Simplicial chiral models

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Principal chiral models on a \((d-1)\)-dimensional simplex are introduced and studied analytically in the large \(N\) limit. The \(d = 0, 2, 4, \text{and } \infty\) models are explicitly solved. The relationship with standard lattice models and with few-matrix systems in the double scaling limit is discussed.

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The importance of understanding the large \(N\) limit of matrix-valued field models cannot be underestimated. Not only is this the basic ingredient of the \(1/N\) expansion in the physically relevant case of QCD, but also the existence of large \(N\) criticalities for finite values of the coupling is the starting point for the approach to two-dimensional quantum gravity known as the double scaling limit. Moreover, we note that solutions of few-matrix systems may have a direct application to more complex systems in the context of strong-coupling expansion, since they may be interpreted as generating functionals for classes of group integrals that are required in strong-coupling calculations [1]. Unfortunately, our knowledge of exact solutions for the large \(N\) limit of unitary matrix models is still impressively poor. After the solution of Gross and Witten [2] of the single link problem, exact results were obtained only for the external field problem and a few toy models \((L = 3, 4 \text{ chiral chains})\) [3,4].

In this Brief Report we introduce a new class of lattice chiral models, whose large \(N\) behavior can be analyzed by solving an integral equation for the eigenvalue distribution of a single Hermitian semidefinite positive matrix. The models we are going to study are principal chiral models, with a global \(U(N) \times U(N)\) symmetry, defined on a \((d-1)\)-dimensional simplex formed by connecting in a fully symmetric way \(d\) vertices by \((d-1)(d-2)/2\) links. The partition function for such a system is defined to be

\[
Z_d = \int \prod_{i=1}^{d} dU_i \exp \left( N\beta \sum_{i \neq j=1}^{d} \text{Tr}[U_i U_j^\dagger + U_j U_i^\dagger] \right). \tag{1}
\]

Despite its apparent simplicity, this class of models includes most of the previously known solvable systems. As a function of the parameter \(d\), which specifies the coordination number of lattice sites, it interpolates between the two-dimensional Gross-Witten model (with third order phase transition) and the infinite-dimensional mean field solution (showing a first order phase transition) of standard infinite-volume lattice models. It also includes a case which, in the double scaling limit, corresponds to that of a \(c = 1\) conformal field theory.

It is possible to eliminate the direct interactions among the unitary matrices \(U_i\) by introducing an identity in the form

\[
1 = \frac{\int dA \exp \left( -N\beta \text{Tr} \left( A - \sum_{i=1}^{d} U_i \right) \left( A^\dagger - \sum_{j=1}^{d} U_j^\dagger \right) \right)}{\int dA \exp(-N\beta \text{Tr}AA^\dagger)} , \tag{2}
\]

where \(A\) is an \(N \times N\) complex matrix. As a consequence we obtain \(Z_d = \tilde{Z}_d/\hat{Z}_0\) where

\[
\tilde{Z}_d = \int \prod_{i=1}^{d} dU_i dA \exp \left\{ -N\beta \text{Tr}AA^\dagger + N\beta \text{Tr}A \sum_i U_i^\dagger + N\beta \text{Tr}A^\dagger \sum_i U_i - N^2\beta d \right\}. \tag{3}
\]

We now introduce the function

\[
F[BB^\dagger] = \frac{1}{N^2} \ln \int dU \exp \left( \frac{N}{2} \text{Tr}[BU^\dagger + UB^\dagger] \right). \tag{4}
\]

\(F\) is a known function of the eigenvalues \(x_i\) of the Hermitian semipositive definite matrix \(BB^\dagger\). More specifically, in the large \(N\) limit, we know that [5,6]

\[
F(x_i) = \frac{1}{N} \sum_i \left( r + x_i \right)^{1/2} \frac{1}{2N^2} \sum_{i,j} \ln \left[ \frac{(r + x_i)^{1/2} + (r + x_j)^{1/2}}{2} \right] - \frac{r - 3}{4} - \frac{3}{4} \tag{5}
\]

and there are two distinct phases: (a) weak coupling \(r = 0\); (b) strong coupling \((1/N) \sum_i (r + x_i)^{-1/2} = 1.\) Up to irrelevant factors, it follows that

\[
\tilde{Z}_d = \int dB \exp \left\{ -\frac{N}{4\beta} \text{Tr}BB^\dagger + N^2dF(BB^\dagger) - N^2\beta d \right\} , \tag{6}
\]

where \(B\) replaces \(2\beta A.\)

Morris [7] has shown that the angular integration can be performed in the case of complex matrices, and, again up to irrelevant factors, we may replace Eq. (6) by
The integrand in Eq. (5) may also be written explicitly in the form
\[
\exp\left\{ -N \frac{1}{4\beta} \sum_{i} x_{i} + N^{2}dF(x_{i}) - N^{2}\beta \right\} .
\]

In the large $N$ limit it is legitimate to evaluate this integral by a saddle-point method. The saddle-point equation resulting from Eqs. (5) and (7) is
\[
\frac{\sqrt{r + x_{i}}}{2\beta} - d = \frac{1}{N} \sum_{j \neq i} \left( 4 - d \right) \sqrt{r + x_{i} + d} \sqrt{r + x_{j}} \frac{x_{i} - x_{j}}{x_{i} + d},
\]

where the integration region is restricted by the condition $0 \leq a \leq b$, with $a$ and $b$ determined dynamically. In particular, the normalization condition
\[
\int_{a}^{b} \rho(z') dz' = 1
\]
must always be satisfied. Furthermore, one has the constraint
\[
\int_{a}^{b} \rho(z') \frac{dz'}{z'} \leq 1,
\]
with the equality holding exactly in the strong-coupling region where $a = \sqrt{r}$.

We introduce a new variable $z_{i} \equiv \sqrt{r + x_{i}}$ and, in the large $N$ limit, we assume that these eigenvalues lie in a single interval $[a, b]$. Denoting the large $N$ eigenvalue density by $\rho(z)$, Eq. (8) becomes an integral equation for $\rho(z)$,
\[
\frac{z}{2\beta} - d = \int_{a}^{b} dz' \rho(z') \left[ \frac{2}{z - z'} - \frac{d}{z + z'} \right],
\]
where $0 \leq a \leq b$, with $a$ and $b$ determined dynamically. In particular, the normalization condition
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Let us begin by first discussing several simple cases where Eq. (9) can be solved readily. For $d = 0$, the problem reduces to one with a pure Gaussian interaction, and, by a more or less standard technique, one finds that
\[
\rho(z) = \frac{1}{4\pi\beta} \sqrt{1 + \beta - z^{2}}.
\]

For $d = 0$, there is no weak-coupling phase. Note also that, since the $F$ term in Eq. (6) vanishes for $d = 0$, Eq. (12) is obtained with $a = 0$. As a consequence, up to a constant,
\[
\tilde{Z}_{0} = \exp(N^{2}\ln \beta),
\]
as expected.

When $d = 2$ we obtain
\[
\rho_{w}(z) = \frac{1}{4\pi\beta} \sqrt{8\beta - (z - 4\beta)^{2}}, \quad \beta \geq \frac{1}{2},
\]
and one may show that all results are consistent with a reinterpretation of the model as a Gross-Witten [2] one-plaquette model, with $\beta_{c} = \frac{1}{2}$.

When $d = 3$ the model can be mapped into the three-link chiral chain, which is known to possess a third order phase transition at $\beta_{c} = \frac{3}{4}$ [3].

The first nontrivial and new situation begins at $d = 4$. We have explicitly solved the $d = 4$ model, both in the weak- and in the strong-coupling phase. The eigenvalue density may be expressed in terms of elliptic integrals, and supplementary conditions allow for the determination of $a$ and $b$.

Introducing the variable $k(\beta) = \sqrt{1 - a^{2}/b^{2}}$, in weak coupling we obtain, in terms of standard elliptic integrals $K, \Pi$, and $E$,
\[
\rho_{w}(z) = \frac{b}{2\pi^{2}\beta} \left( z^{2} - a^{2} \right)^{1/2} \left( b^{2} - z^{2} \right)^{-1/2}
\]
\[
\times \left[ K(k) - \frac{a}{b^{2}} \Pi \left( 1 - \frac{z^{2}}{b^{2}}, k \right) \right],
\]
with the condition $4\pi\beta = bE(k)$. In strong coupling we have
\[
\rho_{s}(z) = \frac{1}{2\pi^{2}\beta} \frac{z^{2}}{b} \left( z^{2} - a^{2} \right)^{-1/2} \left( b^{2} - z^{2} \right)^{-1/2}
\]
\[
\times \left[ \left( b^{2} - a^{2} \right) K(k) - (z^{2} - a^{2}) \Pi \left( 1 - \frac{z^{2}}{b^{2}}, k \right) \right],
\]
(16)

with the condition $4\pi\beta = bE(k) - (a^{2}/b^{2})K(k)$. In both regimes, Eq. (10) must also be satisfied. Closed form solutions for the constraints may be obtained at criticality: when $\beta = \beta_{c} = \frac{1}{d}$, we get $a = 0$, $b = \pi$, and
\[
\rho_{c}(z) = \frac{z}{\pi^{2}} \ln \frac{1 + \sqrt{1 - z^{2}/\pi^{2}}}{1 - \sqrt{1 - z^{2}/\pi^{2}}}.
\]
(17)

Let us finally observe that a large $d$ solution of Eq. (9) may easily be found by assuming $\rho(z) \rightarrow \delta(z - \bar{z})$. The weak-coupling solution is
\[
\bar{z} = \beta d \left[ 1 + \sqrt{1 - \frac{1 - \beta d}{\beta d}} \right], \quad \beta \geq \beta_{c} = \frac{1}{d},
\]
and for strong coupling $\bar{z} = 0$. Amusingly enough, this solution turns out to coincide with the large $D \equiv d/2$ mean field solution [8] of infinite-volume principal chiral models on $D$-dimensional hypercubic lattices with the same coordination number as our corresponding models.

We would like to add a few comments. Solving Eq. (9) is certainly a well defined problem for any value of $d$, and in particular we expect to be able to find explicit solutions for simple cases, such as $d = 1$ and $d = 3$. It is also possible to analyze Eq. (9) numerically; details of our analytical and numerical techniques will be reported elsewhere; we only mention that for sufficiently large $\beta \geq \beta_{c}$, we can get the eigenvalue distribution with desired accuracy, while near criticality convergence is slow: however within 1% accuracy we have evidence that $\beta_{c} = 1/d$ for
all integer values of \(d\) [9]. It would be quite interesting to achieve more information, both qualitative and quantitative, on the \(d\) dependence of the phase transition.

The thermodynamical quantity whose computation is easiest is the internal energy per unit link, \(w_1\), which may be obtained from

\[
d(d - 1)w_1 = \frac{1}{4\beta^2} \int_a^b dz' \rho(z')(z'^2 - r) - d - \frac{1}{\beta}. \tag{19}
\]

One may then extract, in the vicinity of \(\beta_c\), the critical exponent for the specific heat, \(\alpha\). At present we know that when \(d = 2, \alpha = -1\), when \(d = 3, \alpha = -\frac{1}{2}\), when \(d = 4, \alpha = 0\), and for sufficiently large \(d\) the transition is first order, i.e., \(\alpha = 1\).

It is worth observing in this context that a more general model involving four unitary matrices and three couplings, interpolating between our \(d = 4\) case and the four-link chiral chain, can be reexpressed as a model of two coupled complex matrices and admits many solvable limits, all characterized by \(\alpha = 0\), which corresponds to a \(c = 1\) conformal field theory.

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