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Study of the QCD phase diagram with the method of analytic continuation

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Based on works with
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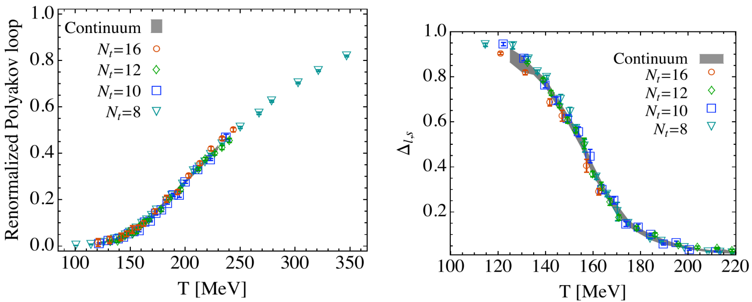
Outline

- The phase diagram for strongly interacting matter
  Theory: the chiral/deconfinement crossovers, Experiments: chemical freeze-out point
- Theory from first principles: Lattice QCD
  Basics, $T \neq 0$,$\mu_B \neq 0 \rightarrow \ldots$
- The sign problem and proposed solutions
  Taylor expansion, Reweighting, Analytic continuation (...)
- The critical line of QCD and Analytic continuation
  Basics, $T \neq 0$,$\mu_B \neq 0 \rightarrow$ the sign problem!
- Renormalized observables and the definitions of $T_c(\mu)$
  Chiral condensate, renormalization (I) and (II), Chiral susceptibility
- Numerical setup
  Discretization used, Parameters, Statistics
- Numerical results
  Finite size effects, Effects of $\mu_s \neq 0$, Effects of different definitions of $T_c(\mu)$,Continuum limit
- Conclusions

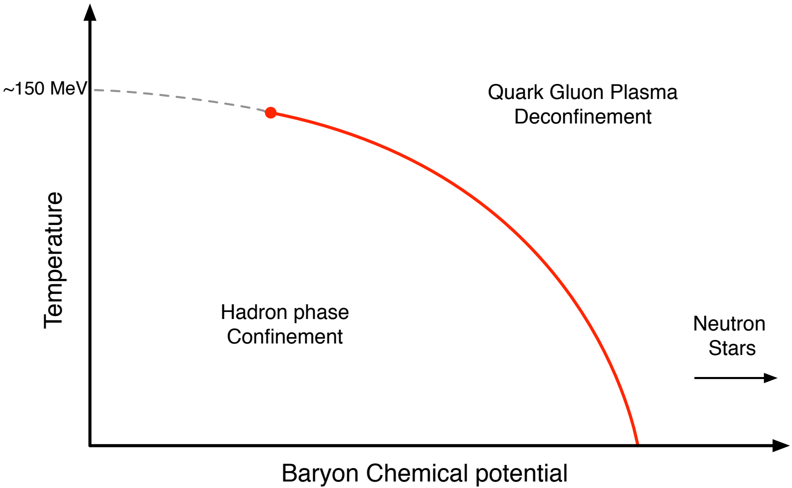
- Low temperature: Confinement, (spontaneous) chiral symmetry breaking
- High temperature: Deconfinement, chiral symmetry restoration



Left: Polyakov loop ($e^{-F_Q/T}$) as a function of temperature.
Right: Chiral condensate ($\sim \langle\bar{\psi}\psi\rangle$) (from JHEP 1009 (2010) 073)

Lattice data suggest no real transitions, only crossovers

...and at nonzero μ_B



QCD

Path Integral formulation: $Z = \int DAD\bar{\psi}D\psi e^{-i\int d^4x\mathcal{L}[A,\bar{\psi},\psi]}$

$$D_\mu = \partial_\mu - ig\hat{A}_\mu, \; (\hat{F}_{\mu\nu} = [D_\mu, D_\nu])$$

$$\mathcal{L} = -\frac{1}{2g^2}\mathrm{Tr}\left\{\hat{F}_{\mu\nu}\hat{F}^{\mu\nu}\right\} + \sum_f \bar{\psi}_f\left(i\gamma^\mu D_\mu - m_f\right)\psi_f$$

Chiral Symmetry: In the vanishing mass limit the Lagrangian is invariant under the transformations

$$\psi'_L = U\psi_L, \; \psi'_R = U^\dagger\psi_R$$

Where ψ_L and ψ_R represent the left- and right-handed parts of all the spinors, and U is a $SU(N_f)$ matrix which mix different flavours. The light quark condensate $\langle\bar{u}u + \bar{d}d\rangle$ is an order parameter for chiral symmetry breaking.

Lattice QCD

- Wick Rotation ($t = -i\tau$), Torus $[L \times t]$ geometry ([anti]periodic boundary conditions[1]), Hypercubic lattice (lattice spacing a)
- The fermion fields live on the vertices, while the gauge fields are replaced by the gauge links (Parallel transport operators, $SU(3)$ matrices $U_\mu(x) \simeq e^{-igaA_\mu(x)}$).
- Finite number of degrees of freedom $\Rightarrow$ The functional integral become a finite dimensional integral, evaluable with Montecarlo and Importance Sampling methods:

$$Z = \int DU e^{-S_G[U]} \prod_f \det M_f[U]$$

Various possible choices for the discretized action, for both S_G and M_f

- The Wick rotation + temporal periodic boundary conditions allow us to study QCD at finite temperature:

$$t = -i\tau \quad \Rightarrow \quad \mathrm{Tr} e^{-iHt} = \mathrm{Tr} e^{-H\tau} = \mathrm{Tr} e^{-H/T} \; [\tau = 1/T]$$

- Continuum and Thermodynamic limits ($a \to 0, L \to \infty$)

[1] Antiperiodic for fermion fields in the temporal direction

Chemical potential and sign problem

In the **continuum theory**, a chemical potential coupled with quark number can be introduced:

$$\mu_f N_f = \mu_f \int d^3x \; \bar{\psi}_f \gamma_0 \psi_f$$

On the lattice, the quark chemical potential associated to the flavour f is introduced by multiplying the gauge links in the fermion matrix $M_f[U]$ in the temporal direction by $e^{-a\mu_f}$.
Unfortunately, this causes the so called **sign problem**. When $\mu_f = 0$,

$$\left(\not{D} + m\right)^\dagger = \gamma_5 \left(\not{D} + m\ \right) \gamma_5 \;\rightarrow \det\left(\not{D} + m\right) \in \mathbb{R}$$

When $\mu_f \neq 0$ this is not true any more:

$$\gamma_5 \left(\not{D} + m - \gamma_0 \mu\right) \gamma_5 = \left(-\not{D} + m + \gamma_0 \mu\right) = \left(\not{D} + m + \gamma_0 \mu^*\right)^\dagger$$

$\Rightarrow$ **The fermion determinant is complex!**[2]

[2]Notice that this is not the case if $\Re\mu = 0$

Sign problem: Ways out

- Analytic Continuation from imaginary μ
- Taylor expansion from $\mu = 0$ [precision issues with higher order derivatives on the lattice]
- Reweighting from the $\mu = 0$ ensemble [scales badly with volume]
- Canonical method [the sign problem is back in a different form]
- ...

At lowest order in μ, the pseudocritical line can be parametrized as:

$$\frac{T_c(\mu_B)}{T_c} = 1 - \kappa \left(\frac{\mu_B}{T_c(\mu_B)} \right)^2 + O(\mu^4)$$

(odd order terms are forbidden by charge conjugation symmetry of QCD)

The sign problem and analytic continuation

For purely imaginary μ, the fermion determinant is real positive, and the sign problem is non existent.

With the transformation $\mu_B = i\mu_{B,I}$, the pseudocritical line parametrization is modified as:

$$\frac{T_c(\mu_{B,I})}{T_c} = 1 + \kappa \left(\frac{\mu_{B,I}}{T_c(\mu_{B,I})} \right)^2 + O(\mu_{B,I}^4)$$

Renormalization of the chiral condensate

$$\langle\bar{\psi}\psi\rangle_{ud} = \frac{T}{V}\frac{\partial \log Z}{\partial m_{ud}} = 2\frac{T}{V}\langle \mathrm{Tr} M_l^{-1}\rangle = \langle\bar{u}u\rangle + \langle\bar{d}d\rangle$$

We have considered two renormalizations:

1. As in [Cheng *et al.*, 08]:

$$\langle\bar{\psi}\psi\rangle^r_{(1)} \equiv \frac{\langle\bar{\psi}\psi\rangle_{ud}(T) - \frac{2m_{ud}}{m_s}\langle\bar{s}s\rangle(T)}{\langle\bar{\psi}\psi\rangle_{ud}(0) - \frac{2m_{ud}}{m_s}\langle\bar{s}s\rangle(0)}$$

2. Alternatively [Endrodi *et al.*, 11]:

$$\langle\bar{\psi}\psi\rangle^r_{(2)} \equiv \frac{m_{ud}}{m_\pi^4}\left(\langle\bar{\psi}\psi\rangle_{ud} - \langle\bar{\psi}\psi\rangle_{ud}(T=0)\right)$$

Renormalized chiral susceptibility

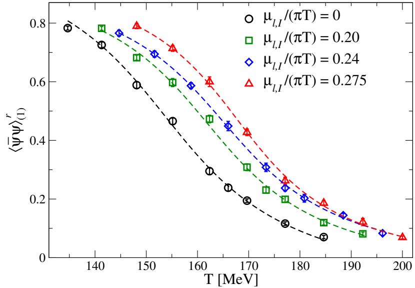
$$\chi_{\bar{\psi}\psi} \equiv \frac{\partial \langle \bar{\psi}\psi \rangle_{ud}}{\partial m_l}$$

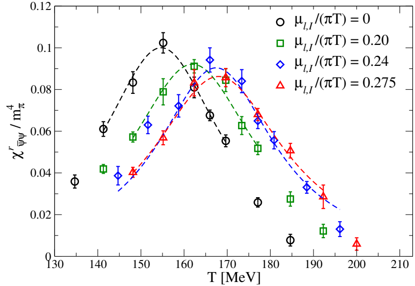
We have chosen this renormalization[Y.Aoki *et al.*, 06]:

$$\chi^r_{\bar{\psi}\psi}(T) \equiv m^2_{ud} \left[\chi_{\bar{\psi}\psi}(T) - \chi_{\bar{\psi}\psi}(0) \right]$$

We use the dimensionless quantity $\chi^r_{\bar{\psi}\psi}(T)/m^4_\pi$.

Defining T_c





Fit for the **chiral condensates (I) and (II)**:

$$\langle\bar{\psi}\psi\rangle^r(T) = A_1 + B_1 \arctan\left[C_1\left(T - T_c\right)\right]$$

Fit at the peak for the **renormalized chiral susceptibility**:

$$\chi^r_{\bar{\psi}\psi}(T) = \frac{A_2}{(T - T_c)^2 + B_2^2}$$

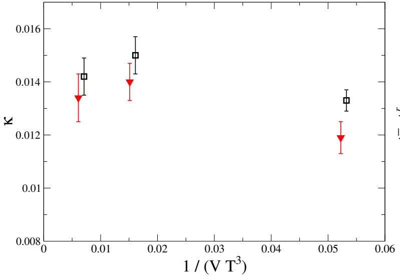
Numerical setup

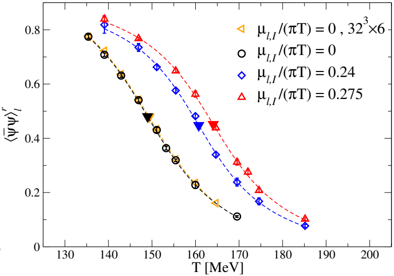
- Study of the $\mu_s = \mu_l \neq 0$ (32^3x8 only) and $\mu_s = 0$ cases.
- Tree level Symanzik improved gauge action with $N_f = 2+1$ flavours of 2-stouted staggered fermions.
- At the physical point (line of constant physics, parameters taken from [Aoki *et al.*, 09]) $N_t = 6, 8, 10, 12$ lattices.
- Also performed simulations at zero temperature for subtractions (32^4,48^3x96).
- Observables evaluated with noisy estimators, with 8 random vectors per quark.

Simulations run on IBM BG-Q at CINECA (Bologna, Italy) and on the Zefiro Cluster (INFN - Pisa).

Lattice	$16^3 \times 6$	$24^3 \times 6$	$32^3 \times 6$
$i\mu/(\pi T)$	0.00 0.20 0.24 0.275	0.00 0.24 0.275	0.00 0.24 0.275
Lattice	$32^3 \times 8$	$40^3 \times 10$	$48^3 \times 12$
$i\mu/(\pi T)$	0.00 0.10 0.15 0.20 0.24 0.275 0.30	0.00 0.20 0.24 0.275	0.00 0.20 0.24 0.275

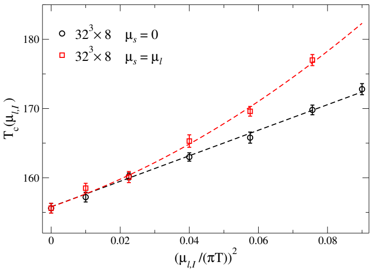
From [Bonati *et al.*, 14] (16^3x6, 24^3x6 and 32^3x6 lattices)





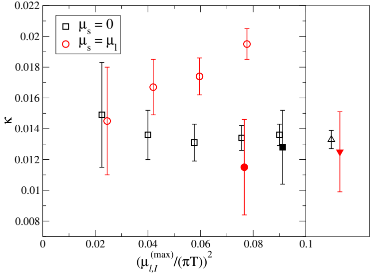
Left: Estimates of κ. Black : Renormalized Chiral Condensate (1), Red : Renormalized Chiral Susceptibility ; Right: The chiral condensate on the $24^3 \times 6$ lattice, with the data for $\mu_I = 0$ on the $32^3 \times 6$ lattice

$\Rightarrow$ **Aspect ratio 4 is enough.**



(Renormalized chiral susceptibility)

Results for κ varying the μ fit range:

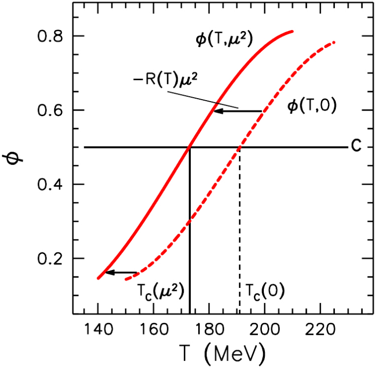


Empty Red: κ, linear fit ($\mu_s = \mu_l$ data)
Full Red: κ, lin+quad fit ($\mu_s = \mu_l$)
Empty Black: κ, linear fit ($\mu_s = 0$)
Empty Black: κ, lin+quad fit ($\mu_s = 0$)
Right: κ from combined (lin+quad) fit

(From [Bonati *et al.*, 15])

In order to better compare our results with those of [Endrodi *et al.*, 11] (same action discretization, but using the Taylor expansion method), we have located $T_c(\mu_B)$ using the chiral condensate (II), using the following equation

$$\langle\bar{\psi}\psi\rangle^r_{(2)}(T_c(\mu_B),\mu_B) = \langle\bar{\psi}\psi\rangle^r_{(2)}(T_c(0),0)$$

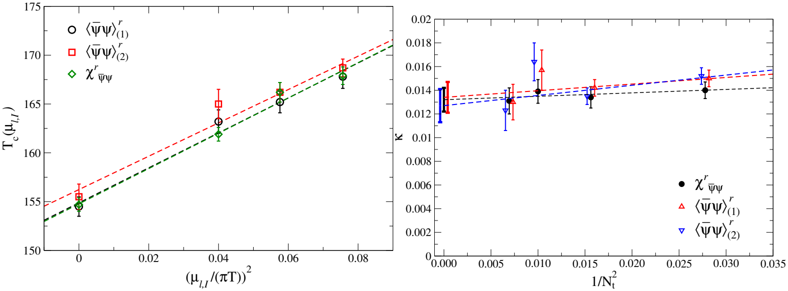


Our result for the curvature using this method is $\kappa = 0.0110(18)$, to be compared with $\kappa = 0.0066(20)$.

Figure from [Endrodi *et al.*, 11]

Critical line and continuum Limit of κ

We evaluated the curvature κ for each N_t (6,8,10,12) and then performed the continuum limit extrapolation on κ itself, assuming finite lattice spacing corrections are of the form $const/N_t^2$



Left: Critical lines obtained from the 48^3x12 lattice, and fits in the form $T_c(\mu_{B,I})/T_c = 1 + \kappa[\mu_{B,I}/\pi T_c(\mu_{B,I})]^2$.
Right: continuum extrapolations of the critical line curvature κ.

Continuum extrapolated results for the curvature:

$$\begin{aligned} \kappa_{\bar{\psi}\psi,1} &= 0.0134(13) \\ \kappa_{\bar{\psi}\psi,2} &= 0.0127(14) \\ \kappa_{\chi} &= 0.0132(10) \end{aligned}$$

Continuum limit of Observables

- For the **renormalized chiral condensates**, we used the formula

$$\langle\bar{\psi}\psi\rangle^r(T) = A_1 + B_1 \arctan\left[C_1\left(T - T_c\right)\right]$$

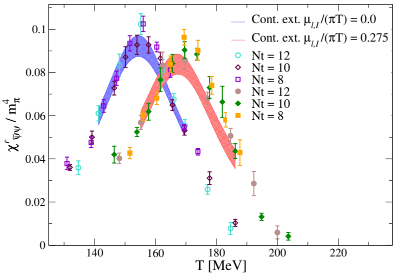
to fit the data from all values of N_t simultaneously. We added a N_t dependency to T_c ($T_c(N_t) = T_c(N_t = \infty) + const./N_t^2$) and a similar one to C_1.

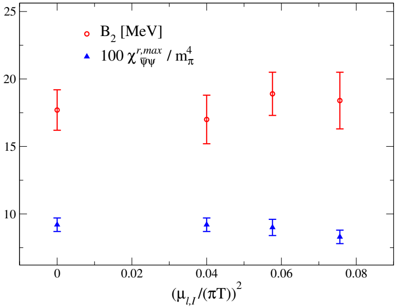
- For the **renormalized chiral susceptibility**, we used the formula

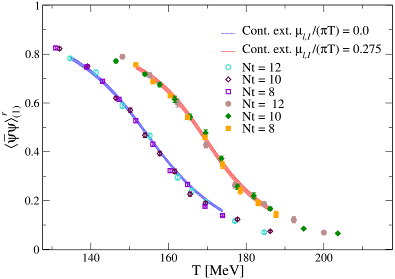
$$\chi^r_{\bar{\psi}\psi}(T) = \frac{A_2}{(T - T_c)^2 + B_2^2}$$

where we added a dependency on N_t similar to $T_c(N_t) = T_c(N_t = \infty) + const./N_t^2$ for all parameters.

Continuum limit of Observables



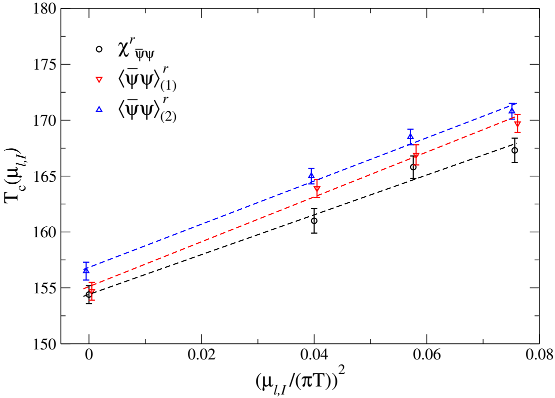




Up Left: Continuum limit on the Renormalized chiral susceptibility.

Right: Width and height of the peak of the renormalized chiral susceptibility (B_2 and $\chi^{r,max}_{\bar\psi\psi}$).

Down Left: Continuum limit of the renormalized chiral condensate (I)



2nd method:

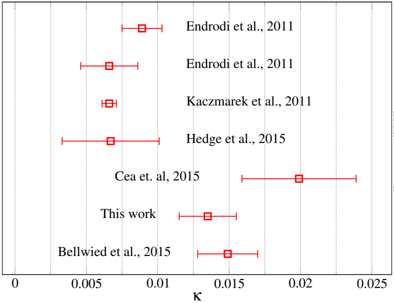
$$\kappa_{\bar\psi\psi,1} = 0.0145(11)$$
$$\kappa_{\bar\psi\psi,2} = 0.0138(10)$$
$$\kappa_{\chi} = 0.0131(12)$$

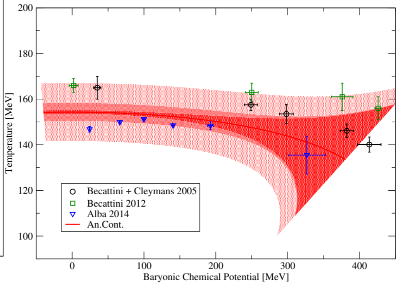
1st method:

$$\kappa_{\bar\psi\psi,1} = 0.0134(13)$$
$$\kappa_{\bar\psi\psi,2} = 0.0127(14)$$
$$\kappa_{\chi} = 0.0132(10)$$

Values of T_c obtained with the continuum limit of the observables, fit with the form $T_c(\mu_{B,I})/T_c = 1 + \kappa[\mu_{B,I}/\pi T_c(\mu_{B,I})]^2$.

Comparison with other determinations





Left: Comparison with other lattice determinations. **Right:** Tentative continuation to real chemical potential, and comparison to the experimental data from chemical freezeout.

Conclusions

- The finite size effects have been studied, and we deemed aspect ratio 4 sufficient for the present level of accuracy.
- We performed an extensive check to compare our determinations with the one of other groups.
- We investigated the effects of including a nonzero strange quark potential ($\mu_s = \mu_l = \mu$). We have confirmed the presence of a quartic contribution. Considering such contribution, the curvature of the critical line for $\mu_s = \mu_l$ or $\mu_s = 0$ is compatible within errors.
- We performed a continuum scaling analysis in two ways, directly on κ and on the observables. The resulting estimates of κ are in agreement. Our prudential estimate is $\kappa = 0.0135(20)$.