ABSENCE OF HIGHER DERIVATIVES IN THE RENORMALIZATION OF PROPAGATORS IN NON-RENORMALIZABLE THEORIES

D. ANSELMI

Dipartimento di Fisica "E. Fermi", Università di Pisa, and INFN

I consider the renormalization of propagators in non-renormalizable theories in arbitrary space-time dimensions. When the space-time manifold admits a metric of constant curvature the propagator is not affected by higher derivatives. More generally, certain lagrangian terms are not turned on by renormalization, if they are absent at the tree level. This property restricts the structure of the action of a non-renormalizable theory, in particular quantum gravity.

Despite decades of strenous effort in the search for approaches "beyond quantum field theory" a satisfactory description of quantum gravity is still elusive. Non-renormalizable quantum field theory is neither less predictive, nor less consistent, than the known alternative approaches, but its structure is more easily classifiable. In this talk I illustrate some results on the renormalization of quantum field theories with infinitely many couplings. Details are contained in ref.[1].

Usually, non-renormalizable theories are used as effective low-energy theories, because the number of parameters that are necessary to remove the divergences is finite at low energies (but grows with the energy). The first step in the task of giving mathematical sense to quantum field theories with infinitely many parameters at arbitrary energies is to show that a unitary propagator is not driven by renormalization into a non-unitary (typically, higher-derivative) propagator. This result can be proved when the space-time manifold admits a metric of constant curvature. Furthermore, it is possible to screen the terms of the lagrangian and prove that, for example, a whole class of terms is not turned on by renormalization, if it is absent at the tree level.

I work in the Euclidean framework and use the background field method[2], where the field redefinitions are gauge-covariant and therefore do not change the BRS transformations of the fields (see for example[3]). The background field method is equivalent to the choice of a specific class of gauge fixings. General gauge invariance can be proved using the formalism of refs.[4,5].

The pure gravity action

$$\mathcal{L} = -\frac{1}{\kappa^2}\int \sqrt{g}\ R \tag{1}$$

is not stable with respect to renormalization, and infinitely many terms containing arbitrarily high powers and derivatives of the curvature tensors need to be introduced. However, the theory (1) is one-loop finite[6]. Indeed, the one-loop divergences $\Delta_1\mathcal{L}$, which are in general a linear combination of

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}, \qquad R_{\mu\nu}R^{\mu\nu}, \qquad R^2, \tag{2}$$

can be simplified using the identity

$$\sqrt{g}\left(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2\right) = \text{total derivative} \tag{3}$$

and become proportional to the vacuum field equations ($R_{\mu\nu} = 0$)[6]:

$$\Delta_1\mathcal{L} = \frac{1}{8\pi^2\varepsilon}\sqrt{g}\left(\frac{1}{120}R^2 + \frac{7}{20}R_{\mu\nu}R^{\mu\nu}\right). \tag{4}$$

Then $\Delta_1\mathcal{L}$ can be removed with a redefinition of the metric tensor, precisely

$$g_{\mu\nu} \rightarrow g_{\mu\nu} + \frac{\kappa^2}{8\pi^2\varepsilon}\frac{1}{20}\left(-7R_{\mu\nu} + \frac{11}{3}g_{\mu\nu}R\right).$$

When gravity is coupled to matter, the counterterms R^2, $R_{\mu\nu}^2$ are nontrivial and can in principle be responsible for the appearance of unphysical singularities in the graviton propagator. These unphysical singularities can be ignored if the perturbative expansion is truncated to a finite power of the energy[7], e.g. in the framework of effective field theory. On the other hand, in a fundamental theory the absence of unphysical singularities in the propagators is a necessary condition for unitarity. Fortunately, in the presence of matter the undesired counterterms R^2, $R_{\mu\nu}^2$ can be traded for a redefinition of the metric tensor plus renormalizations of the vertex couplings, namely there exists a subtraction scheme where the graviton propagator is not affected by higher derivatives. This fact generalizes to every order in the perturbative expansion and in arbitrary space-time dimensions, under some mild assumptions.

The divergences (4) can cured in a different way, by means of parameter redefinitions instead of field redefinitions, but then new coupling constants need to be introduced and (1) is turned into higher-derivative quantum gravity,

$$\mathcal{L}' = \frac{1}{\kappa^2}\sqrt{g}\left(-R(x) + \alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu}\right), \tag{5}$$

which is renormalizable[8], but not unitary. It is therefore important to know that there exists a subtraction scheme where (1) is not driven to (5).

However, the identity (3) is peculiar of four dimensions and the finiteness of one-loop pure quantum gravity is a coincidence, spoiled by the presence of matter[6]. Moreover, Goroff and Sagnotti[3] proved that pure quantum gravity is not finite at the second loop order, but there appears a counterterm proportional to

$$R_{\mu\nu}^{\rho\sigma}R_{\alpha\beta}^{\mu\nu}R_{\rho\sigma}^{\alpha\beta}, \tag{6}$$

which cannot be reabsorbed by means of field redefinitions. Finally, in higher dimensions pure quantum gravity is not even finite at the one-loop order. Therefore it is natural to wonder whether the unphysical singularities of the graviton propagator do survive in higher space-time dimension d. This would be an argument singling out the specialness of four dimensions at the level of effective field theory. At present, only power-counting renormalizability points to the specialness of four dimensions, but this argument does not apply to quantum gravity. The results that I report here imply that in fact higher dimensions are not disfavoured with respect to four dimensions in the realm of effective field theory, because suitable generalizations of (3) do exist.

If the metric is expanded around a flat background,

$$g_{\mu\nu} = \delta_{\mu\nu} + \phi_{\mu\nu},$$

the dimension-independent formula

$$R_{\mu\nu\rho\sigma} = \frac{1}{2}\left(\partial_\rho\partial_\nu\phi_{\mu\sigma} - \partial_\rho\partial_\mu\phi_{\nu\sigma} - \partial_\sigma\partial_\nu\phi_{\mu\rho} + \partial_\sigma\partial_\mu\phi_{\nu\rho}\right) + \mathcal{O}(\phi^2)$$

implies that in the combination

$$\sqrt{g}\ \mathrm{G} \equiv \sqrt{g}\left(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2\right) \tag{7}$$

the sum of the quadratic terms in $\phi_{\mu\nu}$ is a total derivative in every space-time dimension. This ensures that the integral $\int\sqrt{g}\ \mathrm{G}$ is a vertex,

$$\int\sqrt{g}\ \mathrm{G} = \mathcal{O}(\phi^3), \tag{8}$$

and therefore the divergences of the form (2) do not affect the propagator with higher derivatives. It is easy to prove that the right-hand side of (8) does not vanish in dimension greater than four.

This observation can be used, together with a certain amount of tensor algebra, to prove that, in the framework of the dimensional-regularization technique, the lagrangian

$$\mathcal{L}=\frac{1}{\kappa^{d-2}}\sqrt{g}\left[-R+\sum_{n=1}^{\infty}\lambda_n\kappa^{n(d-2)}\Im_n[R,\nabla]\right] \tag{9}$$

preserves its form under renormalization. Here $\Im_n[R,\nabla]$ collectively denote the gauge invariant terms of dimension $n(d-2)+2$ that can be constructed with three or more Riemann tensors $R_{\mu\nu\rho\sigma}$ and the covariant derivative ∇, up to total derivatives. For example, $\Im_1[R,\nabla]$ in six dimensions is a linear combination of terms of the form $R_{\mu\nu\rho\sigma}R_{\alpha\beta\gamma\delta}R_{\varepsilon\zeta\eta\xi}$, with all possible contractions of indices, but does not contain the terms $R_{\mu\nu\rho\sigma}\nabla_\alpha\nabla_\beta R_{\varepsilon\zeta\eta\xi}$, which would affect the graviton propagator with higher derivatives. If d is odd only the even ns contribute to (9). The Ricci tensor $R_{\mu\nu}$ and the scalar curvature R need not appear explicitly in $\Im_n[R,\nabla]$, since they can be removed by means of field redefinitions.

If a cut-off is used, but the subtraction scheme is still such that the renormalized cosmological constant vanishes, in dimension $d>4$ the term (7) can be generated by renormalization, multiplied by a power of the cut-off $\overline{\Lambda}$: $\overline{\Lambda}^{d-4}f(\overline{\Lambda}\kappa^2)\sqrt{g}\,\mathrm{G}$. The appropriate generalization of (9) reads

$$\mathcal{L}=\frac{1}{\kappa^{d-2}}\sqrt{g}\left[-R+\lambda\kappa^2\ \mathrm{G}+\sum_{n=1}^{\infty}\lambda_n^{\prime}\kappa^{2n+2}\Im_n^{\prime}[R,\nabla]\right]. \tag{10}$$

The renormalizability of (9) and (10) does not depend on the expansion $g_{\mu\nu}=\delta_{\mu\nu}+\phi_{\mu\nu}$. However, it is only with respect to this expansion that higher derivatives do not appear in the graviton propagator. If, for example, the metric in (9) or (10) is expanded around an instanton background, then the graviton propagator does contain higher derivatives.

These results generalize immediately to theories containing massless fields of spins 0, 1/2, 1, 3/2 and 2 coupled together. The fields have to be massless to be allowed to set the renormalized cosmological constant to zero.

In the presence of a cosmological term, if the space-time manifold admits a metric $\overline{g}_{\mu\nu}$ of constant curvature,

$$\overline{R}_{\mu\nu\rho\sigma}=\frac{\Lambda}{(d-1)\,(d-2)}\left(\overline{g}_{\mu\rho}\overline{g}_{\nu\sigma}-\overline{g}_{\mu\sigma}\overline{g}_{\nu\rho}\right),$$

$$\overline{g}_{\mu\nu} = \frac{\delta_{\mu\nu}}{\left(1 + \frac{\Lambda x^2}{4(d-1)(d-2)}\right)^2}, \tag{11}$$

it is convenient to expand around this metric,

$$g_{\mu\nu} = \overline{g}_{\mu\nu} + h_{\mu\nu}.$$

Then a further generalization of (8) can be proved, namely

$$\int \sqrt{g}\ \widehat{\mathrm{G}} = \frac{8(d-3)}{(d-1)(d-2)^2}\Lambda^2 \int \sqrt{\overline{g}} + \mathcal{O}\left(h^3\right), \tag{12}$$

where

$$\widehat{\mathrm{G}} = \widehat{R}_{\mu\nu\rho\sigma}\widehat{R}^{\mu\nu\rho\sigma} - 4\widehat{R}_{\mu\nu}\widehat{R}^{\mu\nu} + \widehat{R}^2 + \frac{4(d-3)}{(d-1)(d-2)}\Lambda\left(R - \Lambda\right), \tag{13}$$

and

$$\widehat{R}_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - \frac{\Lambda}{(d-1)\,(d-2)}\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right).$$

Thanks to (12) the theory with lagrangian

$$\mathcal{L} = \frac{1}{\kappa^{d-2}}\sqrt{g}\left[-R + \Lambda + \lambda\kappa^2\ \widehat{\mathrm{G}} + \sum_{n=1}^{\infty}\lambda_n\kappa^{2n+2}\Im_n[\widehat{R}, \nabla, \Lambda]\right] \tag{14}$$

is renormalizable. The object $\Im_n[\widehat{R}, \nabla, \Lambda]$ collectively denotes the gauge-invariant terms of dimension $2n+4$ that can be constructed with three or more tensors $\widehat{R}_{\mu\nu\rho\sigma}$, the covariant derivative ∇, and powers of the cosmological constant Λ, up to total derivatives. The contracted tensor $\widehat{R}_{\mu\nu}$ and the scalar $\widehat{R}$ need not appear explicitly in $\Im_n[R, \nabla, \Lambda]$, since they can be moved away by means of field redefinitions.

Again, the renormalizability of (14) does not depend on the metric $\overline{g}_{\mu\nu}$ around which the expansion is performed. The choice of a vacuum metric $\overline{g}_{\mu\nu}$ with constant curvature is relevant only to the form of the graviton propagator. If the vacuum metric does not satisfy (11), then the graviton propagator of (14) does contain higher derivatives. For example, expanding around a background that satisfies the Einstein field equations with a cosmological term,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \frac{\Lambda}{2}g_{\mu\nu} = 0, \tag{15}$$

but is not of constant curvature, quadratic divergent terms containing arbitrary higher derivatives, such as

$$\overline{\nabla}^{m_1}\overline{R}\cdots\overline{\nabla}^{m_n}\overline{R}\ \overline{\nabla}^{p}h_{\mu\nu}\ \overline{\nabla}^{q}h_{\rho\sigma}, \tag{16}$$

can be constructed. If, on the other hand, the background metric has constant curvature, then the Riemann tensor $\overline{R}_{\mu\nu\rho\sigma}$ is proportional to the metric tensor, the counterterms can be easily classified and the graviton propagator of (14) does not contain higher derivatives.

Finally, the theorem generalizes straightforwardly to the case when fields of spin 0, 1/2, 1, 3/2 and 2 are coupled together.

It is immediate to prove that the metric of constant curvature satisfies the field equations of the complete lagrangian (14), not only (15). This ensures that the background metric is not affected by the radiative corrections, apart from a renormalization of the cosmological constant: the metric of constant curvature is an extremal of (14). However, the metric of constant curvature is not a minimum of the action (14), because the second derivative of (14), calculated on the vacuum metric $\overline{g}_{\mu\nu}$ of constant curvature, receives contributions only from the Einstein and cosmological terms. This means that either the total action (14) is not bounded from below (in the Euclidean framework) or, if it is bounded from below, its absolute minimum is not the metric of constant curvature.

On the other hand, the perturbative calculations (propagator, vertices and Feynman diagrams) of the theory (14) expanded around a metric of constant curvature appear to be well-defined, at least if the curvature is negative (anti-de Sitter space)[9]. When the curvature is positive the Green functions behave less nicely[10]. Thus, the best criterion to choose the perturbative vacuum in a non-renormalizable theory is not necessarily the minimum of the classical action. Strictly speaking, it is not even necessary that the classical action, which has no direct physical significance, be bounded from below. The functional integral can be defined formally.

At the practical level, the renormalization procedure needs to be equipped with the definition of appropriate truncations of the theory. Since the number of parameters is infinite, infinitely many Feynman diagrams contribute at each order of the $\hbar$ expansion. In general, infinitely many diagrams cannot be computed in one shot. To avoid this difficulty, it is convenient to define truncated theories $\Gamma^{(N)}$. In the absence of a cosmological constant, the quantization of a non-renormalizable theory is an expansion in $E\kappa$, where E is the energy scale of the physical process. The truncated theory $\Gamma^{(N)}$ is defined as the theory where only the powers $(E\kappa)^m$ with $m \leq N$ are kept. In the truncated theory finitely many lagrangian terms and coupling constants need to be considered. Moreover, at each order of the $\hbar$ expansion finitely many Feynman diagrams contribute and the divergent terms of order $(E\kappa)^m$ with $m > N$ need not be removed. In the presence of a cosmological constant Λ, there is no Λ-independent definition of the "typical energy" E of a physical process and the $2N$th truncated theory is defined as the theory where only the powers $(E\kappa)^p\left(\Lambda\kappa^2\right)^q$ with $p+2q\leq 2N$ are kept,

$$\mathcal{L}_{2N}=\frac{1}{\kappa^{d-2}}\sqrt{g}\left[-R+\Lambda+\lambda\kappa^2\ \widehat{\mathrm{G}}+\sum_{n=1}^{N-2}\lambda_n\kappa^{2n+2}\Im_n[\widehat{R},\nabla,\Lambda]\right].$$

To conclude, I believe that the true meaning of the identity (3) and its generalizations, such as (8) and (12), is the absence of higher-derivative corrections to the propagators in theories with infinitely many couplings, rather than the finiteness of special truncations of some theories. It would be interesting to see if these results can be further generalized to simplify the vertices containing higher derivatives of the metric. In effective field theory, the results might be useful for a more convenient treatment of the radiative corrections to the low-energy limit, and to properly address the cosmological constant problem. Hopefully, some day we will also build the fundamental theory of quantum gravity from its effective field theory.

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