $\chi$ and $\chi_5$ are defined as

$$\chi = \langle P^2 \rangle / N^4$$

and

$$\chi_5 = \langle P^5 \rangle / N^5$$

where $P$ is the topological charge density.

The finite-size scaling relation for the susceptibility is

$$\chi(N) = \chi_{\infty} \left[ 1 + \frac{1}{2} \left( \frac{N}{\Lambda} \right)^2 \right] + O\left( \frac{1}{N^4} \right)$$

where $\Lambda$ is a scale parameter.

III. SIMULATIONS

We performed Monte Carlo simulations of the CP$^3$ model in the two formulations (1) and (3) for several values of $\beta$ corresponding to correlation lengths $\xi$ up to 30 lattice units. A summary of the runs is presented in Table I. There the integrated autocorrelation time of the magnetic susceptibility $\tau_{int}$ is also reported. In the following we will also show some data for the CP$^9$ model. Most of them were obtained from the simulations presented in Ref. [4], where the details of the runs were given.

The data taken on lattices of different size can be used to extract the finite-size scaling functions of the correlation length $\xi_G$ and of the magnetic susceptibility for the CP$^3$ model. In the scaling region the finite-size scaling functions must be universal, that is, independent of $\beta$ and of the lattice formulation, in that they should reproduce the continuum physics in a periodic box. In Figs. 1 and 2 we plot, respectively, $f_{\xi_G} = \xi_{G,L}/\xi_{G,\infty}$ and $f_{\chi_5} = \chi_{5,L}/\chi_{5,\infty}$ vs $z = L/\xi_G$ for the two actions (1) and (3). For both actions we chose a value of $\beta$ corresponding to a correlation length $\xi \approx 4$. The finite-size scaling

$$\frac{\Lambda_E}{\Lambda_g} = \exp \left( \frac{\pi(2N - 1)}{4N^2} \right)$$

(17)

The same can be done within the action $S_{sym}$. In this case the perturbative expansion of the internal energy is

$$E = \frac{1}{2\beta} + \frac{0.1001N - 0.0589}{N^2\beta^2} + O\left( \frac{1}{\beta^3} \right)$$

(18)

and therefore we find

$$\frac{\Lambda_{sym}}{\Lambda_g} = 1.345 \exp \left( \frac{\pi(0.542N - 0.236)}{N^2} \right)$$

(19)

FIG. 1. Finite-size scaling of the correlation length $\xi_G$ for the CP$^3$ model.
TABLE I. Summary of the simulation runs for the CP$^3$ model. An asterisk marks runs for the Symanzik improved action (3). We use the notation “$m, \gamma$” for a stochastic mixture of microcanonical and over-heat-bath updating with relative weight $\gamma$ (see Ref. [4]).

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$L$</th>
<th>$\xi_G$</th>
<th>$E$</th>
<th>$\chi_m$</th>
<th>$\gamma_{int}^{\chi_m}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.95</td>
<td>12</td>
<td>0.5925(4)</td>
<td>3.987(14)</td>
<td>20.30(8)</td>
<td>6.1(2)</td>
</tr>
<tr>
<td>0.95</td>
<td>15</td>
<td>0.6005(3)</td>
<td>4.239(16)</td>
<td>22.89(8)</td>
<td>7.1(2)</td>
</tr>
<tr>
<td>0.95</td>
<td>18</td>
<td>0.6042(3)</td>
<td>4.277(19)</td>
<td>23.52(11)</td>
<td>7.9(3)</td>
</tr>
<tr>
<td>0.95</td>
<td>21</td>
<td>0.6058(2)</td>
<td>4.272(22)</td>
<td>23.64(13)</td>
<td>8.4(3)</td>
</tr>
<tr>
<td>0.95</td>
<td>24</td>
<td>0.6067(2)</td>
<td>4.142(23)</td>
<td>23.01(12)</td>
<td>7.3(3)</td>
</tr>
<tr>
<td>0.95</td>
<td>27</td>
<td>0.6070(2)</td>
<td>4.129(25)</td>
<td>22.96(11)</td>
<td>7.3(3)</td>
</tr>
<tr>
<td>0.95</td>
<td>30</td>
<td>0.6071(1)</td>
<td>4.108(25)</td>
<td>22.84(11)</td>
<td>7.0(2)</td>
</tr>
<tr>
<td>0.95</td>
<td>33</td>
<td>0.60725(8)</td>
<td>4.094(20)</td>
<td>22.81(7)</td>
<td>6.6(2)</td>
</tr>
<tr>
<td>1.05</td>
<td>60</td>
<td>0.53051(7)</td>
<td>7.59(8)</td>
<td>62.6(4)</td>
<td>15.8(8)</td>
</tr>
<tr>
<td>1.15</td>
<td>120</td>
<td>0.47341(4)</td>
<td>15.1(3)</td>
<td>189.6(2.1)</td>
<td>42(3)</td>
</tr>
<tr>
<td>1.20a</td>
<td>150</td>
<td>0.45018(4)</td>
<td>20.9(4)</td>
<td>328(4)</td>
<td>64(7)</td>
</tr>
<tr>
<td>1.20b</td>
<td>150</td>
<td>0.45011(4)</td>
<td>20.2(4)</td>
<td>317(4)</td>
<td>70(7)</td>
</tr>
<tr>
<td>1.25a</td>
<td>210</td>
<td>0.42944(3)</td>
<td>27.9(8)</td>
<td>549(10)</td>
<td>$\approx$ 110</td>
</tr>
<tr>
<td>1.25b</td>
<td>210</td>
<td>0.42940(3)</td>
<td>28.3(7)</td>
<td>552(11)</td>
<td>$\approx$ 120</td>
</tr>
</tbody>
</table>

TABLE II. Values and ratios of different definitions of correlation length, and correlation function coefficient for the CP$^3$ model.

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$L$</th>
<th>$\xi_G$</th>
<th>$\xi_w$</th>
<th>$\xi_G/\xi_w$</th>
<th>$\xi_d/\xi_w$</th>
<th>$\Lambda_{\omega}/\Lambda_{G}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.95</td>
<td>33</td>
<td>4.094(20)</td>
<td>4.151(21)</td>
<td>0.985(4)</td>
<td>0.999(5)</td>
<td>0.950(13)</td>
</tr>
<tr>
<td>1.05</td>
<td>60</td>
<td>7.59(8)</td>
<td>7.66(14)</td>
<td>0.991(10)</td>
<td>1.014(15)</td>
<td>0.97(4)</td>
</tr>
<tr>
<td>1.15</td>
<td>120</td>
<td>15.1(3)</td>
<td>15.4(4)</td>
<td>0.980(16)</td>
<td>0.980(20)</td>
<td>0.93(5)</td>
</tr>
<tr>
<td>1.20</td>
<td>150</td>
<td>20.5(3)</td>
<td>20.7(5)</td>
<td>0.992(13)</td>
<td>0.992(17)</td>
<td>0.98(4)</td>
</tr>
<tr>
<td>1.25</td>
<td>210</td>
<td>28.1(5)</td>
<td>28.9(12)</td>
<td>0.973(21)</td>
<td>0.98(5)</td>
<td>0.91(7)</td>
</tr>
</tbody>
</table>

TABLE III. Some results for the CP$^3$ model.

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$\xi_G$</th>
<th>$\xi_G/\xi_w$</th>
<th>$\beta^2 Z_P$</th>
<th>$M_G/\Lambda_{\omega}$</th>
<th>$M_G/\Lambda_{\phi_{1,2}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.7</td>
<td>2.35(3)</td>
<td>0.971(6)</td>
<td>0.905(20)</td>
<td>25.73(33)</td>
<td>20.57(27)</td>
</tr>
<tr>
<td>0.75</td>
<td>3.31(4)</td>
<td>0.95(6)</td>
<td>0.86(20)</td>
<td>24.63(31)</td>
<td>20.46(26)</td>
</tr>
<tr>
<td>0.8</td>
<td>4.67(3)</td>
<td>0.95(6)</td>
<td>0.83(80)</td>
<td>23.65(14)</td>
<td>20.30(13)</td>
</tr>
<tr>
<td>0.85</td>
<td>6.44(6)</td>
<td>0.962(10)</td>
<td>0.823(12)</td>
<td>23.18(22)</td>
<td>20.34(19)</td>
</tr>
<tr>
<td>0.9</td>
<td>8.83(8)</td>
<td>0.973(9)</td>
<td>0.816(11)</td>
<td>22.88(21)</td>
<td>20.42(19)</td>
</tr>
<tr>
<td>0.7*</td>
<td>3.78(2)</td>
<td>0.968(6)</td>
<td>0.795(7)</td>
<td>22.53(11)</td>
<td>19.90(9)</td>
</tr>
<tr>
<td>0.75*</td>
<td>5.19(3)</td>
<td>0.974(6)</td>
<td>0.787(7)</td>
<td>22.18(13)</td>
<td>19.94(12)</td>
</tr>
</tbody>
</table>
scaling functions were obtained by approximating infinite lattice quantities with the corresponding values measured on the largest lattice available. The universality with respect to the lattice action is fully satisfied. From Figs. 1 and 2 we also learn that $x \simeq 7$ is a safe value where the finite-size effects are smaller than 1%. The finite-size scaling functions for the CP$^3$ model were shown in Ref. [4].

Regarding the CP$^3$ model, the data for the different definitions of correlation length, for the correlation function coefficient $A_w$, and the ratio $A_w/A_G$ are reported in Table II. The fits to $G_w$ and $G_d$ were performed choosing $x_{\text{min}} \approx 2\xi_w$; fits using a larger $x_{\text{min}}$ gave consistent results. The ratios of these different definitions were analyzed by using the jackknife method. The models defined by $S_g$ and $S_g^{\text{sym}}$ enjoy rotation invariance and stability of dimensionless physical quantities for all values of the correlation length considered. The correlation lengths $\xi_G$ and the ratios $\xi_G/\xi_w$ for the CP$^9$ model are reported in Table III. By fitting with a constant the data of the above dimensionless ratios, we obtained the results in Table IV.

$\xi_w$ should reproduce in the continuum limit the inverse mass of the lowest positive-parity state belonging to the adjoint representation. We also looked for other states, either excited states in the adjoint positive-parity channel, or states in the other channels, the adjoint odd channel, the singlet even and odd channels. We did not find evidence of such states for the CP$^3$ and the CP$^9$ models.

![Image](image_url)

**FIG. 2.** Finite-size scaling of the magnetic susceptibility $\chi_m$ for the CP$^3$ model.

Data for the constant $c$ of Eq. (11) are reported in Tables V and III respectively for the the CP$^3$ and the CP$^9$ models. Data show scaling and the two actions give very close values. The small discrepancies can be ascribed to the nonuniversal terms of order $\beta^{-1}$ in Eq. (11). We note also that the approach to scaling is slower for the action $S_g$.

We checked the asymptotic scaling, according to the two-loop formula $f(\beta) = (2\pi\beta)^{3/2}/N \exp(-2\pi\beta)$, by analyzing the quantity $M_G/\Lambda_g = [\xi_G f(\beta)]^{-1}$.

To begin with, in Fig. 3 we show data for the CP$^3$ model, which were taken by using the action $S_g$ [4]. If this lattice CP$^3$ model belongs to the universality class of the O(3) $\sigma$ model, $M_G/\Lambda_g$ must tend to the asymptotic value 36.5, according to the exact result [21]. $M_G/\Lambda_g$ appears to be constant within the errors for the largest values of $\beta$. Its value (approximately 47) is far from the asymptotic one. However this is not a problem, since field theory predicts an extremely slow approach to asymptopia for quantities such as $M_G/\Lambda_g$. In that region of $\beta$, corresponding to correlation lengths from about 5 to 30 lattice spacings, the $\beta$ function is well approximated by the two loop formula (indeed $M_G/\Lambda_g$ is constant within errors of $\sim 3\%$) but its integral does not (the discrepancy

![Image](image_url)

**FIG. 3.** Asymptotic scaling test for $\xi_G$ in the CP$^3$ model.

**TABLE V.** Some results for the CP$^3$ model.

| $\beta$    | $\beta^2 Z_P$ | $M_G/\Lambda_g$ | $M_G/\Lambda_g|_E$ |
|------------|---------------|------------------|---------------------|
| 0.95       | 1.228(7)      | 39.1(2)          | 26.7(1)             |
| 1.05       | 1.20(2)       | 37.6(4)          | 28.5(3)             |
| 1.15       | 1.10(3)       | 33.9(6)          | 27.6(5)             |
| 1.20       | 1.10(2)       | 33.4(4)          | 27.9(4)             |
| 1.25       | 1.06(3)       | 32.7(6)          | 27.9(5)             |
| 0.85$^*$   | 1.067(9)      | 33.4(2)          | 25.3(2)             |
| 0.95$^*$   | 1.06(2)       | 32.7(4)          | 26.6(3)             |
| 1.00$^*$   | 1.04(2)       | 31.7(4)          | 26.5(3)             |
| 1.05$^*$   | 1.05(3)       | 31.7(8)          | 27.2(6)             |
| 1.07$^*$   | 1.04(2)       | 31.3(4)          | 27.0(4)             |
| 1.12$^*$   | 1.03(2)       | 30.6(4)          | 26.9(3)             |

![Image](image_url)

**TABLE IV.** Tests of scaling by fitting dimensionless ratio data with a constant.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$S$</th>
<th>$\xi_G/\xi_w$</th>
<th>$A_w/A_G$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>$S_g$</td>
<td>0.986(3)</td>
<td>0.952(11)</td>
</tr>
<tr>
<td>4</td>
<td>$S_g^{\text{sym}}$</td>
<td>0.984(4)</td>
<td>0.944(14)</td>
</tr>
<tr>
<td>10</td>
<td>$S_g$</td>
<td>0.965(4)</td>
<td>0.870(13)</td>
</tr>
<tr>
<td>10</td>
<td>$S_g^{\text{sym}}$</td>
<td>0.971(4)</td>
<td>0.900(14)</td>
</tr>
</tbody>
</table>
IV. TOPOLOGICAL SUSCEPTIBILITY

A. Introduction

The topological charge density of a complex spin field \( z \) is

\[
q(x) = \frac{i}{2\pi} \varepsilon_{\mu\nu} D_\mu z D_\nu z.
\]

The topological susceptibility is defined as the correlation at zero momentum of two \( q(x) \) operators:

\[
\chi_t = \int d^2 x \langle q(x) q(0) \rangle.
\]

The large-\( N \) predictions concerning the topological susceptibility are [22]

\[
\chi_t \xi_G^2 = \frac{3}{4\pi N} + O(N^{-5/3}),
\]

and [18]

\[
\chi_t \xi_G^2 = \frac{1}{2\pi N} \left(1 - \frac{0.38}{N}\right) + O \left(\frac{1}{N^3}\right).
\]

Different methods have been proposed to calculate \( \chi_t \) on the lattice. The geometrical definition uses an interpolation among discrete lattice variables to assign an integer topological charge to each lattice configuration. While for large \( N \) this definition is expected to reproduce the physical topological susceptibility, at low \( N \) \( \chi_t^g \) could receive unphysical contributions from exceptional configurations, called dislocations, i.e., topological structures of the size of one lattice spacing. The dislocation contributions may either survive in the continuum limit, as it happens for some lattice formulations of the CP\(^3\) or O(3) \( \sigma \) model, or push the scaling region for \( \chi_t^g \) to very large \( \beta \) values.

Another approach relies on a definition of topological charge density by a local polynomial in the lattice variables. Local operator definitions are subject to mixing with lower and equal dimension operators and to finite renormalizations, which must be evaluated in order to extract \( \chi_t \).

A third method consists in measuring \( \chi_t \) on an ensemble of configurations cooled by minimizing locally the action.

Any sensible definition of a lattice observable must show the correct scaling within each lattice formulation of the theory and universality among the determinations obtained with different lattice actions.

B. The geometrical definition

The geometrical definition of the topological charge is [12]

\[
q_G^\mu = \frac{1}{2\pi} \text{Im} \left\{ \ln[\text{Tr} P_{n+\mu+v} P_{n+\mu} P_n] + \ln[\text{Tr} P_{n+\mu+v} P_{n+\mu} P_n] \right\}, \quad \mu \neq \nu.
\]

Introducing the quantity \( \theta_{n,\mu} = \arg \{z_n z_{n+\mu}\} \), one easily obtains
TABLE VI. Geometric and cooled topological susceptibility for the CP$^3$ model.

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$L$</th>
<th>$10^4 \chi^2$</th>
<th>$\chi^2 \xi_G^2$</th>
<th>$10^4 \chi^2_{\text{cool}}$</th>
<th>$\chi^2_{\text{cool}} \xi_G^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.7</td>
<td>42</td>
<td>50.5(1.1)</td>
<td>0.0279(9)</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>0.75</td>
<td>60</td>
<td>22.6(6)</td>
<td>0.0248(9)</td>
<td>17.8(8)</td>
<td>0.0195(10)</td>
</tr>
<tr>
<td>0.8</td>
<td>60</td>
<td>9.9(3)</td>
<td>0.0216(6)</td>
<td>9.1(5)</td>
<td>0.0197(10)</td>
</tr>
<tr>
<td>0.85</td>
<td>72</td>
<td>5.1(4)</td>
<td>0.0213(17)</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>0.9</td>
<td>81</td>
<td>2.5(3)</td>
<td>0.0198(24)</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>0.9</td>
<td>81</td>
<td>12.3(6)</td>
<td>0.0179(9)</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>0.9</td>
<td>60</td>
<td>13.2(5)</td>
<td>0.0186(7)</td>
<td>12.3(4)</td>
<td>0.0173(7)</td>
</tr>
<tr>
<td>0.75</td>
<td>81</td>
<td>6.5(3)</td>
<td>0.0176(9)</td>
<td>6.1(5)</td>
<td>0.0166(13)</td>
</tr>
</tbody>
</table>

$q^n_g = \frac{1}{4\pi} \varepsilon_{\mu\nu}(\theta_{n,\mu} + \theta_{n+\mu,\nu} - \theta_{n,\nu} - \theta_{n,\mu})$. \hspace{1cm} (26)

The periodic boundary conditions make the geometrical topological charge of each lattice configuration, $Q_g = \sum_n q^n_g$, integer. The topological susceptibility should then be extracted by measuring the following expectation value

\[ \chi^2 = \frac{1}{V} \langle (Q_g)^2 \rangle. \hspace{1cm} (27) \]

In Table VI we report the data of $\chi^2$ for the CP$^3$ model. Using $S_g$ the approach to scaling is slow, instead for $S_0^{\text{Sym}}$ a better behavior is observed. For $S_0$ the leading scaling-violation term must be $O(\ln \xi/\xi^2)$ when $\xi \to \infty$ \cite{15}. Instead for the tree Symanzik improved actions the leading logarithm corrections are absent, and scale violations are $O(\xi^{-2})$ \cite{23}. Assuming that the scaling violation term proportional to $\ln \xi/\xi^2$ is already dominant in our range of correlation lengths, we extrapolate data of $\chi^2 \xi_G^2$ for the action $S_g$. In Fig. 6 we plot $\chi^2 \xi_G^2$ vs $\ln \xi_G/\xi_G^2$. A fit gives

\[ \chi^2 \xi_G^2 = 0.0174(12) \quad b = 0.068(12), \hspace{1cm} (28) \]

where $b$ is the coefficient of the $\ln \xi_G/\xi_G^2$ term. The fitted value of $\chi^2 \xi_G^2$ is in agreement with the value of $\chi^2 \xi_G^2$ obtained with the action $S_0^{\text{Sym}}$, which is $\chi^2 \xi_G^2 = 0.0176(9)$ at $\xi_G = 5.19(3)$. We then conclude that for the CP$^3$ model $\chi^2$ is a good estimator of the topological susceptibility.

TABLE VII. Geometric and cooled topological susceptibility for the CP$^3$ model.

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$L$</th>
<th>$10^4 \chi^2$</th>
<th>$\chi^2 \xi_G^2$</th>
<th>$10^4 \chi^2_{\text{cool}}$</th>
<th>$\chi^2_{\text{cool}} \xi_G^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.95</td>
<td>33</td>
<td>49.5(5)</td>
<td>0.083(1)</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>1.05</td>
<td>60</td>
<td>16.2(3)</td>
<td>0.093(3)</td>
<td>8.6(3)</td>
<td>0.050(2)</td>
</tr>
<tr>
<td>1.15</td>
<td>120</td>
<td>4.01(11)</td>
<td>0.091(6)</td>
<td>2.70(11)</td>
<td>0.062(3)</td>
</tr>
<tr>
<td>1.20</td>
<td>150</td>
<td>2.11(5)</td>
<td>0.089(3)</td>
<td>1.44(5)</td>
<td>0.061(3)</td>
</tr>
<tr>
<td>1.25</td>
<td>210</td>
<td>1.06(3)</td>
<td>0.084(4)</td>
<td>0.77(4)</td>
<td>0.062(4)</td>
</tr>
<tr>
<td>0.95*</td>
<td>36</td>
<td>40.5(6)</td>
<td>0.066(1)</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>0.95*</td>
<td>60</td>
<td>13.4(3)</td>
<td>0.072(2)</td>
<td>9.1(3)</td>
<td>0.049(2)</td>
</tr>
<tr>
<td>1.00*</td>
<td>81</td>
<td>7.0(2)</td>
<td>0.072(2)</td>
<td>5.2(2)</td>
<td>0.054(2)</td>
</tr>
<tr>
<td>1.05*</td>
<td>120</td>
<td>3.7(2)</td>
<td>0.068(4)</td>
<td>3.2(2)</td>
<td>0.058(5)</td>
</tr>
<tr>
<td>1.07*</td>
<td>120</td>
<td>2.83(7)</td>
<td>0.067(2)</td>
<td>2.37(8)</td>
<td>0.056(3)</td>
</tr>
<tr>
<td>1.12*</td>
<td>150</td>
<td>1.50(4)</td>
<td>0.067(2)</td>
<td>1.27(4)</td>
<td>0.057(2)</td>
</tr>
</tbody>
</table>

FIG. 6. Topological susceptibility vs $\ln \xi_G/\xi_G^2$ in the CP$^3$ model.

FIG. 7. Summary of the topological susceptibility determinations with the action $S_g$ and for the CP$^3$ model.

FIG. 8. Summary of the topological susceptibility determinations with the action $S_0^{\text{Sym}}$ and for the CP$^3$ model.
The situation for the CP$^3$ model appears more problematic. Our data of $\chi^2$ for the CP$^3$ model are reported in Table VII and shown in Figs. 7 and 8. For both actions an apparent scaling is observed but data clearly violate universality.

C. The field-theoretical method

The field-theoretical approach relies on a definition of topological charge density by a local polynomial in the lattice variables having the correct classical continuum limit

$$q^L(x) \rightarrow a^2 q(x) + O(a^4)$$ (29)

(a being the lattice spacing). In order to determine $\chi_t$, the correlation at zero momentum of two $q^L(x)$ operators $\chi^L_t$ is calculated

$$\chi^L_t = \left\langle \sum_x q^L(x) q^L(0) \right\rangle = \frac{1}{V} \left\langle (Q^L)^2 \right\rangle,$$ (30a)

$$Q^L = \sum x q^L(x).$$ (30b)

$\chi^L_t$ is connected to $\chi_t$ by a nontrivial relationship, since the presence of irrelevant operators of higher dimension in $q^L(x)$ induces quantum corrections. The classical continuum limit of $q^L(x)$ must be corrected including a renormalization constant $Z(\beta)$ [24]. Other contributions originate from contact terms, i.e., from the limit $x \rightarrow 0$ in Eq. (30). These contact terms appear as mixings with the trace of the energy-momentum tensor $S(x)$ and with the identity operator $I$, which are the only available operators with equal or lower dimension. Therefore the relationship between the lattice and the continuum topological susceptibility takes the form

$$\chi^L_t(\beta) = a^2 Z(\beta)^2 + a^2 A(\beta)(S(x))_{np} + P(\beta)(I) + O(a^4).$$ (31)

np denotes the nonperturbative part (i.e., the perturbative tail must be subtracted), $Z(\beta)$, $P(\beta)$, and $A(\beta)$ are ultraviolet effects, since they originate from the ultraviolet cutoff-dependent modes. They can be computed in perturbation theory as series in $\beta^{-1}$.

The field theoretical method consists in measuring $\chi^L_t(\beta)$ by a standard Monte Carlo simulation, evaluating $Z(\beta)$, $A(\beta)$, and $P(\beta)$, and using Eq. (31) to extract $\chi_t$.

The requirement (29) does not uniquely determine the lattice operator. Different lattice versions can be found and all of them should give the same physical result for $\chi_t$, instead the renormalization functions $Z(\beta)$, $P(\beta)$, and $A(\beta)$ are lattice operator dependent. We considered two versions of lattice topological charge density operator:

$$q^L_t(x) = -\frac{i}{2\pi} \sum_{\mu\nu} \epsilon_{\mu\nu} \text{Tr}[P(x)\Delta^{(2)}_{\mu}P(x)\Delta^{(2)}_{\nu}P(x)],$$ (32)

where $\Delta^{(1)}$ is a symmetrized version of the finite derivative,

$$\Delta^{(1)}_{\mu}P(x) = \frac{1}{2}[P(x+\mu) - P(x-\mu)].$$ (33)

The second lattice operator is

$$q^L_2(x) = -\frac{i}{2\pi} \sum_{\mu\nu} \epsilon_{\mu\nu} \text{Tr}[P(x)\Delta^{(2)}_{\mu}P(x)\Delta^{(2)}_{\nu}P(x)],$$ (34)

where $\Delta^{(2)}$ is another version of finite derivative:

$$\Delta^{(2)}_{\mu}P(x) = \frac{2}{3}[P(x+\mu) - P(x-\mu)] - \frac{1}{12}[P(x+2\mu) - P(x-2\mu)].$$ (35)

$q^L_2(x)$ is a Symanzik tree-improved version of $q^L_t(x)$.

We applied the field-theoretical method to determine the topological susceptibility of the CP$^3$ model.

In the following we will neglect the contribution of the mixing with $S(x)$. This assumption is supported by a perturbative argument: the perturbative series of $A(\beta)$ starts with a $\beta^{-3}$ term. It will be a possible source of systematic error in our calculations.

D. The heating method

In order to estimate the renormalization functions in Eq. (31) nonperturbatively, we applied the method proposed in Ref. [25]. We start from a configuration $C_0$ carrying a definite topological charge $Q_0$ which is an approximate minimum of the lattice action (in this sense we will call it a "classical" configuration). We heat it by a local updating procedure in order to introduce short-ranged fluctuations, taking care to leave intact the background topological structure. We construct ensembles $C_0^{(Q_0)}$ of many independent configurations obtained by heating the starting configuration $C_0$ for the same number $n$ of updating steps, and average the topological charge over the ensembles. If $\xi \gg a$, there should exist an intermediate range of $n$ where fluctuations of length $l \sim a$ are thermalized at the given value of $\beta$ and reproduce the renormalization effects, while fluctuations at the scale $l \sim \xi$ are off equilibrium and still determined by the initial configuration. The average of $Q^L = \sum x q^L(x)$ over the configurations in this range of $n$ should be approximately equal to $Z(\beta)Q_0$.

We can also start from a constant configuration (with $Q_0 = 0$) and construct other ensembles $C_n^{(Q_0)}$ of configurations. We should find an intermediate region of $n$ where the measure of $\chi^L_t$ gives an estimate of the mixing $P(\beta)$ with the identity operator which, being a short-ranged effect (due to the fluctuations at $l \sim a$), is expected to be independent of the physical topological background structure.

If we plot the values $Q^L$ averaged over $C_n^{(Q_0)}$ and the values of $\chi^L_t$ averaged over $C_n^{(0)}$ as functions of $n$, we should observe plateaus in correspondence of the above-mentioned intermediate ranges. The characteristics (starting point and length) of the plateaus are determined by the phenomenon of critical slowing down. The renormalization functions are determined by short-ranged fluctuations, which we do not expect to be critically slowed down; therefore the starting point of the plateaus should be independent of $\beta$. On the other hand, the end point of the plateaus is reached when the Monte Carlo procedure changes the long-ranged modes that de-
To determine the topological properties, and critical slowing down should strongly affect these modes; therefore the length of the plateaus should be $\beta$ dependent. This behavior is essential for the existence of an intermediate range of $n$ where the renormalization effects can be measured: indeed the success of the present method for estimating $Z(\beta)$ and $P(\beta)$ strongly relies on the distinction between the fluctuations at distance $l \sim a$, contributing to the renormalizations, and those at $l \sim \xi$ determining the relevant topological properties. The fluctuations at $l \sim a$ are soon thermalized, whereas the topological charge thermalization is much slower.

In order to check that heating does not change the background topological structure of the initial configuration, after a given number of heating sweeps we cool the configurations (by locally minimizing the action) and verify that the cooled configurations have topological charge equal to $Q_0$.

We used as an updating procedure a 20-hit Metropolis algorithm (tuned to 50% acceptance), which gives a sufficiently mild heating.

This method has been already applied to determine the topological susceptibility of the CP$^1$ or O(3) $\sigma$ model [4, 14]. Consistency of the direct measures of $Z(\beta)$ and $P(\beta)$ with the corresponding perturbative computation has been shown in Ref. [14] within a lattice formulation of the O(3) $\sigma$ model.

We construct the initial configuration carrying topological charge $Q_0 = 1$ ("lattice instanton") starting from a discretization of the continuum SU(2) instanton:

$$z_1(x) = \frac{x_1 - \bar{x}_1 - i(x_2 - \bar{x}_2)}{[\rho^2 + (x_1 - \bar{x}_1)^2 + (x_2 - \bar{x}_2)^2]^{1/2}},$$

$$z_2(x) = \frac{\rho}{[\rho^2 + (x_1 - \bar{x}_1)^2 + (x_2 - \bar{x}_2)^2]^{1/2}},$$

$$z_i(x) = 0, \quad i = 3, \ldots, N,$$

$$\lambda_\mu(x) = \frac{\bar{z}(x+\mu)z(x)}{|\bar{z}(x+\mu)z(x)|}. \tag{36}$$

The parameter $\rho$ controls the size of the instanton, and $\bar{x}$ is its center, which we always place at the lattice center: $\bar{x} = (L/2, L/2)$. Starting from the configuration (36), we performed a few cooling steps in order to smooth over the configuration at the lattice periodic boundary. After this procedure, we end up with a smooth configuration $C_0^{(1)}$ with topological charge $Q^L \approx 1$. The geometrical topological charge of this configuration is exactly equal to 1.

In Fig. 9 we plot $Q_1(C^{(1)}_n)/Q_{0,1}$ and $Q_2(C^{(1)}_n)/Q_{0,2}$, where $Q_1(C^{(1)}_n)$ ($i = 1, 2$) is the lattice topological charge $Q^L = \sum x_q^L(x)$ averaged over the ensemble $C_n^{(1)}$, $Q_{0,i}$ is the topological charge of the starting configuration measured by the operator $Q^L$. The data in Fig. 9 were taken at $\beta = 1.12$ and for the tree Symanzik improved action. We see clearly a plateau starting from $n = 7$ for both operators. For $n = 12$ we also cooled the sample of configurations finding $Q^L \approx Q_{0,i}$ after a few cooling steps. This value of $n$ is marked by a dashed line in Fig. 9. According to the above considerations, the value of $Q^L$ at the plateau gives an estimate of $Z_1(\beta)$. We repeated this procedure for other values of $\beta$, and for both actions (1)

\begin{table}
\centering
\caption{Measure of the multiplicative renormalization of $\chi^L$, starting from an instanton of size $\rho$ or two instantons of size $\rho$ and distance $d$. $Q_G$ is the geometrical charge of the initial configuration. The estimate of $Z^L_i$ are taken by averaging the data in the range of $n$ reported in the column "Plateau."}
\begin{tabular}{cccccccccc}
$\beta$ & $L$ & $Q_G$ & $\rho$ & $d$ & Stat & Plateau & $Q_{1,0}$ & $Z^L_1$ & $Q_{2,0}$ & $Z^L_2$
\hline
1.15 & 48 & 1 & 10 & \ldots & 1000 & 8–10 & 0.9908 & 0.375(7) & 0.9997 & 0.420(12)
1.15 & 60 & 2 & 8 & 12 & 400 & 8–10 & 1.9165 & 0.375(8) & 1.9912 & 0.417(11)
1.15 & 60 & 2 & 8 & 16 & 400 & 8–10 & 1.9517 & 0.377(7) & 1.9971 & 0.418(10)
1.20 & 48 & 1 & 10 & \ldots & 2000 & 7–10 & 0.9908 & 0.413(5) & 0.9997 & 0.461(7)
1.20 & 60 & 2 & 8 & 20 & 400 & 7–10 & 1.9590 & 0.413(7) & 1.9983 & 0.459(10)
1.20 & 48 & 1 & 10 & \ldots & 1000 & 7–10 & 0.9908 & 0.432(6) & 0.9997 & 0.481(10)
1.07 & 48 & 1 & 10 & \ldots & 1000 & 7–10 & 0.9905 & 0.442(6) & 0.9997 & 0.494(9)
1.12 & 48 & 1 & 10 & \ldots & 1000 & 7–12 & 0.9905 & 0.464(5) & 0.9997 & 0.518(8)
1.17 & 48 & 1 & 10 & \ldots & 500 & 7–12 & 0.9905 & 0.487(7) & 0.9997 & 0.541(11)
\end{tabular}
\end{table}
and (3). We checked also the dependence of the measure on the size of the instanton $\rho$ (in the range of $\rho \sim \xi$), and of the value of the topological charge of the initial configuration. The behavior of $Q_i(C^{(1)})/Q_{0,i}$ is always very similar to the case reported in Fig. 9. The results are presented in Table VIII. Configurations with topological charge two were constructed by allocating two instantons of size $\rho$ at a distance $d$.

We now proceed to the analysis of the ensembles $C^{(0)}_{\rho}$ of configurations obtained by heating the constant configuration $C^{(0)}$, defined by $z(x) = (0, 0, 0, 1)$ and $\lambda_{\mu}(x) = 1$, for several values of $\beta$. In Figs. 10 and 11 we plot the average value of $\chi^L_{\beta}$ as a function of the number $n$ of heating steps for the operator $q^L_{\beta}$ and respectively for the actions (1) and (3). For every value of $\beta$ we observe a plateau starting from $n \simeq 25$; the plateau is longer for higher values of $\beta$, as expected. After $n_c$ heating sweeps (see Table IX) we cooled the sample of configurations and found vanishing $Q^L_{\beta}$ in a few cooling steps. After the plateau, $\chi^L_{\beta}$ increases to reach the true equilibrium value.

We identify the topological susceptibility measured at the plateau $\chi^L_{\beta,p}$ with the perturbative tail at the given value of $\beta$. Since $Z(\beta)$ and $P(\beta)$ have their origin in the fluctuations at $t \sim a$, finite-size corrections are of the order of $L^{-2}$ and therefore negligible on our lattice. Results are reported in Table IX.

The values of $Z(\beta)$ and $P(\beta)$ obtained by this procedure can be inserted in Eq. (31) to extract the physical value of the topological susceptibility. The results are summarized in Table X and shown in Figs. 7 and 8.

All four classes of measures (2 operators $\times$ 2 actions) show scaling within the errors. The small discrepancies among them should be explained by the mixing with the trace of the energy-momentum tensor $S(x)$ in Eq. (31). Indeed the function $A(\beta)$ is operator and action dependent and should vary slowly in the relative small range of $\beta$ considered. The fact that the discrepancies are small give further support to the initial assumption of neglecting the contribution of the mixing with $S(x)$ in our calculations. From it we also get an idea of the systematic error of our calculations.

E. Heating and geometrical charge

To clarify the origin of the failure of the geometrical method, we followed the behavior of the geometrical topological susceptibility $\chi^g_{\beta}$ during the heating proce-
TABLE X. Measure of $\chi_t$ by the field-theoretical method.

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$10^4\chi_t^{l}$</th>
<th>$10^4\chi_t^{n.1}$</th>
<th>$\chi_t^{n.1}\xi_G$</th>
<th>$10^4\chi_t^{l}$</th>
<th>$10^4\chi_t^{n.2}$</th>
<th>$\chi_t^{n.2}\xi_G$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.15</td>
<td>0.647(18)</td>
<td>2.52(17)</td>
<td>0.058(4)</td>
<td>1.265(36)</td>
<td>3.1(3)</td>
<td>0.071(7)</td>
</tr>
<tr>
<td>1.20</td>
<td>0.485(12)</td>
<td>1.41(8)</td>
<td>0.060(4)</td>
<td>0.952(22)</td>
<td>1.68(14)</td>
<td>0.071(5)</td>
</tr>
<tr>
<td>1.25</td>
<td>0.345(8)</td>
<td>0.76(5)</td>
<td>0.060(5)</td>
<td>0.706(15)</td>
<td>0.92(9)</td>
<td>0.072(8)</td>
</tr>
<tr>
<td>1.07*</td>
<td>0.646(16)</td>
<td>2.21(11)</td>
<td>0.052(4)</td>
<td>1.147(28)</td>
<td>2.62(16)</td>
<td>0.062(4)</td>
</tr>
<tr>
<td>1.12*</td>
<td>0.442(10)</td>
<td>1.24(6)</td>
<td>0.055(3)</td>
<td>0.800(16)</td>
<td>1.44(9)</td>
<td>0.064(4)</td>
</tr>
</tbody>
</table>

The idea behind the cooling method is that local changes should not modify the topological properties of a configuration, and its topological content can be extracted from the cooled configuration, where the short-ranged fluctuations responsible of the renormalization effects, have been eliminated.

The cooling algorithm consists in assigning to each lattice variable $z_n$ ($\lambda_{n,\mu}$) a new value $z_n'$ ($\lambda_{n,\mu}'$) (keeping all other variables fixed) that minimizes the action.

To determine the topological charge of the cooled configurations, we used the operator $q^2(x)$, which turns out to be better than $q_1^l(x)$ in estimating the topological content of a smooth configuration. This can be seen by comparing the values of $Q_{0,1}$ and $Q_{0,2}$ in Table VII. The topological susceptibility measured on cooled configurations by Eq. (30), $\chi_t^{cool}$, is seen to gradually rise up to an extended plateau. Our averages and errors are estimated on the plateau measurements.

Table VI reports also data of some measurements of $\chi_t^{cool}$ by cooling method for the CP$^3$ model. These measures are quite consistent, especially those corresponding to the longest correlation lengths, with the geometrical determinations.

For the CP$^3$ model the results are reported in Table VII and plotted in Figs. 7 and 8. $\chi_t^{cool}$ shows scaling and the test of universality is satisfactory. Furthermore, they are consistent with the measurements obtained by the field theoretical method with the operator $q_1^l$.

G. Conclusions

The geometrical approach gives a good definition of lattice topological susceptibility for $N = 10$. On the other hand, we showed that $N = 4$ is not large enough to suppress the unphysical configurations contributing to $\chi_t^{l}$, at least for $\xi \leq 30$. The other methods, field theoretical and cooling, give consistent measures of $\chi_t$. We finally quote for the CP$^3$ model $\chi_t \xi_G \simeq 0.06$ with an uncertainty of 10–20%.

V. THE STRING TENSION

A. Wilson loops and finite-size effects

In the CP$^{N-1}$ models it is possible to define the (Abelian) Wilson loop

$$W(C) = \prod_{n,\mu \in C} \lambda_{n,\mu}. \quad (37)$$

The large-$N$ expansion predicts an exponential area law
behavior for sufficiently large Wilson loops [3],

\[ W(C) \sim e^{-\sigma A(C) - \rho P(C)} \text{ for } A(C) \gg \xi^2, \]

where \( \sigma \) is the Abelian string tension and \( \rho \) is a (renormalization-dependent) perimeter term. This implies also that the dynamical matter fields do not screen the linear potential at any distance. The large-\( N \) prediction for \( \sigma \) is \( \sigma \xi^2 = \pi/N \).

Starting from the rectangular Wilson loops, we can define the Creutz ratios as

\[ \chi(l, m) = \frac{W(l, m-1) W(l-1, m)}{W(l, m) W(l-1, m-1)}. \]

The double ratio takes care of renormalization effects [constant and perimeter terms in \( \ln W(l, m) \)]. It is therefore easier to extract the string tension from \( \chi(l, m) \).

In principle one can also define the Polyakov line and study the correlation of two such lines, thus extracting the particle-antiparticle potential. In practice the signal is so small and noisy that one can hardly extract a physically meaningful number.

In order to extract the string tension from our simulations, we should understand the behavior of the large Abelian Wilson loop of a confining theory in a 2D finite lattice with periodic boundary conditions.

To this purpose, consider a simple 2D model whose lattice gauge field propagator \( \Delta_{\lambda} \) is

\[ \Delta_{\lambda}(k) = \frac{1}{k^2}, \quad \hat{k}_\mu = 4 \sin^2 \frac{k_\mu}{2}. \]

For this model we find that a rectangular \( R \times T \) Wilson loop in a \( L \times L \) lattice with periodic boundary conditions has the form

\[ \ln W(R, T) = -\frac{1}{2} \int \frac{d^2k}{(2\pi)^2} \frac{\sin^2(k_1 R/2)}{\sin^2(k_1/2)} \frac{\sin^2(k_2 T/2)}{\sin^2(k_2/2)} \hat{k}^2 \Delta_{\lambda}(k), \]

where the integral must be evaluated by eliminating the zero mode. The result of the integral is

\[ \ln W(R, T) = \frac{1}{2} RT \left( 1 - \frac{RT}{L^2} \right). \]

As expected the propagator defined in Eq. (40) gives rise to a linear confining potential with a string tension \( \sigma = \frac{1}{4} \), but the finite-size corrections are not small. For the Creutz ratios \( \eta(R) = \chi(R, R) \) we obtain (in the following we will consider only Creutz ratios with equal arguments)

\[ \eta(R) = \frac{1}{2} \left[ 1 - \left( \frac{2R - 1}{L} \right)^2 \right]. \]

From this analysis we learn that, if there are no screening effects in the \( CP^{N-1} \) models, for a sufficiently large \( R \) the behavior of the Creutz ratios \( \eta(R) \) should be

\[ \eta(R) \approx \sigma \left[ 1 - \left( \frac{2R - 1}{L} \right)^2 \right]. \]
B. Monte Carlo results

The gauge degrees of freedom are strongly fluctuating in the numerical simulation; therefore large Wilson loops are hard to measure. The action $S_g$ allows us to define improved estimators for operators that are linear with respect to each $\lambda_{n,\mu}$ variable, such as the Wilson loops. Improved estimators can be obtained by replacing each $\lambda_{n,\mu}$ with its average $\lambda_{n,\mu}^{imp}$ in the field of its neighbors:

$$
\lambda_{n,\mu}^{imp} = \frac{\int d\lambda_{n,\mu} \lambda_{n,\mu} \exp[2\beta N \text{Re}(\bar{z}_{n+\mu} z_{n,\mu})]}{\int d\lambda_{n,\mu} \exp[2\beta N \text{Re}(\bar{z}_{n+\mu} z_{n,\mu})]}
= \bar{z}_{n+\mu} z_n \frac{I_1(2\beta N |\bar{z}_{n+\mu} z_n|)}{|\bar{z}_{n+\mu} z_n| \frac{I_0(2\beta N |\bar{z}_{n+\mu} z_n|)}{I_0(2\beta N |\bar{z}_{n+\mu} z_n|)}}
$$

where $I_0$ and $I_1$ are modified Bessel functions.

Another way of reducing the noise is measuring the Wilson loops on cooled configurations [27]. Few cooling steps should leave intact the long-range physical quantities, such as the string tension, reducing the noise coming from the short-ranged modes. Cooling as other similar techniques, smearing and fuzzy operators, provides a sequence of approximate improved estimators. The Creutz ratios measured on cooled configurations as function of the cooling step are seen to reduce the errors and give origin to a plateau, whose length depends on the size of the involved Wilson loops. Then the cooling procedure starts to destroy the physical signal. Our averages and errors are estimated on the plateau measurements.

To begin with we present data for the $CP^3$ model. In Fig. 14 we show the ratios $\eta_{\sigma}(R) \xi_0^2$ and $\eta_{\sigma}(R) \xi_0^2$ vs $r = R/\xi_0$. Data were obtained by using the action $S_g$ and at $\beta = 0.8$ on a $60 \times 60$ lattice. The improved estimators defined in Eq. (47) allow good mean up to $r \approx 2$, for larger $r$ the signal becomes too noisy. By using cooling we found clear signals up to $r \approx 3$. As Fig. 14 shows, the two sets of data are in perfect agree-