Phase transitions in topological string theory and matrix models

Based on

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Motivations

- Recent results about large N phase transitions in models that should describe some black hole microstates counting. The validity of OSV conjecture $Z_{BH} = |Z_{top}(g_s, t)|^2$, in these cases, seems to depend on these phase transitions

- Phase transitions are known to occur in topological string theory as one decreases the size of the CY threefold. The critical point is tipically (in Kähler moduli space) the mirror of a conifold point, and there is a universal behavior described by $c = 1$ strings at self-dual radius (BCOV, GV)

- We have now representations of topological string partition functions based on sums over tableaux (topological vertex), which are known to undergo phase transitions as well

- Topological strings and matrix models: matrix models describe at large N 2D gravity $\Rightarrow$ relation between 2D gravity and topological string amplitudes

Results

- From our previous investigations on the relation between q-deformed YM theory in 2D and topological strings it emerged the possibility to study topological string amplitudes on bundles over $\mathbb{P}^1$ in terms of matrix models $\Rightarrow$ in that case it appears a sort of non perturbative completion for topological strings (D-branes degrees of freedom) from the sum over representations $\Rightarrow$ third order phase transitions

- Surprisingly we find that also a finite N formulation, reproducing the perturbative topological string on $X_p = \mathcal{O}(-p) \oplus \mathcal{O}(p-2) \rightarrow \mathbb{P}^1$, exists: q-deformation of the Kostov, Staudacher, Wynter (KSW) model (ordinary KSW is recoverd in a suitable double-scaling limit $p \rightarrow \infty$)

- We present a closed expression for the prepotential for any p at planar level and its conjectured generalization at all genus (checked explicitly till genus 4) $\Rightarrow$ relation between Gromov-Witten invariants and Hurwitz numbers

- We find phase transitions at small area, but with different critical behavior (pure gravity rather than $c = 1$

Plan of the talk

- Short review of q-deformed YM theory on the sphere: relation with topological strings and black-holes. Deformed Douglas-Kazakov phase transition.

- Small distances and phase transitions in topological string theory

- Topological strings on $X_p = \mathcal{O}(-p) \oplus \mathcal{O}(p-2) \to \mathbb{P}^1$: matrix model formulation, relation with the KSW model and Hurwitz theory. Exact results for the free-energy.

- Phase transitions and new critical behavior in topological string theory

Our study was originally motivated by the relation between q-deformed YM on the sphere and the OSV conjecture

$$Z^q_{\mathrm{YM}} = \sum_R \mathrm{dim}^2_q(R)\ q^{\frac{p}{2}C_2(R)}\ e^{i\theta\, C_1(R)}$$

where R runs through the unitary irreps of $U(N)$, $C_1(R)$ and $C_2(R)$ are respectively its first and second Casimir invariants and the *quantum* dimension

$$\mathrm{dim}_q(R) = \prod_{1\le i<j\le N} \frac{\left[R_i - R_j + j - i\right]_q}{\left[j-i\right]_q}$$

with $[x]_q = q^{\frac{x}{2}} - q^{-\frac{x}{2}}$ and $q = e^{-g_s}$. As $N \to \infty$, taking into account coupled $U(N)$ representations:

$$Z^q_{\mathrm{YM}} = \sum_{l=-\infty}^{\infty} \sum_{\widehat{R}_1,\widehat{R}_2} Z^+_{\widehat{R}_1,\widehat{R}_2}(t + p\, g_s l)\, Z^-_{\widehat{R}_1,\widehat{R}_2}(\bar{t} - p\, g_s l)$$

The parameter t is related to the gauge theory data as

$$t = (p-2)\,\frac{N g_s}{2} + i\theta$$

When $\widehat{R}_1 = \widehat{R}_2 = \cdot$ are the trivial representations

$$Z^{\pm}_{\cdot,\cdot} = Z^{X_p}_{top}(t)$$

$Z^{X_p}_{top}(t)$ is the topological string partition function on $X_p = \mathcal{O}(-p) \oplus \mathcal{O}(p-2) \to \mathbb{P}^1$ with Kähler parameter t.

- AOSV intended to check OSV conjecture for type IIA superstrings compactified on X_p, $Z_{BH} = |Z_{top}(g_s,t)|^2$

- Z^q_{YM} is claimed to compute the exact entropy associated to black hole solutions of IIA SUGRA compactified on X_p. At large N (large BH charges) $|Z_{top}(X_p)|^2$ should emerge!

Remark: Notice that they do not have in any case $|Z_{top}(X_p)|^2$ but the object they got is $Z^{\pm}_{\hat{R}_1,\hat{R}_2}$ summed over the external "D-branes" $\Rightarrow$ claimed to represent the non perturbative completion.

Later it was observed that, at large N, Z^q_{YM} has a third order phase transition for $p > 2$ similar to the Douglas-Kazakov phase transition of 2DYM on the sphere at

$$t_c = \frac{1}{2} p(p-2) \log[1 + \tan^2(\frac{\pi}{p})]$$

It is expected that this critical value gives the radius of convergence of the strong coupling expansion (on which is based the check of OSV)

The "small area phase" is instead described by the perturbative closed topological string on the resolved conifold $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$.

It appears interesting to explore the phase structure of this family of models, particulary in relation with topological strings and (possibly) black hole physics $\Rightarrow$ what happens to geometry at small distances?

We expect that matrix models be relevant in discussing tableaux sums (on which are based topological string amplitudes on toric manifolds).

$$Z_{\text{top}}(X_1) =_{N\to\infty} \int DM \exp[-\frac{N}{2t} \text{Tr} (\log M)^2]$$

Phase transitions in topological strings

It has been recognized for a long time that string theory provides in a natural way a deformation of classical geometry. The deformation parameter is given by $\frac{\ell_s}{R}$ where ℓ_s is the string length and R is the characteristic size of the target space.

- $\frac{\ell_s}{R} << 1$ the target geometry can be regarded as a classical background corrected by stringy effects.

- $\frac{\ell_s}{R} >> 1$ the classical geometric intuition breaks down, use some notion of stringy or quantum geometry.

In the context of type A topological string theory on CY manifolds (where R is set by the Kähler moduli) the breakdown of classical geometry can be made precise quantitatively by looking at the behavior of topological string amplitudes.

Consider the genus zero free energy at large radius:

$$F_0(t) = F_0^{Cl}(t) + F_0^{St}(t) = \frac{C}{6}t^3 + \sum_{k=1}^{\infty} N_{0,k} e^{-kt}$$

$F_0^{Cl}(t)$ is the classical contribution, depending on the triple intersection number C. $F_0^{St}(t)$ is the stringy part that is exponentially suppressed at large Kähler parameter t: it comes from world-sheet instantons. $N_{0,k}$ are the infamous Gromov-Witten invariants "counting" holomorphic maps into the CY.

Remark: $F_0(t)$ is a power series in e^{-t} and it will have in general a finite radius of convergence t_c, related to the asymptotic growth of $N_{0,k}$.

Critical behavior in topological strings

The behavior of the series can be parameterized by a critical exponent γ:

$$F_0(t) \sim \sum_{k=1}^{\infty} \sim k^{\gamma-3} e^{k(t_c-t)} \sim (e^{-t_c} - e^{-t})^{2-\gamma}$$

By mirror symmetry computations (local $\mathbb{P}^2$, the quintic) it has been found that

$$N_{0,k} \sim \frac{e^{kt_c}}{k^3 \log^2 k}, \qquad k \to \infty.$$

this indicates $\gamma = 0$ and the critical behavior

$$F_0(t) \sim (e^{-t_c} - e^{-t})^2 \log(e^{-t_c} - e^{-t}).$$

Higher genus Gromov–Witten invariants have the asymptotic behavior [BCOV]:

$$N_{g,k} \sim k^{(\gamma-2)(1-g)-1} e^{kt_c}, \qquad k \to \infty,$$

thus

$$\begin{aligned} F_1(t) &\sim c_1 \log\,(e^{-t_c} - e^{-t}), \\ F_g(t) &\sim c_g (e^{-t_c} - e^{-t})^{(1-g)(2-\gamma)}, \qquad g \geq 2. \end{aligned}$$

the phase transition at $t = t_c$ is common for all $F_g(t)$ and there is a coherent behavior. $t > t_c$ is called Calabi-Yau phase. For $t < t_c$ the nonlinear sigma model is not well defined and classical geometry is misleading$\Rightarrow$ it is a "non-geometric phase", tipically a CFT (LG orbifolds+perturbations), obtained by mirror symmetry (or by linear sigma approach).

Double-scaling limit

Let us consider the total free energy F as a perturbative expansion in powers of the string coupling constant g_s:

$$F(g_s, t) = \sum_{g=0}^{\infty} F_g(t) g_s^{2g-2}.$$

One can define the double-scaled string coupling as

$$\kappa = a g_s (e^{-t_c} - e^{-t})^{\gamma/2-1}$$

and consider the limit

$$t \longrightarrow t_c, \qquad g_s \longrightarrow 0, \qquad \kappa \text{ fixed}$$

The most singular part of $F_g(t)$ survives at each genus, and the total free energy becomes the double-scaled free energy

$$F_{\rm ds}(\kappa) = f_0 \kappa^{-2} + f_1 \log \kappa + \sum_{g \geq 2} f_g \kappa^{2g-2},$$

where $f_g = a^{2-2g} c_g$.

For the known cases the double-scaled theory coincides with the free energy of the $c = 1$ string at the self–dual radius [GV].

Topological strings on $X_p = \mathcal{O}(-p) \oplus \mathcal{O}(p-2) \to \mathbb{P}^1$

X_p has a single Kähler parameter t measuring the $\mathbb{P}^1$. The partition can be computed at all genus by

- Topological vertex (AKMV)
- Gromov-Witten theory (BP)

$$Z_{X_p} = \sum_R W_R^2(q)\, q^{(p-2)\kappa_R/2} (-1)^{l(R)\,p} e^{-l(R)\,t}$$

- l_i are the lengths of the rows and $l(R) = \sum_i l_i$ the total number of boxes
- $k_R = \sum_i l_i(l_i - 2i + 1)$
- $W_R = q^{\kappa_R/4} \prod_{1 \le i < j \le d(l)} \frac{[l_i - l_j + j - i]_q}{[j-i]_q} \prod_{i=1}^{d(l)} \prod_{k=1}^{l_i} \frac{1}{[k-i+d(l)]_q}$

We notice that as $g_s \to 0$, $W_R \to g_s^{-l(R)} \frac{d_R}{l(R)!}$: W_R can be seen as a q-deformation of the dimension d_R of the representation R of the symmetric group $S_{l(R)}$.

This suggests to consider a double-scaling limit

$$g_s \to 0, \quad t \to +\infty, \quad p \to \infty$$

taking $p\, g_s = \tau_2/N$, $(-1)^p e^{-t} = (g_s N)^2 e^{-\tau_1}$ fixed. In this limit topological strings partition function reduces to

$$Z_{X_p} \to Z_{Hurwitz} = \sum_R \left(\frac{d_R}{l(R)!}\right)^2 N^{2l(R)} e^{-\tau_2 \frac{k_R}{2N}} e^{-\tau_1 l(R)}$$

that is the generating functional of simple Hurwitz numbers of $\mathbb{P}^1$ at all genus and degrees!

Simple Hurwitz numbers $H^{\mathbb{P}^1}_{g,d}(1)$ are the number of branched covers of $\mathbb{P}^1$ by genus g of degree d having only simple branched points.

By matching $F_{Hurwitz} = \log[Z_{Hurwitz}]$ and the topological string free-energy F_{X_p}

$$F_{X_p}(g_s, t) = \sum_{g=0}^{\infty}\sum_{d=1}^{\infty} N_{g,d}(p) e^{-dt} g_s^{2g-2}$$

we get a non-trivial relation between Gromov-Witten invariants and simple Hurwitz numbers

$$\lim_{p\to\infty} p^{2-2g-2d} N_{g,d}(p) = (-1)^p \frac{H^{\mathbb{P}^1}_{g,d}(1)_c}{(2g-2+2d)!}$$

Remark: Topological string on X_p can be considered as a q-deformation of the Hurwitz model. It is quite interesting that $Z_{Hurwitz}$ can be written as the large N limit of a finite N model (KSW)

$$Z_{KSW} = \sum_R \left(\frac{\mathrm{dim}(R)}{\Omega_R}\right) e^{-\tau_2 \frac{k_R}{2N}} e^{-\tau_1 l(R)}$$

with $\Omega_R = N^{l(R)} \prod_{i=1}^N \frac{h_i!}{(N-i)!}$. At planar level Z_{KSW} can approximated by a matrix model (as Douglas and Kazakov did for QCD_2) and studied through saddle-point technique, leading to exact results for the phase structure: for large τ_1 the planar free energy is

$$F^0_{KSW} = \sum_{k=1}^{\infty} \frac{k^{k-3}}{k!} \tau_2^{2k-2} e^{-k\tau_1}$$

leading to the exact answer $H^{\mathbb{P}^1}_{0,d}(1)_c = \frac{d^{d-3}}{d!}(2d-2)!$.

Remark: Mathematicians generalized later the answer at all genus (GJV) but the the general structure is still controlled by genus zero saddle-point equation!

Matrix model description of topological strings on X_p

To extract the asymptotic behavior of the GW invariants and to study the free energy we construct a matrix model for the perturbative closed topological string amplitude on X_p at genus zero

$$Z_{X_p} = \sum_R W_R^2 \, q^{(p-2)\kappa_R/2} e^{-l(R)t}.$$

Let us introduce at finite N

- ${}_q\Omega_R = \prod_{i=1}^N \frac{[h_i]!}{[N-i]!}$

- $\dim_q(R) = \prod_{1\le i<j\le N} \frac{\left[l_i - l_j + j - i\right]_q}{\left[j-i\right]_q}$

We have

$$Z_{X_p} = Z_{qKSW} = \sum_R \left(\frac{\dim_q R}{{}_q\Omega_R}\right)^2 q^{(p-1)\kappa_R/2} e^{-l(R)t}$$

This model reduces in the limit $g_s \to 0$, $p \to \infty$ with $g_s p = \tau_2/N$ to the KSW model and its leading order in the large N expansion can be studied in the saddle-point approximation. Introduce the auxiliary 't Hooft parameter $T = N g_s$ and continuous variables in the standard way:

$$\frac{h_i}{N} = \frac{l_i}{N} - \frac{i}{N} + 1 \to \ell(x) - x + 1 = h(x),$$

The delicate point is to evaluate the large N limit of the deformed measure. The numerator of ${}_q\Omega_R$ leads to

$$\log \prod_{i=1}^{N} \prod_{j=1}^{h_i} (q^{\frac{h_i-j}{2}} - q^{-\frac{h_i-j}{2}})^2 = 2 \sum_{i=1}^{N} \sum_{j=1}^{h_i} \log 2 \sinh g_s \frac{h_i - j}{2}$$

which becomes in the large N limit

$$\frac{2N^2}{T} \int_0^1 dx \left(\frac{T^2 h^2}{4} - \frac{\pi^2}{6} + \mathrm{Li}_2(e^{-Th}) \right).$$

Then we can write the effective action controlling the leading large N contribution as follows

$$S = -\int_0^1 \int_0^1 dx dy \log \left| 2 \sinh \frac{T}{2}(h(x) - h(y)) \right| + \frac{2}{T} \int_0^1 dx \mathrm{Li}_2(e^{-Th})$$

$$+ \int_0^1 dx h(x)(t - (p-1)T) + \frac{pT}{2} \int_0^1 dx h^2(x) + (p-1)\frac{T}{3} - \frac{\pi^2}{3T} - \frac{1}{2}t.$$

The planar theory can be, thus, understood as coming from a matrix model: the effective action can be derived from a Chern–Simons–like matrix model with a potential $V(h)$ of the form

$$V(h) = \frac{2}{T}\mathrm{Li}_2(e^{-Th}) + (t - (p-1)T)h + \frac{pT}{2}h^2,$$

and the saddle–point equation is simply

$$\int dh' \rho(h') \coth \frac{T}{2}(h - h') = ph + \frac{2}{T}\log(1 - e^{-Th}) + \frac{t}{T} - (p-1),$$

where the density $\rho(h)$ is defined in terms of the inverse function $x(h)$ as follows $\rho(h) = -\frac{dx(h)}{dh}$. Because of the positivity constraint $h_1 > h_2 > \cdots h_N \geq 0 \Rightarrow h(x) \geq 0$, which the Young tableaux variables h_i must satisfy, the support of $\rho(h)$ will be chosen in the interval $[0, a]$.

The saddle-point equation can be related to a standard Riemann-Hilbert problem:

$$\int_{e^{-\beta}}^{e} \frac{dy}{y}\rho(y)\frac{s+y}{s-y} = p \log s - (t+1) + (p-1)(T-1) - 2\log(1-e^{-1}s)$$

with $-\beta = 1 - Ta$. The normalization of ρ is now

$$\int_{e^{-\beta}}^{e} dy \frac{\rho(y)}{y} = T$$

The support of the density $\rho(s)$ comes from the original tableau variables h, i.e. $[0, a]$.

To solve the saddle point equation we need a further ingredient, namely we have to choose an ansatz for the density $\rho(s)$.

To recover the large radius expansion in e^{-t} the analogy with QCD_2 suggests to choose a chiral, one–cut ansatz: for $x \in [-\beta, -\gamma]$ ($-\gamma < 1$) the $\rho(s)$ is arbitrary, while for $x \in [-\gamma, 1]$ we require ρ to be equal to 1. The effective equation to be solved is

$$\int_{e^{-\beta}}^{e^{-\gamma}} \frac{dy}{y}\frac{\rho(y)}{s-y} = \frac{p}{2s}\log s - \frac{t - p(T-1)}{2s} - \frac{1}{s}\log(1 - e^{\gamma}s)$$

Remark: It can be shown that the free energy is T independent (up overall scaling): in the following $T = 1$.

Let us introduce the variable (defined in terms of the end-points of the cut)

$$w = \left(\frac{e^{-\frac{\gamma}{2}} + e^{-\frac{\beta}{2}}}{2}\right)^{-\frac{2}{p}}$$

the planar free energy can be exactly computed in terms of w. It is given by:

$$\begin{aligned} F_0(w(t),p) &= p(p-2)\mathrm{Li}_3\left(1-\frac{1}{w}\right) + (p-1)^2\mathrm{Li}_3(1-w) \\ &- \frac{p}{6}(p-2)(p-1)^2\log^3(w). \end{aligned}$$

The original Riemann-Hilbert problem is encoded into the algebraic equation (equivalent to the end-points equations)

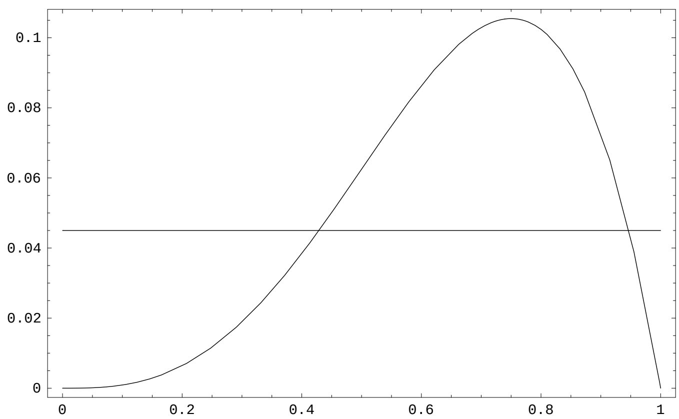
$$e^{-t} = w^{(p-1)^2-1} - w^{(p-1)^2}$$

depending on the Kähler parameter t (with $w > e^{-\frac{t}{p(p-2)}}$).

Remark: For $p = 1$ there is solution for any t and we obtain the expected resolved conifold free-energy

$$F(t) = \mathrm{Li}_3(e^{-t})$$

For $p = 2$ we have still solution for any t leading to $F(t) = -\mathrm{Li}_3(e^{-t})$ (as expected by mathematicians (FJ)). The situation drastically changes for $p > 2$



We want to discuss the large area/CY phase $\Rightarrow t > 0$ and large. The solution

$$w = \sum_{m=0}^{\infty} \frac{(m(p-1)^2 - 2)!}{(mp(p-2)-1)!m!} e^{-mt}$$

exists only for $t > t_c$ with:

$$t_c = \log\left((p(p-2))^{p(2-p)}(p-1)^{2(p-1)^2}\right)$$

Remark: The topological string theory on X_p undergoes a phase transition at t_c.

We can rewrite the free energy as

$$F_0(t) = \sum_{k=1}^{\infty} N_{0,k} e^{-kt}$$

with

$$N_{0,k} = \frac{1}{k!k^2} \frac{((p-1)^2 k - 1)!}{(((p-1)^2 - 1)k)!},$$

By using Stirling's formula, we obtain

$$N_{0,k} \sim e^{kt_c} k^{-7/2}$$

- We have obtained the genus-zero Gromov-Witten invariants in closed form (checked with topological vertex)
- the convergence radius of the string series coincides with the phase transition point t_c
- we have a critical exponent $\gamma = -\frac{1}{2}$ different from all previously known cases!

Higher-genus and double-scaling

- To define a double–scaled theory we need to understand the behavior at higher genus: $F_g(t)$ should have the same critical point

- Important to have a control on higher-genus amplitudes to understand how to cross the transition point (mirror symmetry is not well established in this case (equivariant case)).

- Important because there are very few cases in which higher genus are known in closed form

In principle we should compute $F_g(t)$ from summing over representations. Unfortunately, there is no systematic way to compute corrections to the saddle–point from sums over partitions.

$\Rightarrow$ An ansatz suggested by the undeformed case [GJV] (that can be checked by a vertex computation) allows us to guess the genus 1 formula as function of w:

$$F_1(t) = -\frac{1}{24}\log(w-w_c) - \frac{1}{12}\log(p-1) + \frac{1}{24}(p^2-2p+3)\log w$$

Here w_c is the genus zero critical point and the request of reproducing the (known) critical behavior of Hurwitz theory (given by the $p \to \infty$ limit) fixes the functional form (up two coefficients determined by topological vertex comparison).

For the higher genus case $F_g(t)$ are conjectured to be rational functions of the variable w

$$F_g = \frac{\mathcal{P}_g(w,p)}{(w-w_c)^{5(g-1)}}, \quad \mathcal{P}_g(w,p) = \sum_{i=1}^{5(g-1)} a_{g,i}(p)(w-1)^i.$$

The $a_{g,i}(p)$ have the form

$$a_{g,i}(p) = \frac{b_{g,i}(p)}{(p-1)^n},$$

$n \in \mathbb{N}$ and $b_{g,i}$ polynomials in p with rational coefficients.

The conjecture is verified by direct computation up to $g=4$; the "free" parameter are the $5(g-1)$ coefficients $a_{g,i}(p)$, uniquely determined from the genus g Gromov–Witten invariants up to degree $5(g-1)$. For $g=2$:

$$a_{2,5}(p) = \frac{1}{2880}\frac{p(p-2)}{(p-1)^2}$$

$$a_{2,4}(p) = -\frac{1}{2880}\frac{12-14p+7p^2}{(p-1)^4}$$

$$a_{2,3}(p) = \frac{36-106\,p+161\,p^2-204\,p^3+171\,p^4-72\,p^5+12\,p^6}{2880(p-1)^8}$$

$$a_{2,2}(p) = -\frac{36-90\,p+121\,p^2-60\,p^3-5\,p^4+12\,p^5-2\,p^6}{2880(p-1)^{10}}$$

$$a_{2,1}(p) = -\frac{1}{240}\frac{1}{(p-1)^{10}}.$$

We get closed formulae for the Gromov–Witten invariants $N_{g,d}$ for all d by using Lagrange inversion (here the case $g=1$ and $f=(p-1)^2$)

$$N_{1,k} = \frac{1}{24k}\sum_{\ell=0}^{k-1}\frac{f^{k-\ell}}{\ell!}\prod_{j=1}^{\ell}(k(f-1)+j-1) - \frac{1}{24}\frac{(kf-1)!}{k!(k(f-1))!}(f+2).$$

Assuming the critical behavior in w it is easy to compute the double-scaling. The planar part suggests

$$\mu^{5/2} = g_s^{-2}\frac{(p-1)^8}{4w_c^3}(w-w_c)^5 = g_s^{-2}\frac{(p-1)^8}{4w_c^3}A^5(e^{-t_c}-e^{-t})^{\frac{5}{2}}$$

where $A = \frac{\sqrt{2}}{p-1}w_c^{1-(p-1)^2/2}$ and now consider the limit

$$t \to t_c, \quad g_s \to 0, \quad \mu \text{ fixed.}$$

The total free energy of the model becomes the double–scaled free energy $F_{X_p} \to F_{\rm ds}(z)$. Up to genus one

$$F_{\rm ds}(\mu) = -\frac{4}{15}\mu^{5/2} - \frac{1}{48}\log\mu + \cdots .$$

This is, at this order, the free energy of 2d gravity, $F_{(2,3)}(\mu)$. We recall that $F_{(2,3)}(\mu)$ is determined as a function of μ by the following equation,

$$F''_{(2,3)}(\mu) = -u(\mu),$$

where $u(\mu)$ is a solution of the Painlevé I equation

$$u^2 - \frac{1}{6}u'' = \mu$$

with $u(\mu) = \mu^{\frac{1}{2}} + \cdots, \quad \mu \to \infty$. This leads to the expansion

$$F_{(2,3)}(\mu) = -\frac{4}{15}\mu^{5/2} - \frac{1}{48}\log\mu + \sum_{g\geq 2} a_g \mu^{-5(g-1)/2}$$

with

$$a_2 = \frac{7}{5560}, \quad a_3 = \frac{245}{331776}, \quad a_4 = \frac{259553}{159252480},$$

an so on. In view of the above results for genus $g = 0, 1$, it is natural to conjecture that the double–scaled free energy of topological string theory on X_p equals the free energy of 2d gravity and in fact the numbers a_g are reproduced by our results (up to genus 4).

The non-perturbative proposal

Coming back to the OSV conjecture (AOSV formulation): Z^q_{YM} has been proposed to provide a non-perturbative definition of Z_{X_p}.

Z^q_{YM} undergoes a large N third-order phase transition for $p > 2$ when the Kähler parameter reaches the critical point:

$$t_{\text{np}}(p) = \frac{1}{2}p(p-2)\log\left(1+\tan^2\left(\frac{\pi}{p}\right)\right).$$

We expect that this critical value $t_{\text{np}}(p)$ gives the convergence radius of the strong coupling expansion of Z^q_{YM}.

Remark: Both the perturbative theory Z_{X_p} and the non-pertubative definition Z^q_{YM} undergo a phase transition at small radius for $p > 2$. $\Rightarrow$ It is interesting to compare the critical behaviors as a further probe of the proposal.

- The first things to compare are the radii of convergence $t_c(p)$ and $t_{\text{np}}(p)$, as a function of p. In the perturbative theory, one has

  $$t_c \to 2\log p, \quad p \to \infty,$$

  while

  $$t_{\text{np}} \to \frac{\pi^2}{2}, \quad p \to \infty.$$

  Therefore, at large p, the proposed nonperturbative completion has better convergence properties as a function of t at small radius.

- We have some evidence that double–scaled free energy $F(\mu)$ of Z^q_{YM} is determined by

$$F''(\mu) = v(\mu)^2/4$$

where $v(\mu)$ satisfies the Painlevé II equation

$$2v'' - v^3 + \mu v = 0.$$

This also describes the universality class of the Gross–Witten–Wadia unitary matrix model and of pure 2d supergravity.

Interestingly, it is well–known that the Painlevé I equation does not define 2d gravity beyond the perturbation regime, since the resulting series for the specific heat is not Borel summable. In contrast, the Painlevé II equation has a unique real, pole–free solution with the right asymptotic properties which therefore gives the non–perturbative solution of the theory. This is consistent with the proposal that q-deformed 2d Yang–Mills leads to a nonperturbative definition of topological string theory on X_p.