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Current Algebra of Principal Chiral Model Coupled to World-Sheet Gravity

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Abstract:

Keywords: .

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Outline of Talk:

- **Principal Chiral Model-Definition**
- **Hamiltonian analysis**
- **Current Algebra-Poisson bracket of Lax Connection**
- **Nambu-Gotto form of Principal Chiral Model**
- **Conclusion**

Hamiltonian formalism and Current algebra

Action for the Principal Chiral Model

$$\begin{aligned} S &= -\frac{\sqrt{\lambda}}{4\pi}\int d\tau d\sigma\sqrt{-\gamma}\gamma^{\alpha\beta}J_\alpha^A J_\beta^B K_{AB} = \\ &= -\frac{\sqrt{\lambda}}{4\pi}\int d\tau d\sigma\sqrt{-\gamma}\gamma^{\alpha\beta}E_M^A\partial_\alpha x^M E_N^B\partial_\beta x^B K_{AB} = \\ &= -\frac{\sqrt{\lambda}}{4\pi}\int d\tau d\sigma\sqrt{-\gamma}\gamma^{\alpha\beta}G_{MN}\partial_\alpha x^M\partial_\beta x^B \end{aligned} \tag{1}$$

- $\frac{\sqrt{\lambda}}{2\pi}$... is the effective string tension.
- Coordinates σ, τ... parametrise the string world-sheet, σ to be $-\pi \leq \sigma \leq \pi$.
- The current J_α^A.... defined through the group element g that belongs to the group G

$$J_\alpha = g^{-1}\partial_\alpha g \equiv J_\alpha^A T_A \ , \tag{2}$$

- T_A.... basis of the algebra $\mathbf{g}$

$$\mathrm{Tr}(T_A T_B) = K_{AB} \ , \quad [T_A, T_B] = f_{AB}^C T_C \tag{3}$$

- K_{AB} ...invertible matrix, $f_{AB}^C = -f_{BA}^C$... structure constants of the algebra $\mathbf{g}$.

- Indices $A, B...$ label components of the basis T_A.
- x^M:... parametrise the group element q. Then

$$J^A_\alpha = E^A_M \partial_\alpha x^M \ . \tag{4}$$

- metric on some target manifold labelled with x^M:...

$$G_{MN} = E^A_M K_{AB} E^B_N \tag{5}$$

- E^A_M.... vielbeins of the target manifold

Hamiltonian Formalism

Conjugate momenta to x^M:

$$p_M = -\frac{\sqrt{\lambda}}{2\pi}\sqrt{-\gamma}\gamma^{\tau\alpha}E^A_M K_{AB} J^B_\alpha \tag{6}$$

Canonical Poisson brackets

$$\left\{x^M(\sigma), p_N(\sigma')\right\} = \delta^M_N \delta(\sigma - \sigma') \tag{7}$$

Parametrisation of metric variables $\gamma_{\alpha\beta}$:

$$\lambda^{\pm} = \frac{\sqrt{-\gamma} \pm \gamma_{\tau\sigma}}{\gamma_{\sigma\sigma}} \ , \quad \xi = \ln \gamma_{\sigma\sigma} \ , \tag{8}$$

Hamiltonian density $\mathcal{H}_0$:

$$\mathcal{H}_0 = \lambda^+ T_+ + \lambda^- T_- \ ,$$
$$T_+ = \frac{1}{2}(T_0 + T_1) \ , T_- = \frac{1}{2}(T_0 - T_1) \ , \tag{9}$$

where

$$T_1 \equiv p_M \partial_\sigma x^M \ ,$$
$$T_0 \equiv \frac{\pi}{\sqrt{\lambda}} p_M G^{MN} p_N + \frac{\sqrt{\lambda}}{4\pi} \partial_\sigma x^M G_{MN} \partial_\sigma x^N \ . \tag{10}$$

Absence of time-derivative of $\gamma_{\alpha\beta}$:

$$\pi^\pm = \frac{\delta S}{\delta \partial_\tau \lambda^\pm} = 0 \ , \quad \pi^\xi = \frac{\delta S}{\delta \partial_\tau \xi} = 0 \ . \tag{11}$$

Primary constraint of the theory

Stability of the primary constraints: existence of two secondary constraints

$$T_+ = T_- = 0 \ . \tag{12}$$

Constraints T_0, T_1 *do not generate any additional ones.*

Notation:All constraints:

- Φ_α

- $\alpha = (\lambda^{\pm}, \xi, \pm)$ ...label the first class constraints

- $\pi^{\lambda}_{\pm} = \pi_\xi = 0$
- $T_\pm = 0.$

- ρ^A... Lagrange multipliers

- Generalised Hamiltonian density:

$$\mathcal{H}_T = \mathcal{H}_0 + \rho^\alpha(p, x, t)\Phi_\alpha \ , \tag{13}$$

Remark:

In principle ρ^α depend on the phase space variables p_M, x^M, π, λ.

Hamiltonian equation of motion:

$$\partial_\tau F = \partial_\tau F + \left\{ J^A_\tau, H \right\} = \partial_\tau F + \{F, H_0\} + \rho^\alpha \{F, \Phi_\alpha\} \ . \tag{14}$$

Then:

$$J^A_\tau = \frac{2\pi}{\sqrt{\lambda}}\left(-\frac{1}{\sqrt{-\gamma}\gamma^{\tau\tau}} + \rho^0\right)K^{AB}E^M_B p_M + \left(-\frac{\gamma^{\tau\sigma}}{\gamma^{\tau\tau}} + \rho^1\right)E^A_M \partial_\sigma x^M \ , \tag{15}$$

Current Algebra

Important:

The current algebra has closed form for $\rho^{\pm} = \rho(\tau, \sigma)$

Define:

$$-\frac{1}{\sqrt{-\tilde{\gamma}}\tilde{\gamma}^{\tau\tau}} = -\frac{1}{\sqrt{-\tilde{\gamma}}\tilde{\gamma}^{\tau\tau}} + \rho^{0} \ , -\frac{\tilde{\gamma}^{\tau\sigma}}{\tilde{\gamma}^{\tau\tau}} = -\frac{\gamma^{\tau\sigma}}{\gamma^{\tau\tau}} + \rho^{1} \ . \quad (16)$$

Lax connection:

$$\hat{J}_{\alpha} = \frac{1}{1-\Lambda^2}(J_{\alpha} - \Lambda\tilde{\gamma}_{\alpha\beta}\tilde{\epsilon}^{\beta\gamma}J_{\gamma}) \quad (17)$$

is flat:

$$\partial_{\alpha}\hat{J}^{A}_{\beta} - \partial_{\beta}\hat{J}^{A}_{\alpha} + \hat{J}^{B}_{\alpha}\hat{J}^{C}_{\beta}f^{A}_{BC} = 0 \quad (18)$$

for general world-sheet metric.

Sign of Integrability of Principal chiral model on world-sheet with general metric

Further step: Determine the form of Poisson bracket of Lax connection:

To do this we need to find the form of Poisson brackets between currents

Then the Poisson bracket of Lax connection takes the form:

$$\left\{\hat{J}^A_\sigma(\sigma,\Lambda),\hat{J}^B_\sigma(\sigma',\Gamma)\right\} = -\frac{2\pi}{\sqrt{\lambda}}\Big[\frac{K^{AB}\partial_\sigma\delta(\sigma-\sigma')}{(1-\Lambda^2)(1-\Gamma^2)}(\Gamma+\Lambda) -$$
$$-\frac{\Gamma^2}{(\Gamma^2-1)(\Lambda-\Gamma)}\hat{J}^C_\sigma(\Lambda)f^A_{CD}K^{DB}\delta(\sigma-\sigma') -$$
$$-\frac{\Lambda^2}{(\Lambda^2-1)(\Lambda-\Gamma)}\hat{J}^C_\sigma(\Gamma)f^A_{CD}K^{DB}\delta(\sigma-\sigma')\Big] \tag{19}$$

Comments:

- It is desirable that this Poisson bracket does not depend on $\tilde{\gamma}$.
- *Implication:*
  Poisson brackets of the monodromy matrices do not depend on the world-sheet metric and gauge parameters

Monodromy Matrix

$$\mathcal{T}(\Lambda) = P\exp\left(\int_{x_1}^{y_1} d\sigma' \hat{J}_\sigma(\tau,\sigma')\right) . \tag{20}$$

Poisson brackets:

$$\{\mathcal{T}_{\alpha\beta}(x_1,y_1,\Lambda),\mathcal{T}_{\gamma\delta}(x_2,y_2,\Gamma)\} =$$
$$= \int_{x_1}^{y_1} d\sigma \int_{x_2}^{y_2} d\sigma' \mathcal{T}_{\alpha\omega_1}(x_1,\sigma,\Lambda)\mathcal{T}_{\omega_2\beta}(\sigma,y_1,\Lambda) \times$$
$$\times \{L_{\omega_1\omega_2}(\sigma,\Lambda),L_{\rho_1\rho_2}(\sigma',\Gamma)\}\,\mathcal{T}_{\gamma\rho_1}(x_2,\sigma',\Gamma)\mathcal{T}_{\rho_{2},\delta}(\sigma',y_2',\Gamma) . \tag{21}$$

Comments

- The calculation depends on the Poisson bracket of Lax connection:
- It can be shown that for the Poisson bracket of the Lax connection given above the Poisson brackets of the transition matrix defines an infinite number of integrals of motion that are in involution (Dorey, Maillet)
- Strong implication if integrability
- This result holds on the world-sheet with general metric with all gauge freedom
- It can be shown that theory has infinite number of charges in involution for any fixed world-sheet metric but with unfixed Virasoro constraints
- Open problem: Try to determine the Poisson bracket of the Lax connection for completely fixed theory (uniform light-cone gauge):work in progress

Nambu-Gotto Principal Model

$$S = \frac{\sqrt{\lambda}}{2\pi} \int d\sigma d\tau \mathcal{L} \ , \mathcal{L} = -\sqrt{-\det \mathbf{A}_{\alpha\beta}} \quad (22)$$

where

$$\mathbf{A}_{\alpha\beta} = J^A_\alpha J^B_\beta K_{AB} \ . \quad (23)$$

J^A_α obey the equations of motion

$$[\partial_\alpha [J^A_\beta \left(\mathbf{A}^{-1}\right)^{\beta\alpha} \sqrt{-\det \mathbf{A}}] = 0 \quad (24)$$

Definition of Lax connection:

$$L_\sigma = \frac{1}{1-\Lambda^2}[J^A_\sigma + \Lambda \left(\mathbf{A}^{-1}\right)^{\tau\alpha} J^A_\alpha \sqrt{-\det \mathbf{A}}] \ ,$$
$$L_\tau = \frac{1}{1-\Lambda^2}[J^A_\tau - \Lambda \left(\mathbf{A}^{-1}\right)^{\sigma\alpha} J^A_\alpha \sqrt{-\det \mathbf{A}}] \ . \quad (25)$$

It can be show that it is flat:

$$\partial_\alpha L^A_\beta - \partial_\beta L^A_\alpha + L^B_\alpha L^C_\beta f^A_{BC} = 0 \quad (26)$$

Hamiltonian formalism

Define:

$$J^A_\alpha = E^A_M \partial_\alpha x^M \quad (27)$$

Momentum conjugate to x^M:

$$p_M = \frac{\delta S}{\delta \partial_\tau x^M} = -\frac{\sqrt{\lambda}}{2\pi} G_{MN} \partial_\alpha x^M \left(\mathbf{A}^{-1}\right)^{\alpha\tau} \sqrt{-\det \mathbf{A}} \quad (28)$$

Two primary constraints:

$$\Phi_0 = \frac{2\pi}{\sqrt{\lambda}} p_M G^{MN} p_N + \frac{\sqrt{\lambda}}{2\pi} \partial_\sigma x^M g_{MN} \partial_\sigma x^N = 0$$
$$\Phi_1 = p_M \partial_\sigma x^M = 0 \tag{29}$$

Original Hamiltonian density:

$$\mathcal{H} = p_M \partial_\tau x^M - \mathcal{L} = 0 \tag{30}$$

Total Hamiltonian density:

$$\mathcal{H}_T = \lambda^0 \Phi_0 + \lambda^1 \Phi_1 \tag{31}$$

Then it can be shown that the Poisson bracket of the Lax connection for the Nambu-Gotto string takes the same form as above

Implication of the classical integrability of the theory

Conclusion

- Principal chiral model on world-sheet with general metric is classically integrable

- Open important problem: Calculate the Poisson bracket of the gauge fixed theory

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