Exact Spectrum of CHL Dyons

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References

- **Dijkgraaf, Verlinde, Verlinde;**
- **David, Jatkar, Sen;**
- **Cardoso, de Wit, Kappelli , Mohaupt**
- **Kawai;**
- **Gaiotto, Strominger, Xi, Yin**

Plan

- Proposal for dyon degeneracies in N=4
- Questions
- Answers
- Derivation

Motivation

- There has been considerable progress in understanding the quantum structure of black holes in string theory. In many examples one can understand subleading corrections to Bekenstein-Hawking entropy precisely. For example, small black hole that are half-BPS.

- BPS spectrum gives valuable information about the strong coupling structure of the theory both for duality (small charges) and black holes (large charges). Exact spectrum for all intermediate charges.
- Quarter BPS dyons that we consider here are not weakly coupled in any frame (naively), hence more interesting.

Heterotic on T^4 £ T^2

- Total rank of the four dimensional theory is

16 ($\mathrm{E_8 \times E_8}$) **+ 12** ($\mathrm{g}_{\mu\mathrm{m}}$, $\mathrm{B}_{\mu\mathrm{m}}$) = **28**

- **N=4** supersymmetry in **D=4**

- Duality group

$$\mathbf{SO(22,6;Z) \times SL(2;Z)}$$

CHL Orbifolds D=4 and N=4

- Models with smaller rank but same susy.
- For example, a $\mathbf{Z_2}$ orbifold by $\mathbf{\{1, \alpha\ T\}}$

- α flips $\mathbf{E_8}$ factors so rank reduced by **8**.
- T is a shift along the circle, X ! X + π R
  so twisted states are massive.
- Fermions not affected so **N =4** susy.

Why CHL Orbifolds?

- S-duality group is a **subgroup of SL(2, Z)** so counting of dyons is quite different.
- **Wald entropy is modified** in a nontrivial way and is calculable.
- Nontrivial but tractable generalization with interesting physical differences for the spectrum of black holes and dyons.

S = log (Ω)

- A full function worth of impressive agreement to subleading order keeping four-derivative terms and using Wald entropy.

$$
\begin{aligned}
S &= \pi\sqrt{Q_e^2 Q_m^2 - (Q_e \cdot Q_m)^2} \\
&+ F(\frac{Q_e^2}{Q_m^2}, \frac{Q_e \cdot Q_m}{Q_m^2}) + O(1/Q_m^2)
\end{aligned}
$$

Entropy function

- Both microscopic and macroscopic entropy computed from totally different sources equals the extremum of

$$
\begin{aligned}
S_{stat} \;&=\; \frac{\pi}{2}[\frac{a^2+S^2}{S}Q_m^2+\frac{1}{S}Q_e^2 \\
&\quad -2\,\frac{a}{S}\,Q_e\cdot Q_m+128\,\pi\,\phi(a,S)] \\
&\quad +\mathsf{constant}+\mathsf{O}(\mathsf{Q}^{-2})\,,
\end{aligned}
$$

where

$$\phi(a,S) \equiv -\frac{1}{64\pi^2}\{(k+2)\ \ln S + \ln f^{(k)}(a+iS) + \ln f^{(k)}(a+iS)^*\}$$

with

$$f^{(k)}(\tau) \equiv \eta(\tau)^{k+2}\,\eta(N\tau)^{k+2}$$

S-duality group = Γ_1(N)

Because of the shift, there are 1/2 quantized electric charges (winding modes)

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1/2 \\ n \end{pmatrix} = \begin{pmatrix} 1/2 \\ n \end{pmatrix}$$

This requires that c = 0 mod 2
which gives Γ_0(2) subgroup of SL(2, Z).

Spectrum of Dyons

- For a $\mathbf{Z_N}$ orbifold, the dyonic degeneracies are encapsulated by a Siegel modular form $\Phi_k(\Omega)$ of **Sp(2, Z)** of **level N** and **index k** as a function of period matrices Ω of a ***genus two Riemann surface***..

$$\mathrm{k} = \frac{24}{\mathrm{N}+1} - 2$$

Sp(2, Z)

- 4 £ 4 matrices ***g*** of integers that leave the symplectic form invariant:

$$\mathbf{g} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \qquad \mathbf{J} = \begin{pmatrix} 0 & \mathbf{I} \\ -\mathbf{I} & 0 \end{pmatrix}$$

$$\mathbf{g}^{\mathbf{t}}\mathbf{J}\mathbf{g} = \mathbf{J}$$

where A, B, C, D are 2£ 2 matrices.

Genus Two Period Matrix

- Like the τ parameter at genus one

$$\Omega = \begin{pmatrix} \rho & v \\ v & \sigma \end{pmatrix}$$

$$\Omega \to (A\Omega + B)(C\Omega + D)^{-1}.$$

Siegel Modular Forms

- $\mathbf{\Phi_k(\Omega)}$ is a Siegel modular form of weight k and level N if

$$\boldsymbol{\Phi}_{\mathbf{k}}(\boldsymbol{\Omega}') = \{\det(C\Omega + D)\}^k \boldsymbol{\Phi}_{\mathbf{k}}(\boldsymbol{\Omega})$$

$$\Omega' = (A\Omega + B)(C\Omega + D)^{-1},$$

under elements $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ of $G_0(N)$.

Fourier Coefficients

- Define T-duality invariant combinations

$$Q_e^2/2, \quad Q_m^2/2, \quad Q_e \cdot Q_m$$

- Degeneracies d(Q) of dyons are given by the Fourier coefficients of the inverse of an image $\tilde{\Phi}_{\mathrm{k}}(\tilde{\Omega})$ of $\Phi_{\mathrm{k}}(\Omega)$.

$$\frac{\mathbf{1}}{\tilde{\mathbf{\Phi}}_{\mathbf{k}}(\tilde{\mathbf{\Omega}})} = \sum_{\mathbf{m,n,p}} \tilde{\mathbf{q}}^{\mathbf{m}}\, \tilde{\mathbf{p}}^{\mathbf{n}}\, \tilde{\mathbf{y}}^{\mathbf{l}}\, \mathbf{g}(\mathbf{m}, \mathbf{n}, \mathbf{l})$$

$$\mathbf{d}(\mathbf{Q}) = \mathbf{g}(\mathbf{Q}_{\mathbf{m}}^{\mathbf{2}}/\mathbf{2}, \mathbf{Q}_{\mathbf{e}}^{\mathbf{2}}/\mathbf{2}, \mathbf{Q}_{\mathbf{e}} \cdot \mathbf{Q}_{\mathbf{m}}).$$

$$\tilde{\mathbf{q}} \equiv \mathbf{e}^{\mathbf{2}\pi\mathbf{i}\tilde{\rho}},\ \ \tilde{\mathbf{y}} \equiv \mathbf{e}^{\mathbf{2}\pi\mathbf{i}\tilde{\mathbf{v}}},\ \ \tilde{\mathbf{p}} \equiv \mathbf{e}^{\mathbf{2}\pi\mathbf{i}\tilde{\sigma}}$$

Big Black Holes

- Define the discriminant which is the unique quartic invariant of **SL(2) £ SO(22, 6)**

$$\Delta = Q_e^2 Q_m^2 - (Q_e \cdot Q_m)^2$$

- For positive discriminant, big black hole exists with entropy given by

$$S = \pi \sqrt{\Delta}$$

Three Consistency Checks

- All $d(Q)$ are integers.
- Agrees with black hole entropy including sub-leading logarithmic correction,

$$\log d(Q) = S_{BH}$$

- $d(Q)$ is S-duality invariant under $\Gamma_1(N)$

Questions

1) Why does genus-two Riemann surface play a role in the counting of dyons? The group Sp(2, Z) cannot fit in the physical U-duality group. Why does it appear?

2) Is there a microscopic derivation that makes modular properties manifest?

3) Are there restrictions on the charges for which genus two answer is valid?

4) States with negative discriminant predicted but there are no corresponding black holes. Do these states exist?

5) Are there curves of marginal stability? How then to interpret the partition function with no moduli dependence?

1) Why genus-two?

- Dyon partition function can be mapped by duality to genus-two partition function of the left-moving heterotic string on CHL orbifolds.
- Makes modular properties under subgroups of **Sp(2, Z)** manifest.
- Suggests a new derivation of the formulae.

2) Microscopic Derivation

Required twisted determinants can be explicitly evaluated using orbifold techniques (**N=1,2** or **k=10, 6**) to obtain

$$2^{12}\Phi_{10} = \prod_{\alpha=even} (\vartheta_\alpha)^2$$

$$2^{12}\Phi_6 = (\vartheta_{0100}\vartheta_{0110}\vartheta_{1000}\vartheta_{1001}\vartheta_{1100}\vartheta_{1111})^2$$

3) Irreducibility Criteria

- For electric and magnetic charges Q^i_e and Q^i_m that are SO(22, 6) vectors, define

$$\mathrm{I} = \mathrm{gcd}(\mathrm{Q}^i_e \mathrm{Q}^j_m - \mathrm{Q}^j_e \mathrm{Q}^i_m)$$

- Genus-two answer is correct only if I=1.
- **In general I+1 genus will contribute.**

4) Negative discriminant states

- States with negative discriminant are realized as multi-centered configurations.
- In a simple example, the supergravity realization is a two centered solution with field angular momentum $(Q_e \cdot Q_m)/2$
- The degeneracy is given by (2J + 1) in agreement with microscopics.

5) Curves of Marginal Stability

- Quarter-BPS dyons do have curves of marginal stability. For example, we find that the distance between the two centers goes to infinity at the curve of marginal stability and the state decays. Straight lines in the axion-dilaton plane.
- This implies that the degeneracies are valid in certain regions of moduli space.

Φ_k is a complicated beast

- **Fourier representation (Maass lift)**

  Makes integrality of d(Q) manifest

- **Product representation (Borcherds lift)**

  Relates to 5d elliptic genus of D1D5P

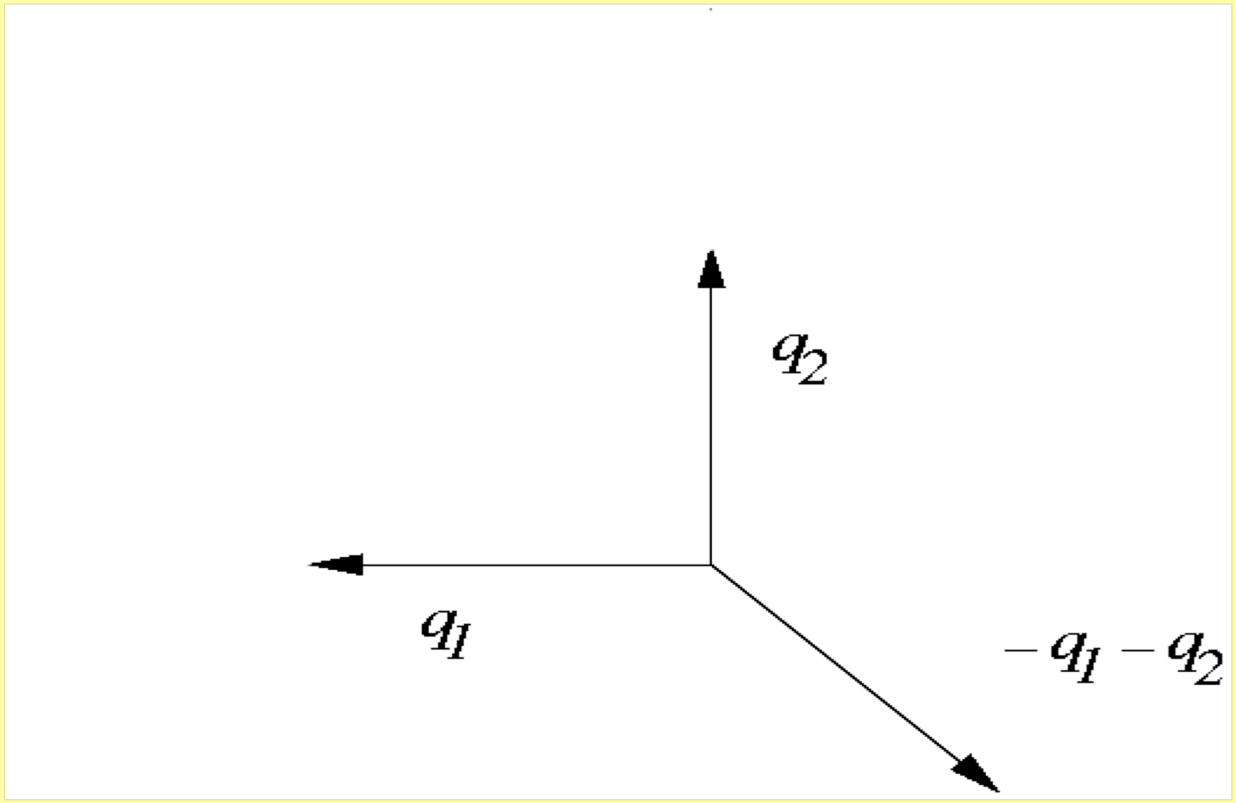
- **Determinant representation (Genus-2)**

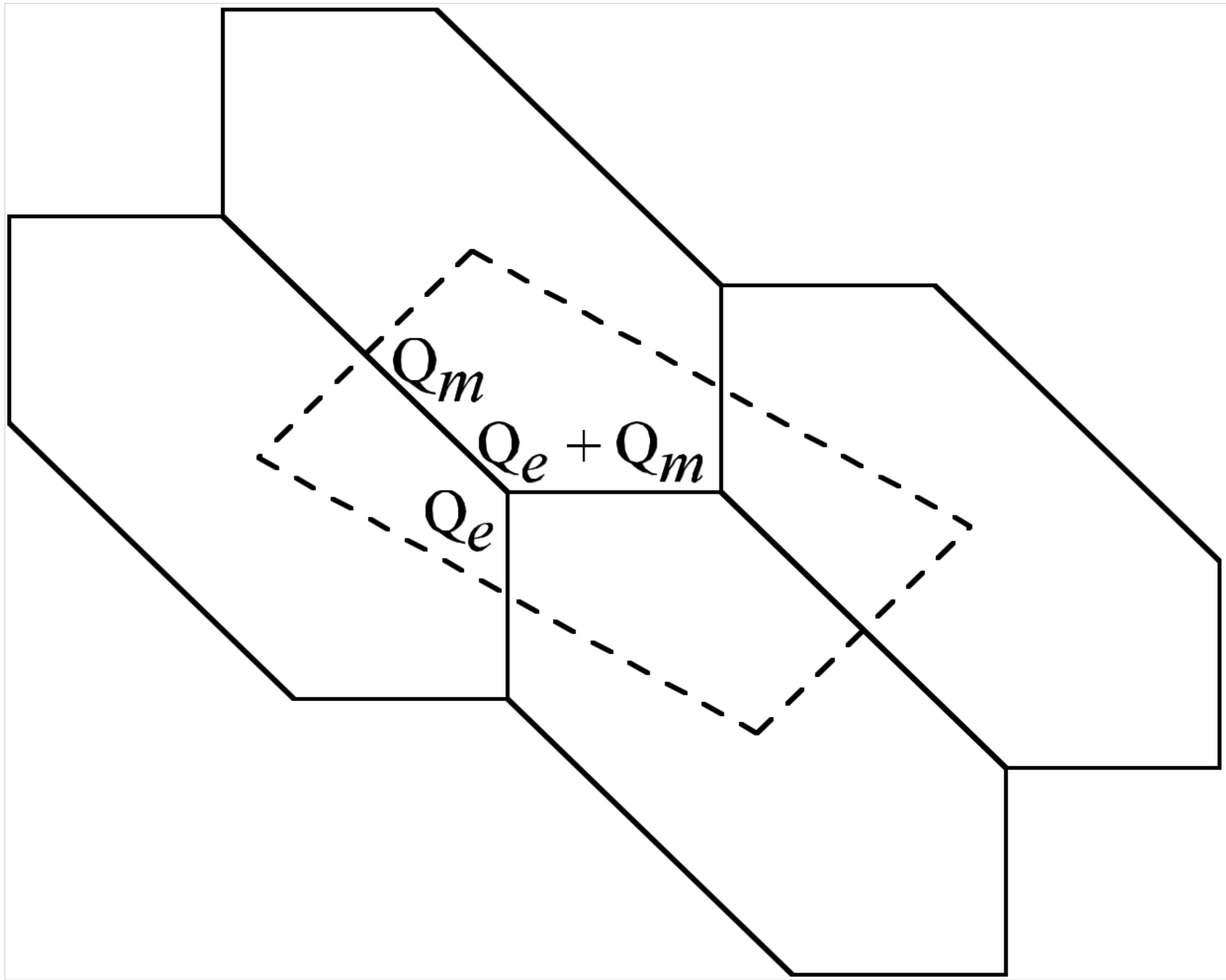
  Makes the modular properties manifest.

String Webs

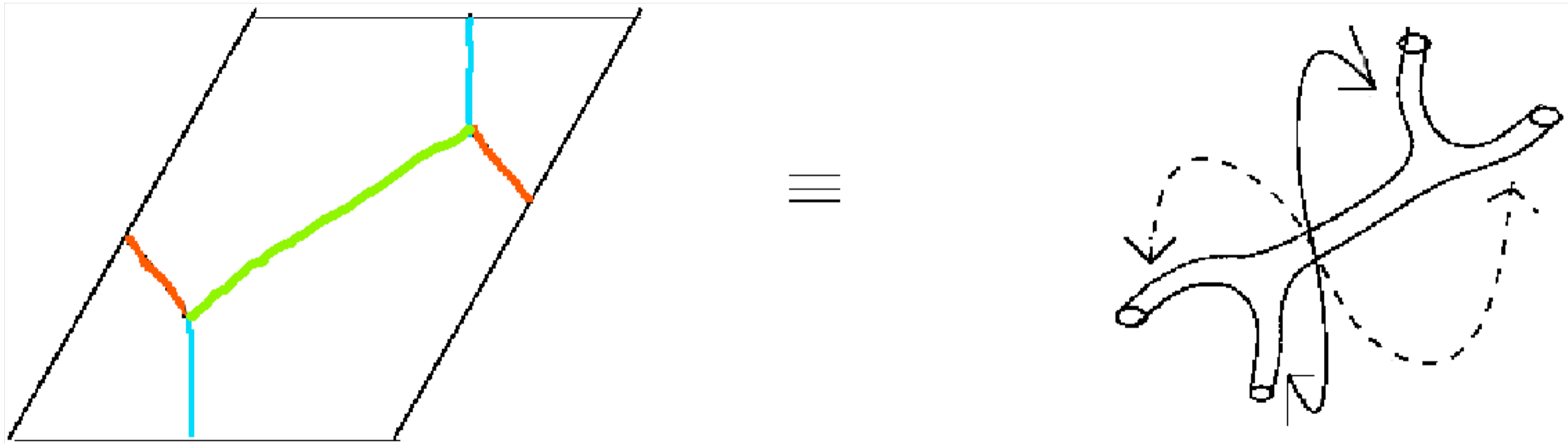
- Quarter BPS states of heterotic on $\mathbf{T}^4$ **£** $\mathbf{T}^2$ is described as a string web of (p, q) strings wrapping the $\mathbf{T}^2$ in Type-IIB string on **K3 £** $\mathbf{T}^2$ and left-moving oscillations.
- The strings arise from wrapping various D3, D5, NS5 branes on cycles of **K3**

String junction tension balance




Q_m
$Q_e + Q_m$
Q_e

M-lift of String Webs



- Genus-2 worldsheet is worldvolume of Euclidean M5 brane with various fluxes turned on wrapping **K3 £ T^2**. The **T^2** is holomorphically embedded in **T^4** by Abel map. It can carry left-moving oscillations.
- K3-wrapped M5-brane is the heterotic string. So genus-2 chiral partition fn of heterotic counts its left-moving BPS oscillations.

Genus one gives electric states

- Degeneracies of electric states are given by the Fourier coefficients of the genus-one partition fn. In this case the string web are just 1-dimensional strands.

$$\frac{1}{\eta^{24}(\tau)} = \sum c(n) q^n$$

$$d(Q) = c(Q_e^2/2)$$

Genus two gives dyons

- Genus-2 determinants are complicated. One needs determinants both for bosons and ghosts. But the *total partition function* of 26 left-moving bosons and ghosts can be deduced from modular properties.

$$\mathbf{Z}(\Omega) = \frac{1}{\Phi_{10}(\Omega)}$$

- Φ_{10} (Igusa cusp form) is the unique weight ten cusp form of **Sp(2, Z).**
- Just as at genus one, η^{24} (Jacobi-Ramanujan function) is the unique weight 12 form of **Sp(1, Z) » SL(2, Z).** Hence the one-loop partition function is

$$\mathrm{Z}(\tau) = \frac{1}{\eta^{24}(\tau)}$$

Z_2 Orbifold

- Bosonic realization of E_8 £ E_8 string

$$(\mathbf{X}^{\mathbf{I}}, \mathbf{Y}^{\mathbf{I}}), \qquad \mathbf{I} = \mathbf{1}, \mathbf{2}, \ldots, \mathbf{8}$$

- Orbifold action flips **X** and **Y**.

$$\frac{1}{\sqrt{2}}(\mathbf{X} - \mathbf{Y}) \text{ odd} \qquad \frac{1}{\sqrt{2}}(\mathbf{X} + \mathbf{Y}) \text{ even}$$

Twisted Partition Function

- We need to evaluate the partition function on a genus two surface with twisted boundary conditions along one cycle.
- Consider a genus-g surface. Choose A and B-cycles with intersections

$$\#(A_i, A_j) = \#(B_i, B_j) = 0;$$

$$\#(A_i, B_j) = -\#(B_i, A_j) = \delta_{ij}$$

Period matrix

- Holomorphic differentials $\omega_i(z)$

$$\oint_{A_i} \omega_j = \delta_{ij} \qquad \oint_{B_i} \omega_j = \Omega_{ij}.$$

- Higher genus analog of dz on a torus

$$\oint_A dz = 1 \qquad \oint_B dz = \tau.$$

Sp(g, Z)

- Linear relabeling of A and B cycles that preserves the intersection numbers is an Sp(g, Z) transformation.
- The period matrix transforms as

$$\Omega \to (A\Omega + B)(C\Omega + D)^{-1}.$$

- Analog of

$$\tau \to (a\tau + b)(c\tau + d)^{-1}$$

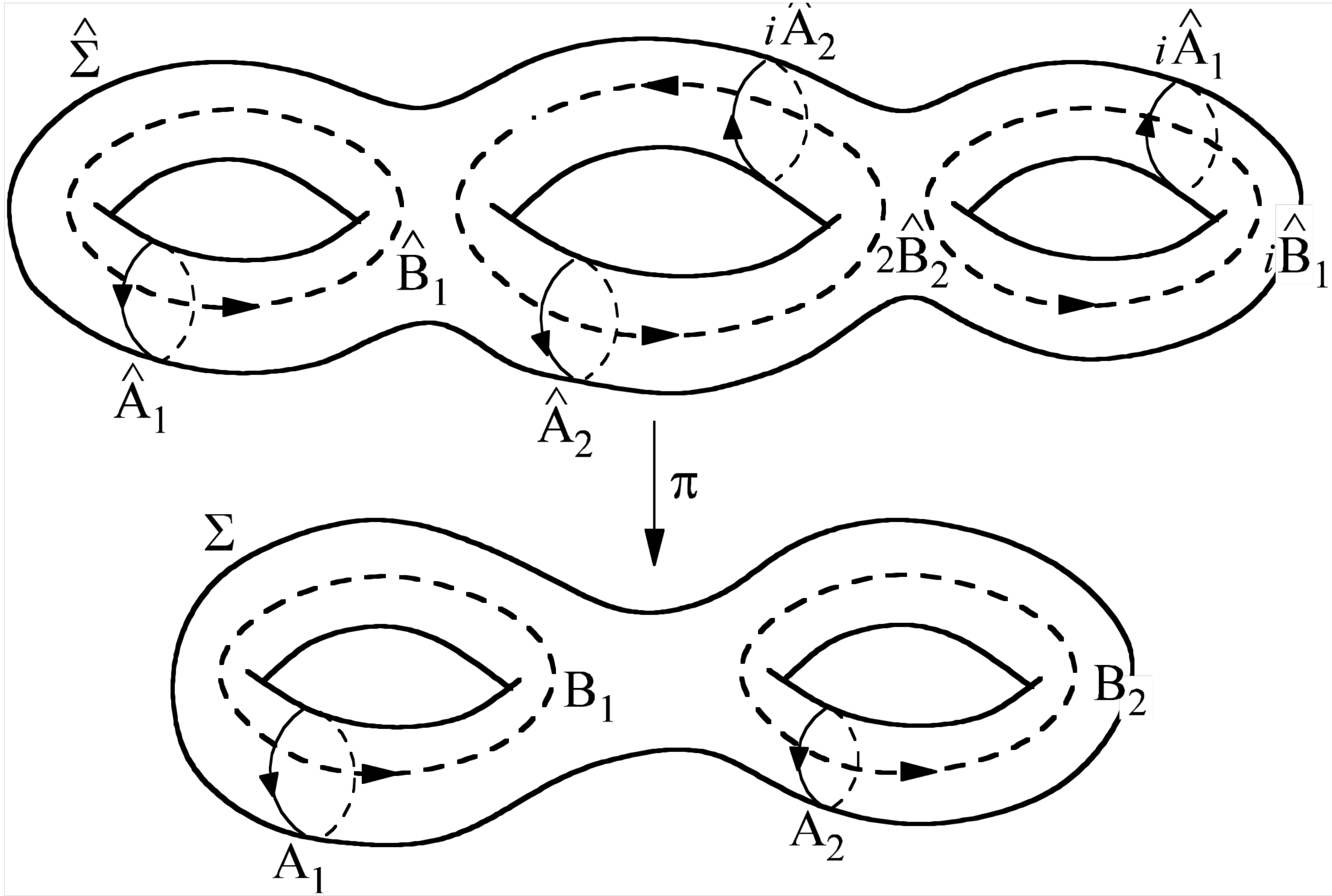
Boson Partition Function

- Period matrix arise naturally in partition fn of bosons on circles or on some lattice.

$$Z = Z^{qu} \sum_{(p_L, p_R)} e^{\pi i (p_L \cdot \Omega \cdot p_L - p_R \cdot \bar{\Omega} \cdot p_R)}$$

- Z^{qu} is the quantum fluctuation determinant.

Double Cover



Prym periods

- Prym differentials are differentials that are odd across the branch cut

$$\nu_l(z) \quad l = 1, \ldots, (g-1)$$

- Prym periods

$$\oint_{\hat{A}_1} \nu = -\oint_{i(\hat{A}_1)} \nu = 1;$$

$$\oint_{\hat{B}_1} \nu = -\oint_{i(\hat{B}_1)} \nu = \tilde{\tau}$$

Twisted determinants

- We have 8 bosons that are odd. So the twisted partition function is

$$Z^{qu}_{twisted} = Z^{qu}_{untwisted} \left(\frac{\Delta_u}{\Delta_t}\right)^8.$$

$$c^{-2} = \left(\frac{\Delta_u}{\Delta_t}\right)^8$$

X ! –X and X ! X + π R

- Boson X » X + 2π R at self-radius

  Exploit the enhanced SU(2) symmetry

  $(J_x, J_y, J_z) = (\cos X, \sin X, \partial X)$

- X ! –X

  (J_x, J_y, J_z) ! $(J_x, -J_y, -J_z)$

- X! X + π R

  (J_x, J_y, J_z) ! $(-J_x, -J_y, J_z)$

Orbifold = Circle

- $$Z_{\text{orbifold}}(R=1) = Z_{\text{circle}}(R=\frac{1}{2}).$$

- This trick allows us to express the quantum determinants of twisted bosons in terms of quantum determinants of untwisted bosons and rations of theta functions.

- Express the twisted determinant in terms of the untwisted determinant and ratios of momentum lattice sums.
- Lattice sums in turn can be expressed in terms of theta functions.
- This allows us to express the required ratio of determinants in terms of ratio of theta functions.

Orbifold = Circle

- $$Z_{\mathsf{orbifold}}(R = 1) = Z_{\mathsf{circle}}(R = \frac{1}{2}).$$
- $$\frac{\vartheta(\tilde{\tau})}{\Delta_t} = \frac{\vartheta(\Omega)}{\Delta_u}$$
- $$(\frac{\Delta_t}{\Delta_u})^2 = (\frac{\vartheta(\tilde{\tau})}{\vartheta(\Omega)})^2$$

- Express the twisted determinant in terms of the untwisted determinant and ratios of momentum lattice sums.
- Lattice sums in turn can be expressed in terms of theta functions.
- This allows us to express the required ratio of determinants in terms of ratio of theta functions.

- Using these CFT techniques one can exactly evaluate the genus-two partition function for the $\mathbf{Z_2}$ CHL orbifold. One obtains precisely the answer obtained by Ibukiyama et al by different means.

$$2^{12}\Phi_{10} = \prod_{\alpha=even} (\vartheta_\alpha)^2$$

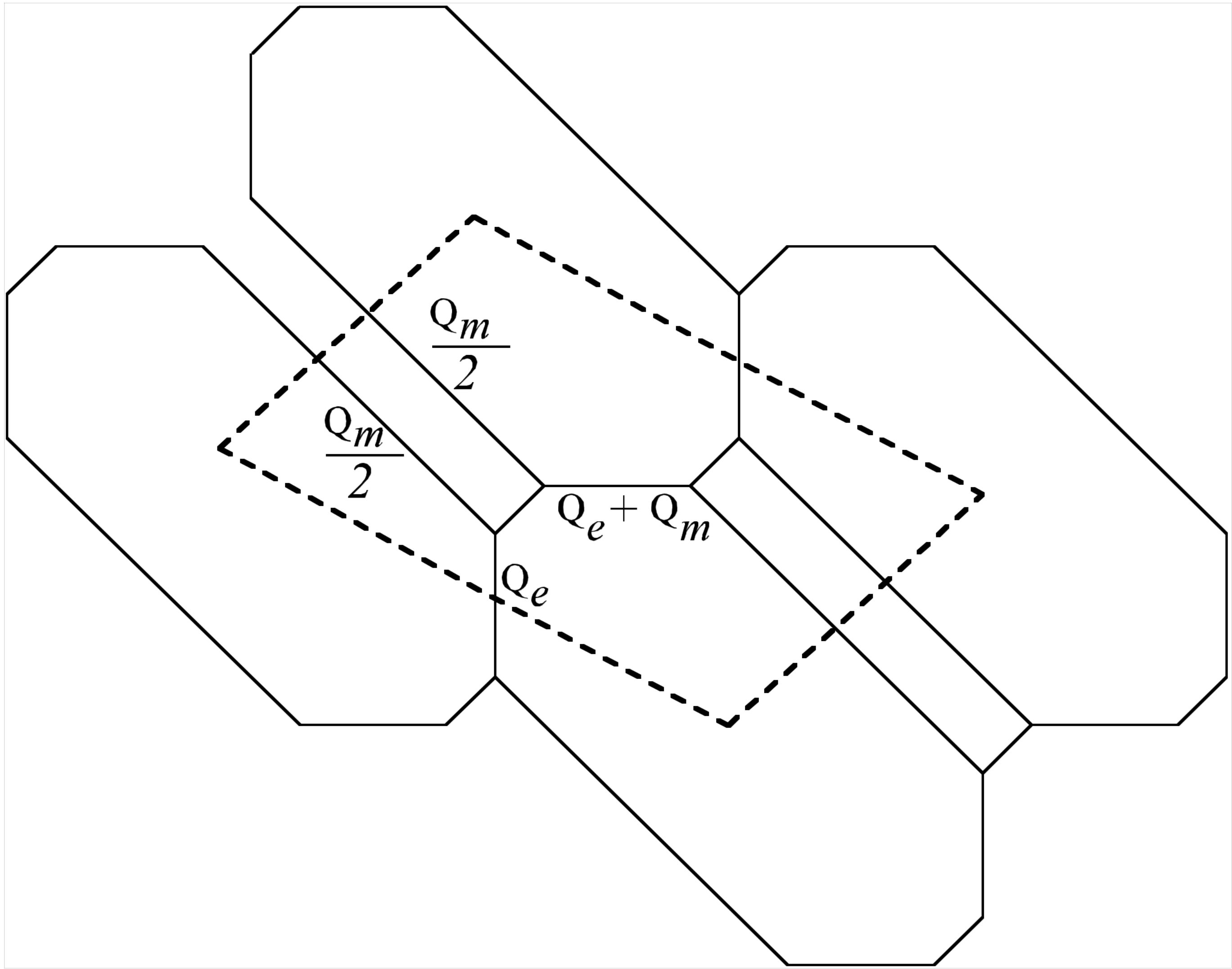
$$2^{12}\Phi_6 = (\vartheta_{0100}\vartheta_{0110}\vartheta_{1000}\vartheta_{1001}\vartheta_{1100}\vartheta_{1111})^2$$

Higher genus contributions

- For example if $Q_m = 2(Q'_m)$ then

$$\mathbf{I = gcd(Q_e^i Q_m^j - Q_e^j Q_m^i) = 2}$$

- Now genus three contribution is possible.
- The condition gcd =1 is equivalent to the condition Q_1 and Q_5 be relatively prime.


$\frac{Q_m}{2}$
$\frac{Q_m}{2}$
$Q_e + Q_m$
Q_e

Dual graph

- Face goes to a point in the dual graph.
- Two points in the dual graph are connected by vector if they are adjacent.
- The vector is equal in length but perpendicular to the common edge.
- String junction goes to a triangle.

- If one can insert a triangle at a string junction then the junction can open up and a higher genus web is possible.
- Adding a face in the web is equivalent to adding a lattice point in the dual graph.
- If the fundamental parellelogram has unit area then it does not contain a lattice point.
- $Q_e \wedge Q_m$ is the area tensor. Unit area means gcd of all its components is one.

Negative discriminant states

- Consider a charge configuration

$$\left(\frac{1}{2}Q_m^2,\ \frac{1}{2}Q_e^2,\ Q_e\cdot Q_m\right) = (-1,-1,N)$$

- Degeneracy d(Q) = N

$$\frac{y}{pq(1-y)^2} = \sum_{N=1}^{\infty} N p^{-1} q^{-1} y^N$$

Two centered solution

- One electric center with

$$Q_e = (n, w; \tilde{n}, \tilde{w}) = (1, -1; 0, N)$$

- One magnetic center with

$$Q_m = (W, N; \tilde{W}, \tilde{N}) = (0, 0; 1, -1)$$

- Field angular momentum is N/2

Supergravity Analysis

- The relative distance between the two centers is fixed by solving Denef's constraint.
- Angular momentum quantization gives (2 J +1) » N states in agreement with the microscopic prediction.
- Intricate moduli dependence.

S-duality

- Different expansion for different charges.
  Consider a function with Z_2 symmetry.

$$(y^{1/2} - y^{-1/2})^{-2}$$

$$\frac{y^{-1}}{(1-y^{-1})^2} = \sum_{N=1}^{\infty} N y^{-N}$$

$$\frac{y}{(1-y)^2} = \sum_{N=1}^{\infty} N y^{N}$$

Conclusions

- Completely different derivation of the dyon partition function using M-lift of string webs that makes the modular properties manifest.
- Higher genus contributions are possible.
- Physical predictions such as negative discriminant states seem to be borne out.