The Music of Shapes

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Joint work with **Ali Chamseddine**

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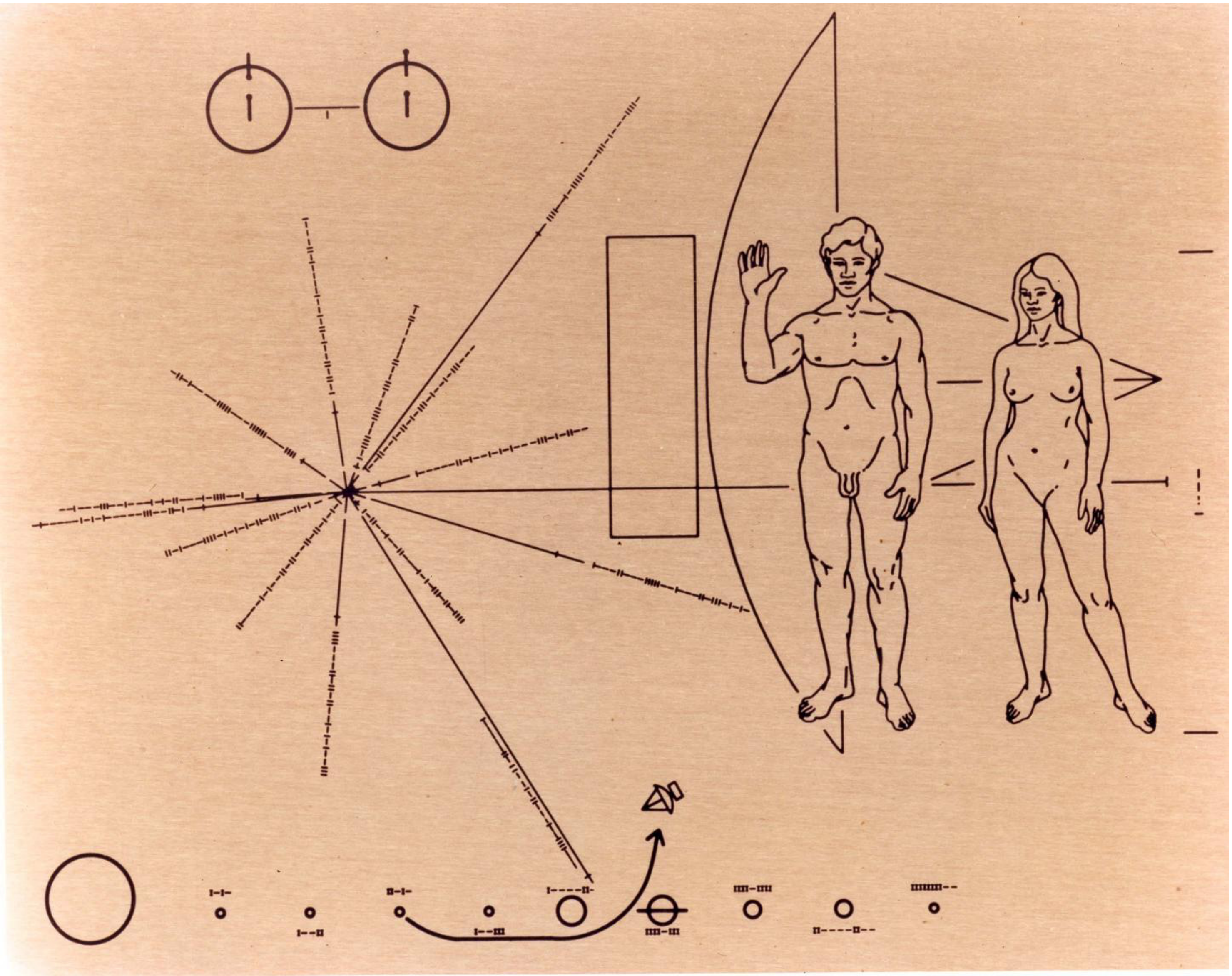
Our goal is to understand the very elaborate Lagrangian given by gravity coupled with the Standard Model, with all its subtleties (V-A, BEH, seesaw, etc etc...) from basic geometric principles. This requires rethinking completely what Geometry is, and the simplest manner is to start with the simplest question :

"Where are we ?"



Two questions arise :

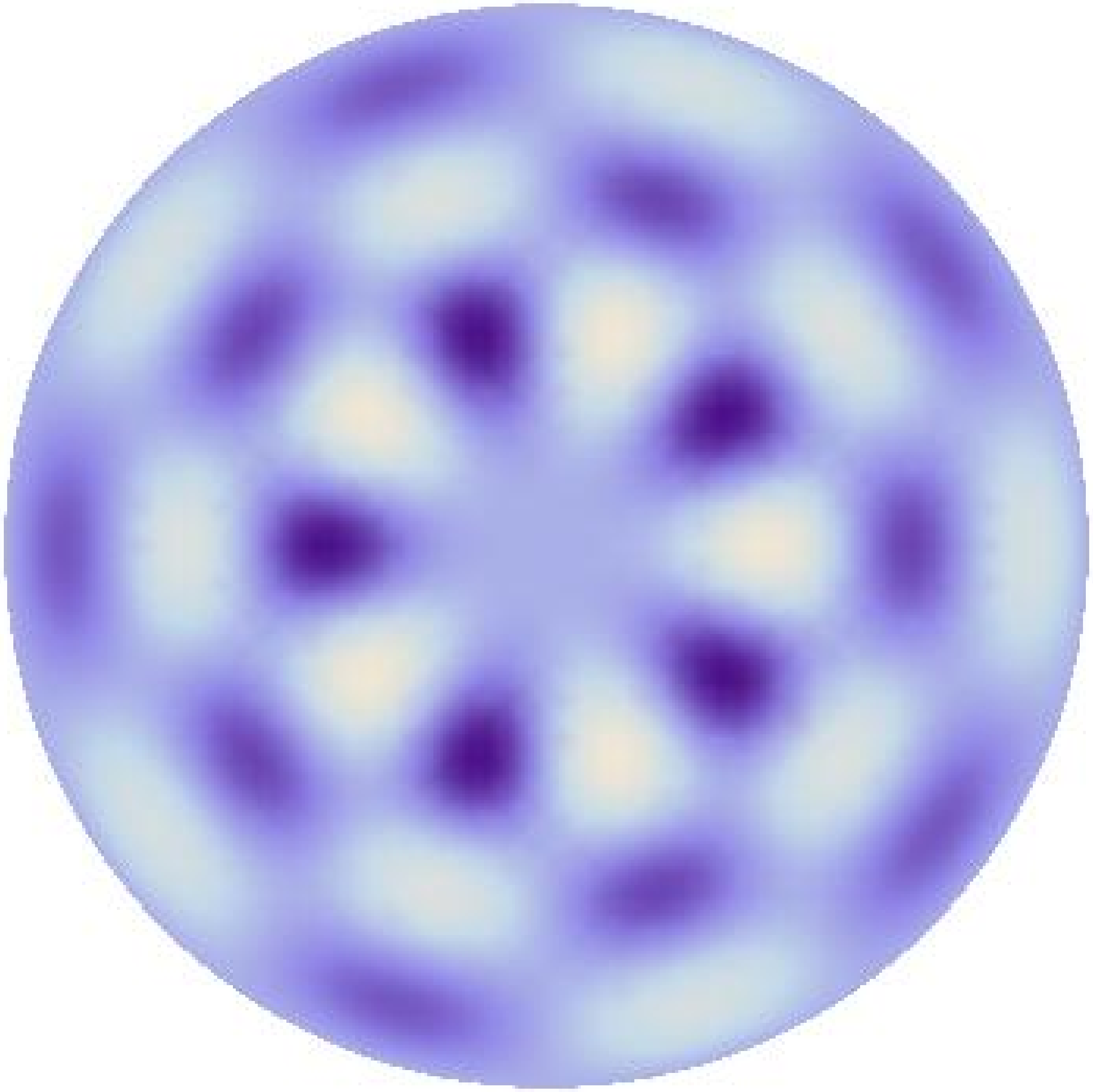
Find complete invariants of geometric spaces, of "shapes"

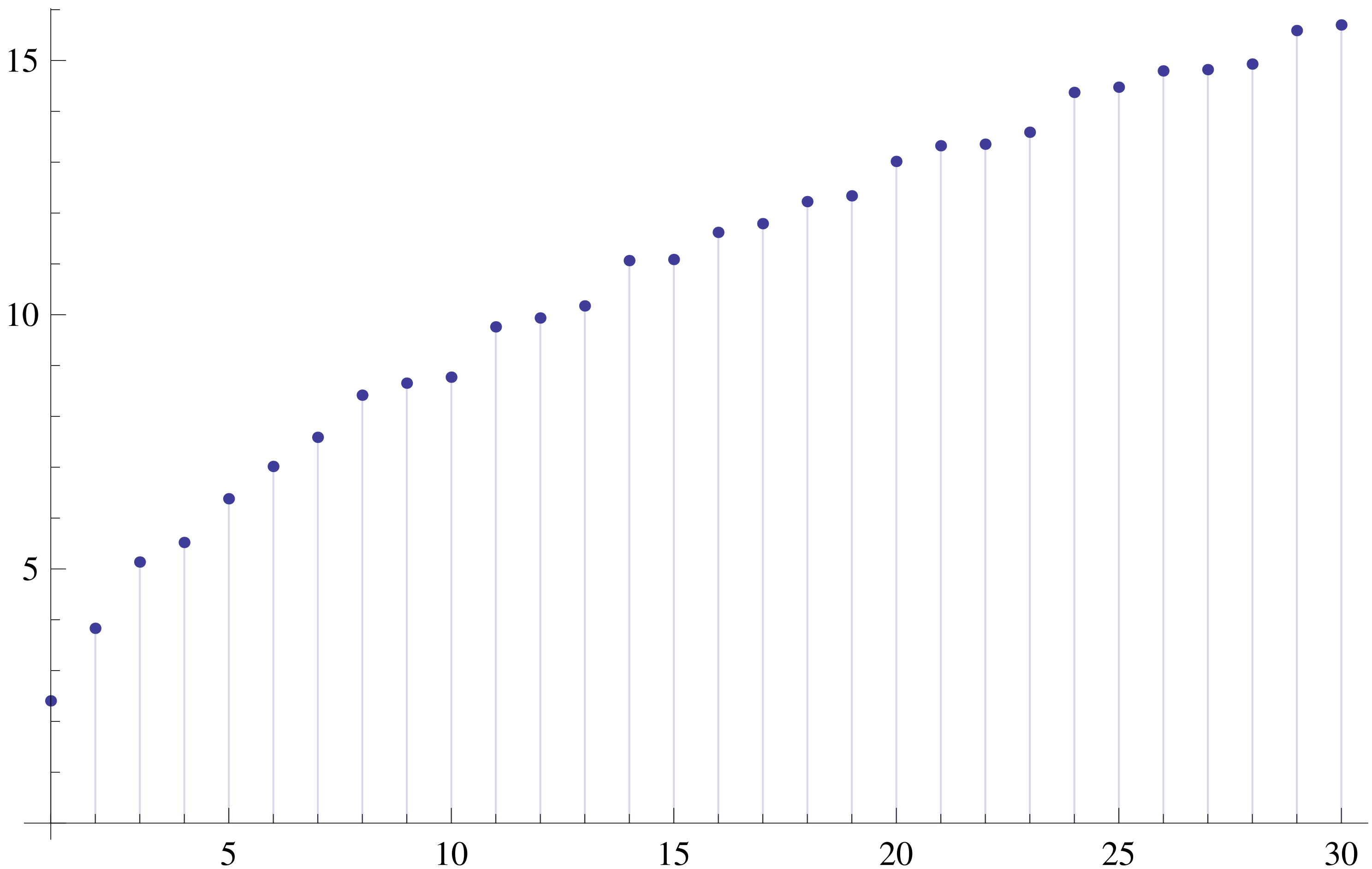
How can we invariantly specify a point in a geometric space ?

The music of shapes

Milnor, John (1964), ”Eigenvalues of the Laplace operator on certain manifolds”, Proceedings of the National Academy of Sciences of the United States of America 51

Kac, Mark (1966), ”Can one hear the shape of a drum ?”, American Mathematical Monthly 73 (4, part 2) : 1–23





Eigenfrequencies of the disk

Spectrum of disk

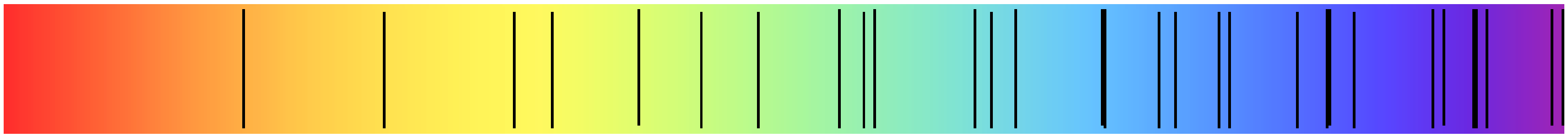
2.40483, 3.83171, 5.13562, 5.52008, 6.38016, 7.01559,

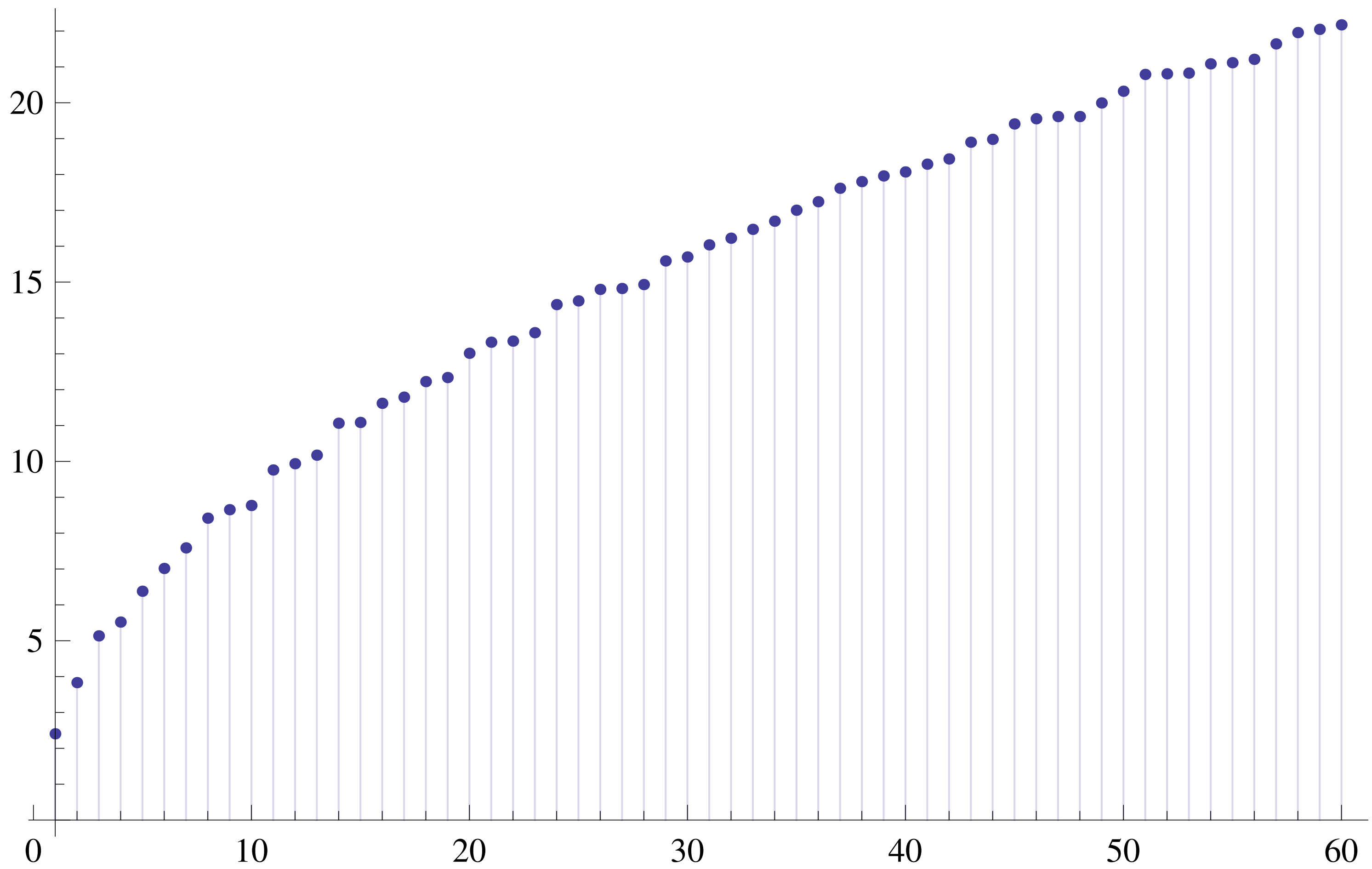
7.58834, 8.41724, 8.65373, 8.77148, 9.76102, 9.93611,

10.1735, 11.0647, 11.0864, 11.6198, 11.7915, 12.2251,

12.3386, 13.0152, 13.3237, 13.3543, 13.5893, 14.3725,

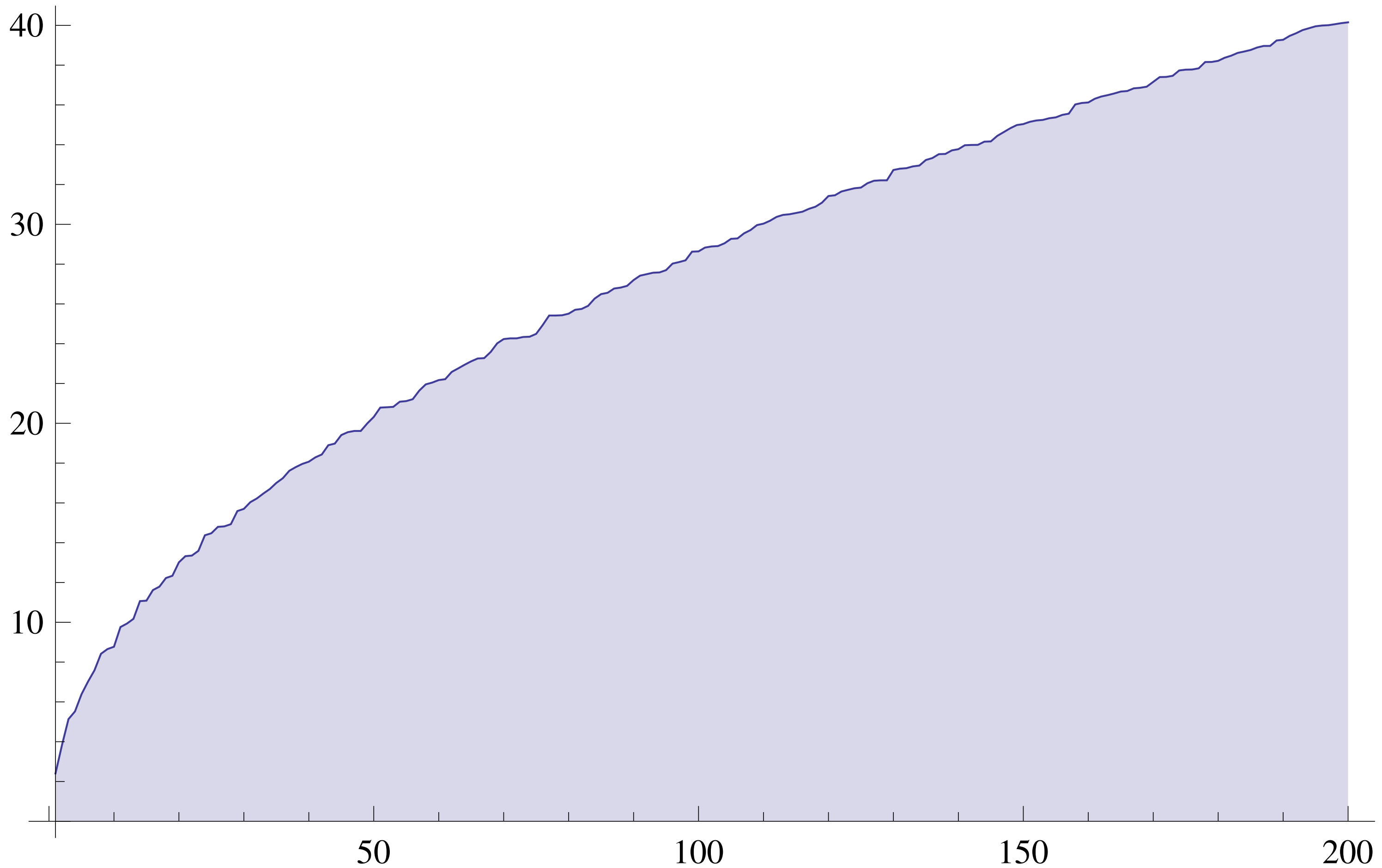
14.4755, 14.796, 14.8213, 14.9309, 15.5898, 15.7002 ...



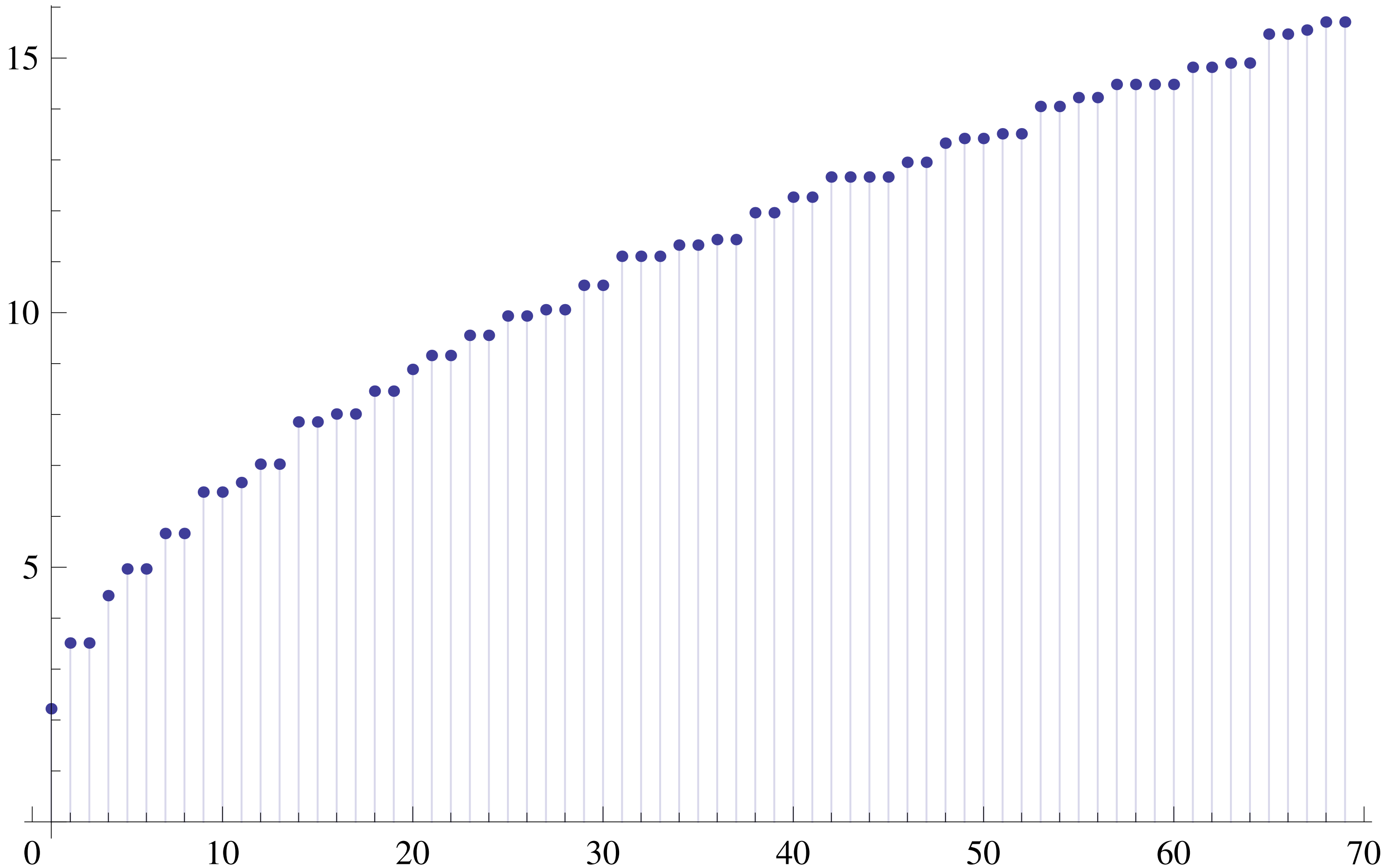


Eigenfrequencies of the disk

High frequencies

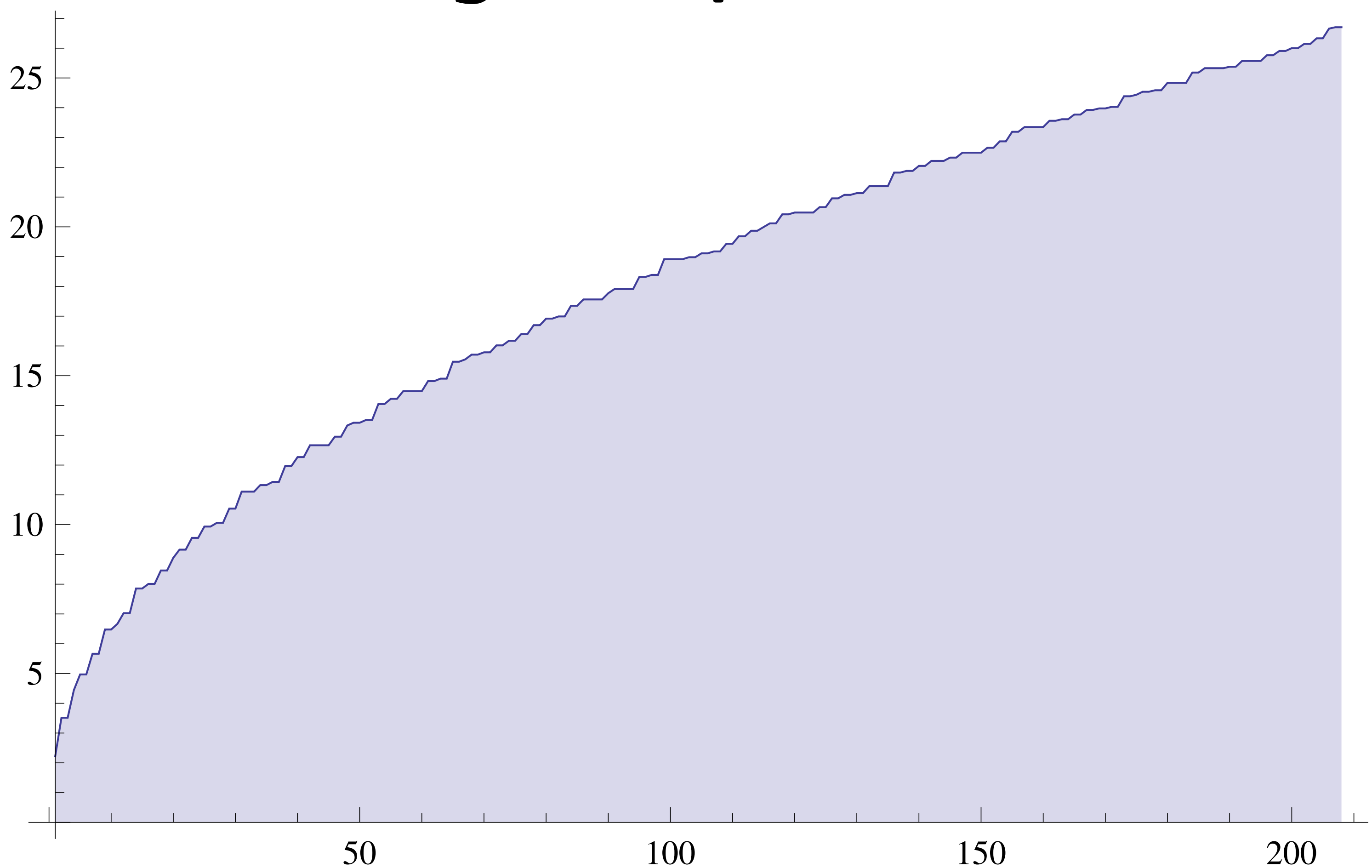


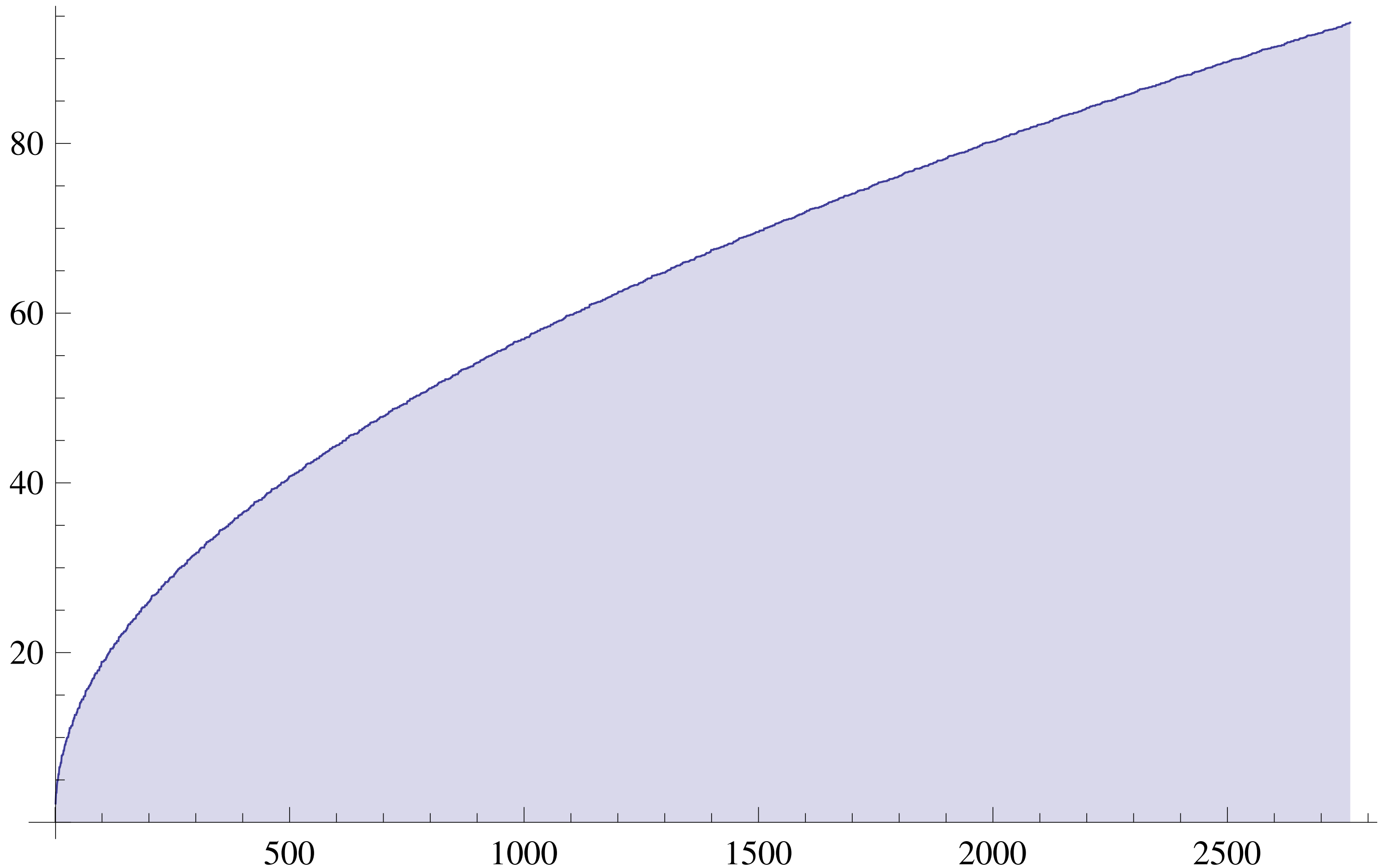
Vibrations of the square



Eigenfrequencies of square

High frequencies





80
60
40
20
500
1000
1500
2000
2500

It is well known since a famous one page paper of John Milnor that the spectrum of operators, such as the Laplacian, does not suffice to characterize a compact Riemannian space. But it turns out that the missing information is encoded by the relative position of two abelian algebras of operators in Hilbert space. Due to a theorem of von Neumann the algebra of multiplication by all measurable bounded functions acts in Hilbert space in a unique manner, independent of the geometry one starts with. Its relative position with respect to the other abelian algebra given by all functions of the Laplacian suffices to recover the full geometry, provided one knows the spectrum of the Laplacian. For some reason which has to do with the inverse problem, it is better to work with the Dirac operator.

The unitary (CKM) invariant of Riemannian manifolds

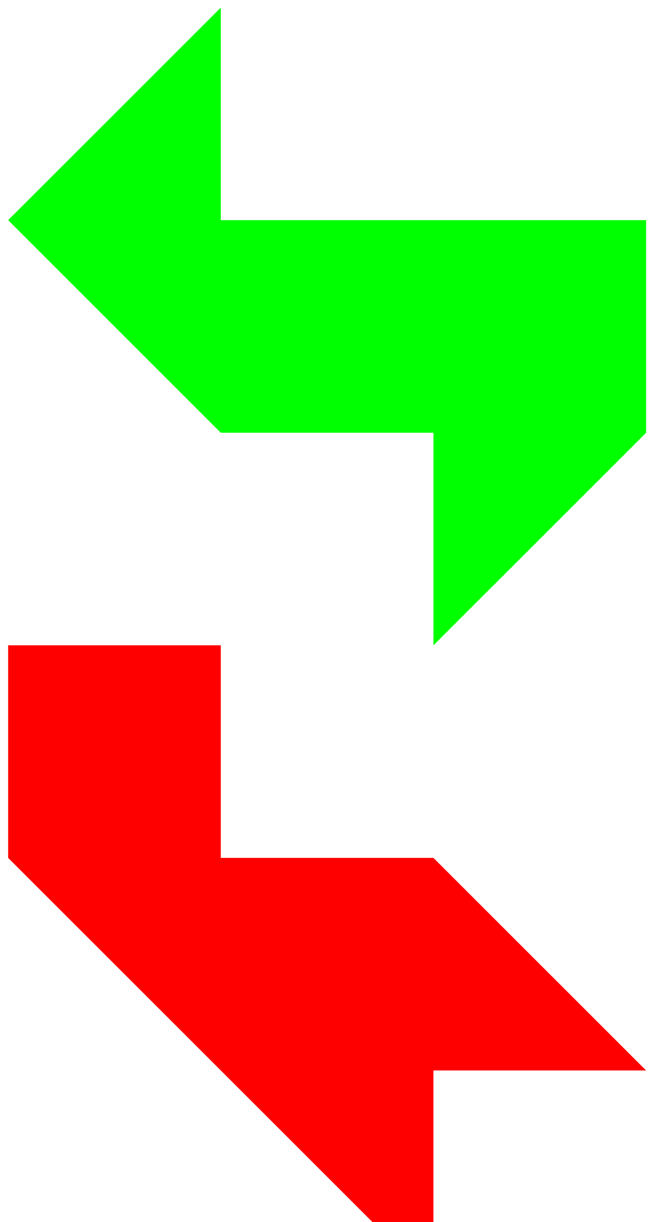
The invariants are :

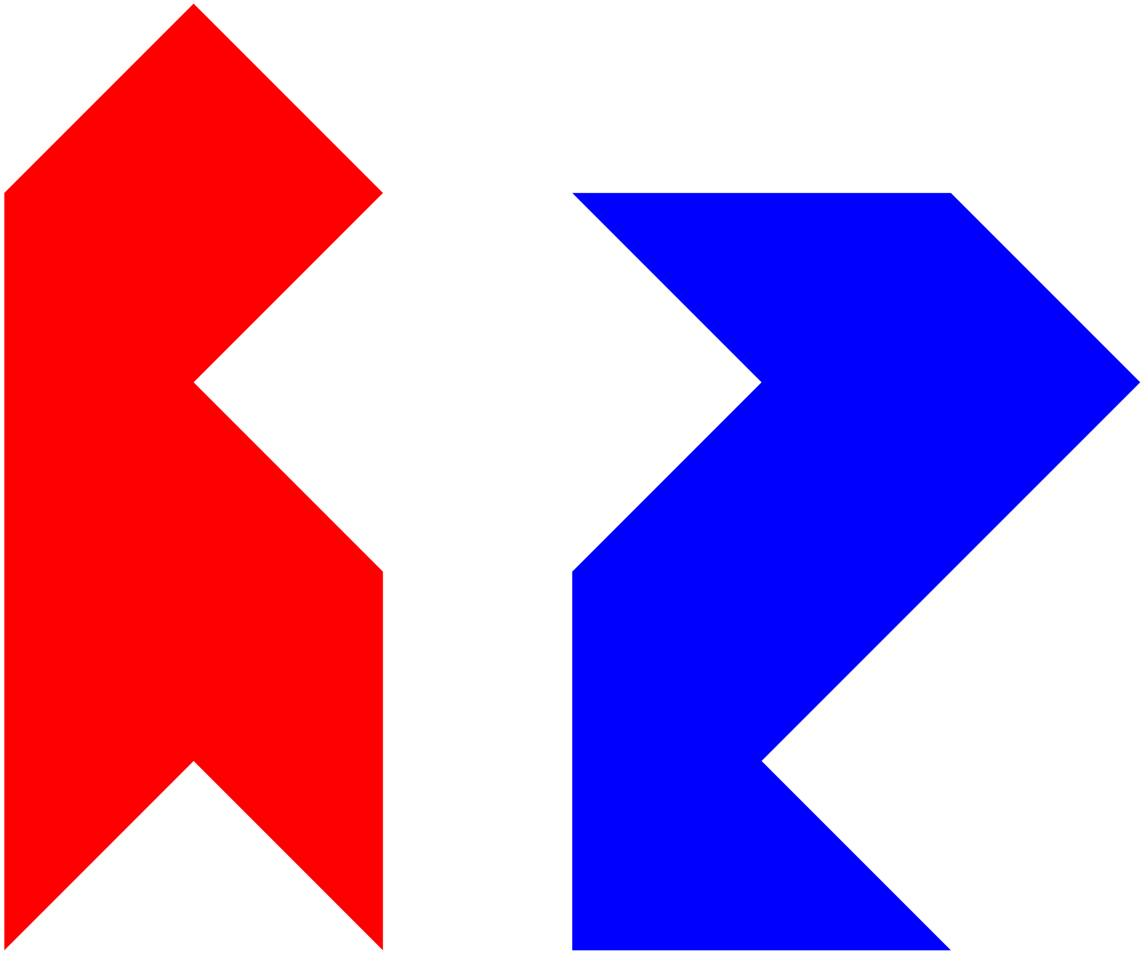
— The spectrum Spec(D).

— The relative spectrum $\text{Spec}_N(M)$ $(N = \{f(D)\})$.

Gordon, Web, Wolpert

Gordon, C. ; Webb, D. ; Wolpert, S. (1992), "Isospectral plane domains and surfaces via Riemannian orbifolds", Inventiones mathematicae

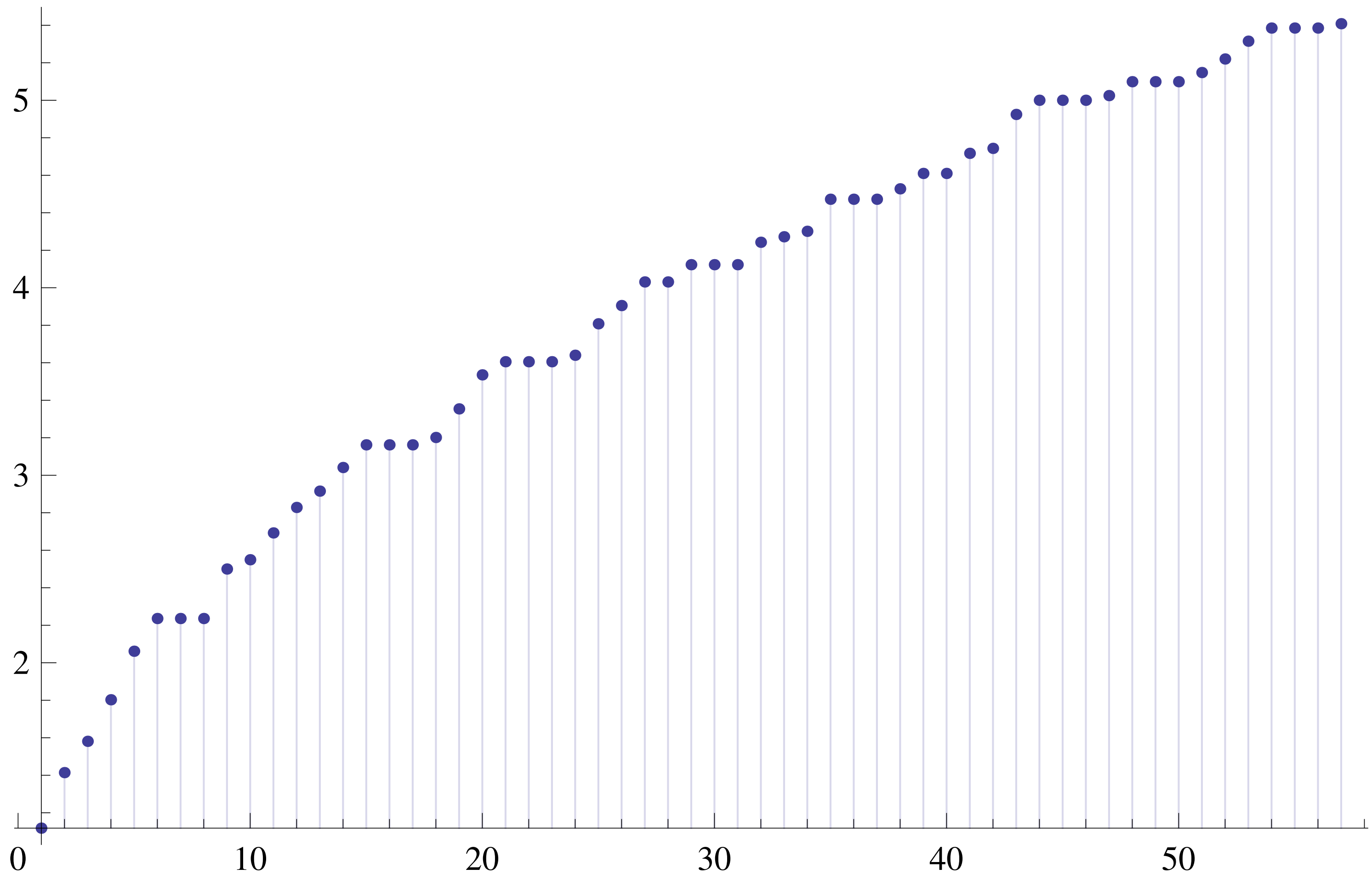




Two shapes with same spectrum (Chapman).

Shape I

Shape II


5
4
3
2
0
10
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50

Spectrum $= \{\sqrt{x} \mid x \in S\}$,

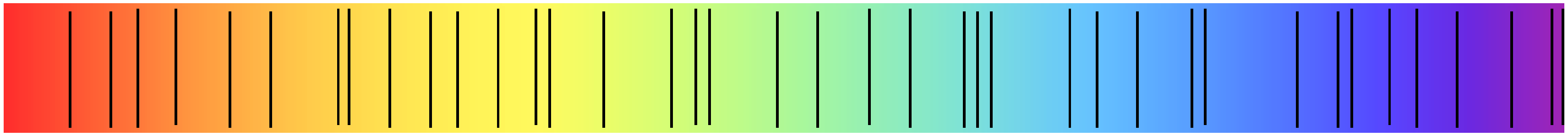
$$S = \{\frac{5}{4}, 2, \frac{5}{2}, \frac{13}{4}, \frac{17}{4}, 5, 5, 5, \frac{25}{4}, \frac{13}{2}, \frac{29}{4}, 8, \frac{17}{2}, \frac{37}{4}, 10, 10, 10, \frac{41}{4}, \frac{45}{4}, \frac{25}{2},$$
$$13, 13, 13, \frac{53}{4}, \frac{29}{2}, \frac{61}{4}, \frac{65}{4}, \frac{65}{4}, 17, 17, 17, 18, \frac{73}{4}, \frac{37}{2}, 20, 20, 20,$$
$$\frac{41}{2}, \frac{85}{4}, \frac{85}{4}, \frac{89}{4}, \frac{45}{2}, \frac{97}{4}, 25, 25, 25, \frac{101}{4}, 26, 26, 26, \frac{53}{2},$$
$$\frac{109}{4}, \frac{113}{4}, 29, 29, 29, \frac{117}{4}, \ldots$$

Same spectrum

$$\{a^2 + b^2 \mid a, b > 0\} \cup \{c^2/4 + d^2/4 \mid 0 < c < d\}$$

$$=$$

$$\{e^2/4 + f^2 \mid e, f > 0\} \cup \{g^2/2 + h^2/2 \mid 0 < g < h\}$$



Three classes of notes

One looks at the fractional part

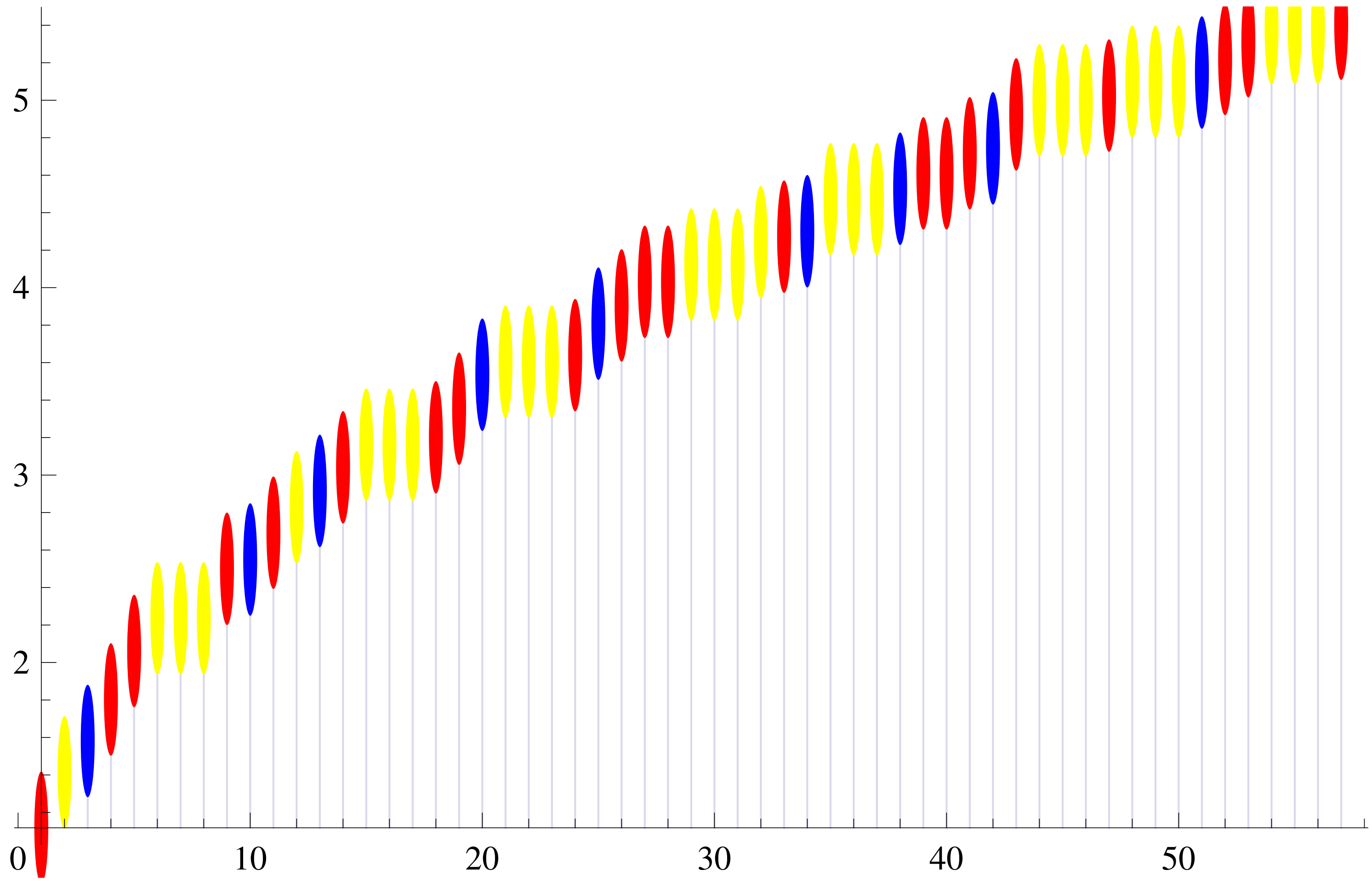
$\frac{1}{4}$: $\{e^2/4 + f^2\}$ with $e, f > 0 = \{c^2/4 + d^2/4\}$ with $c + d$ odd.

$\frac{1}{2}$: The $c^2/4 + d^2/4$ with c, d odd and $g^2/2 + h^2/2$ with $g + h$ odd.

0 : $\{a^2 + b^2 \mid a, b > 0\} \cup \{4c^2/4 + 4d^2/4 \mid 0 < c < d\}$ et $\{4e^2/4 + f^2 \mid e, f > 0\} \cup \{g^2/2 + h^2/2 \mid 0 < g < h\}$ with $g + h$ even.

Possible chords

The possible chords are not the same. Blue–Red is not possible for shape II the one which contains the rectangle.



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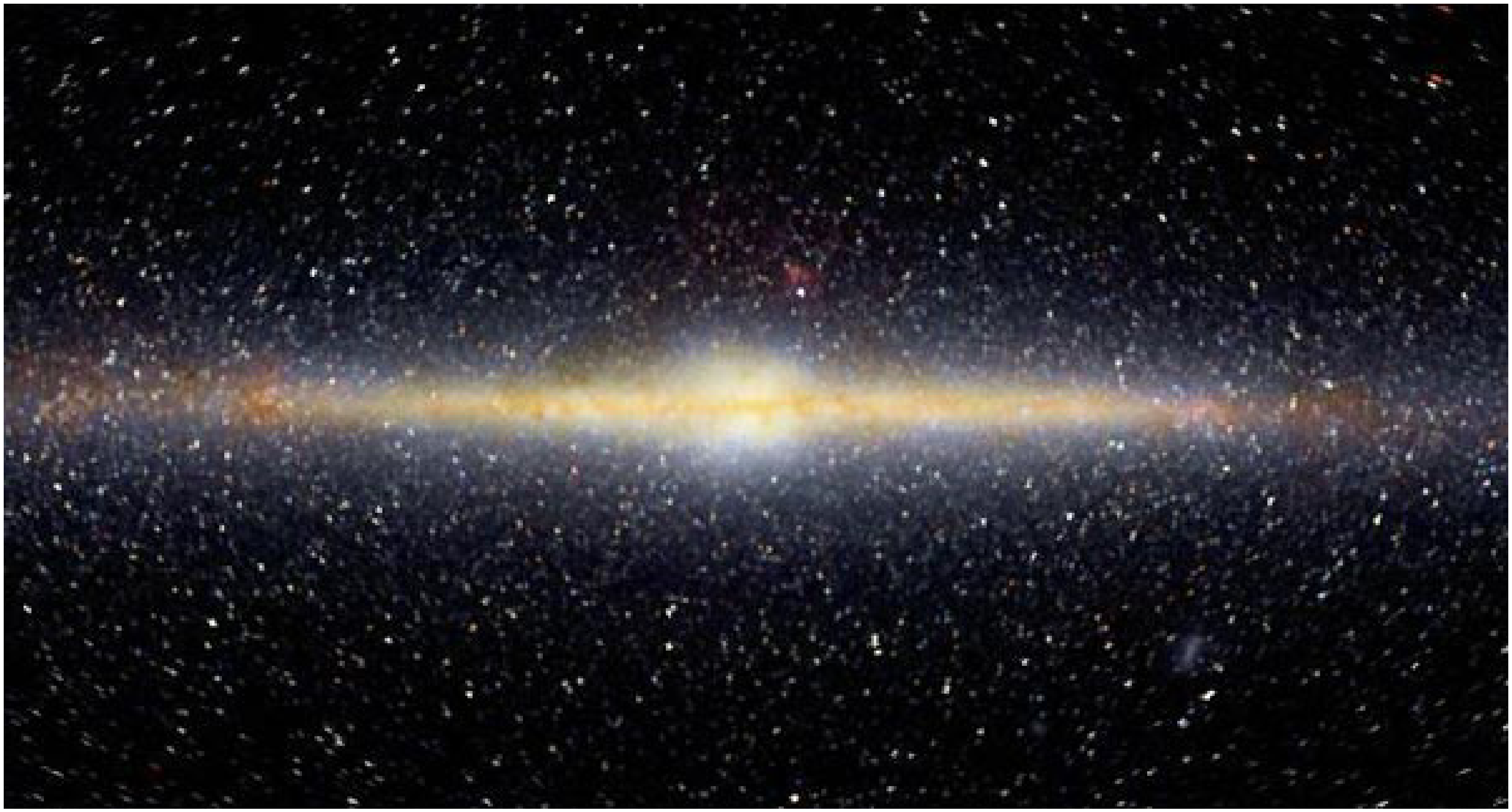
Points

The missing invariant should be interpreted as giving the probability for correlations between the possible frequencies, while a “point” of the geometric space X can be thought of as a correlation, *i.e.* a specific positive hermitian matrix $\rho_{\lambda\kappa}$ (up to scale) which encodes the scalar product at the point between the eigenfunctions of the Dirac operator associated to various frequencies *i.e.* eigenvalues of the Dirac operator.

It is rather convincing that our faith in outer space is based on the strong correlations that exist between different frequencies, as encoded by the matrix $g_{\lambda\mu}$, so that the picture in infrared of the milky way is not that different from its visible light counterpart, which can be seen with a bare eye on a clear night.



Musical shape ?

The ear is sensitive to *ratios* of frequencies.

The two sequences

$$\{440, 440, 440, 493, 552, 493, 440, 552, 493, 493, 440\}$$

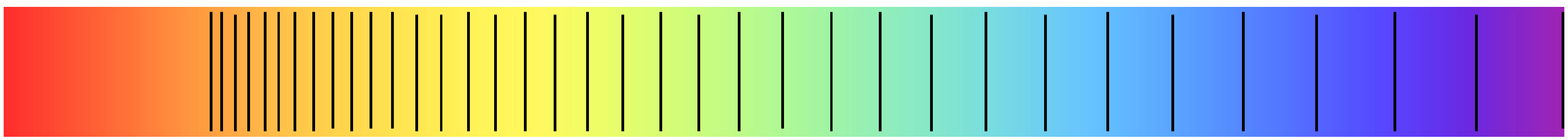
$$\{622, 622, 622, 697, 780, 697, 622, 780, 697, 697, 622\}$$

are in the ratio $\sim \sqrt{2}$.

$$\frac{\log 3}{\log 2} \sim 1 + \frac{1}{1 + \frac{1}{1+\frac{1}{2+\frac{1}{2}}}} = \frac{19}{12}$$

Towards a musical shape

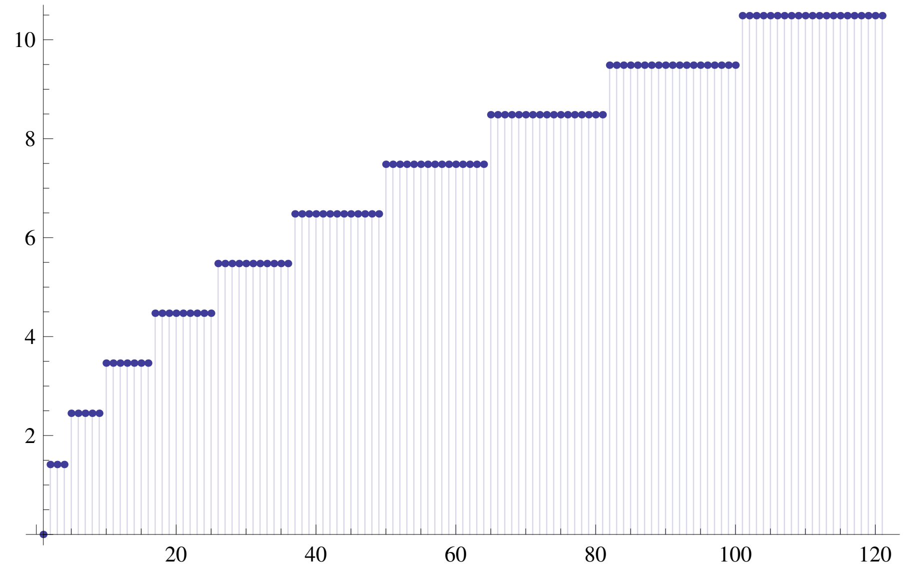
$$\{q^n \mid n \in \mathbb{N}\}, \quad q = 2^{\frac{1}{12}}$$

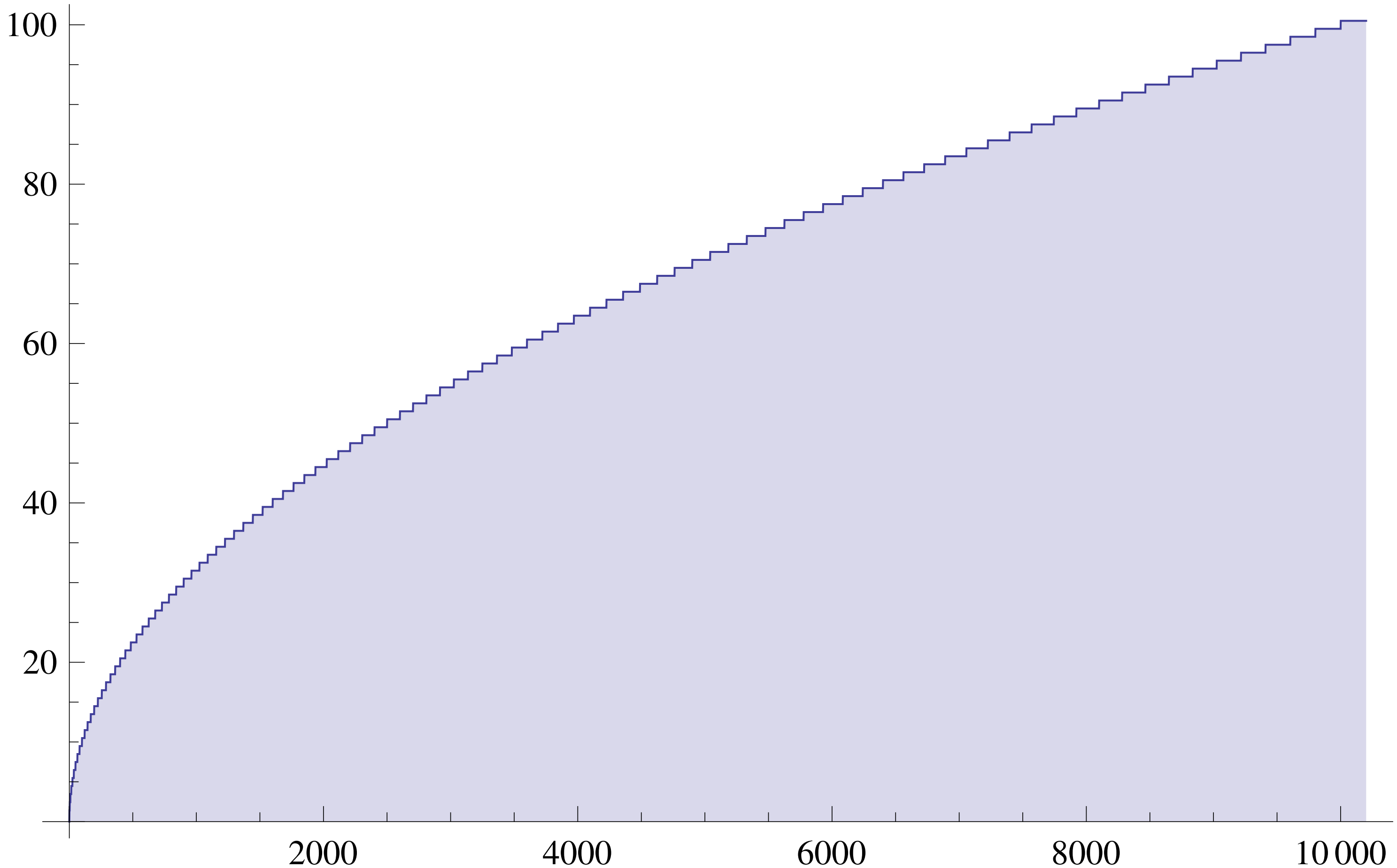


$$2^{1/12} = 1.05946..., \quad 3^{1/19} = 1.05953...$$

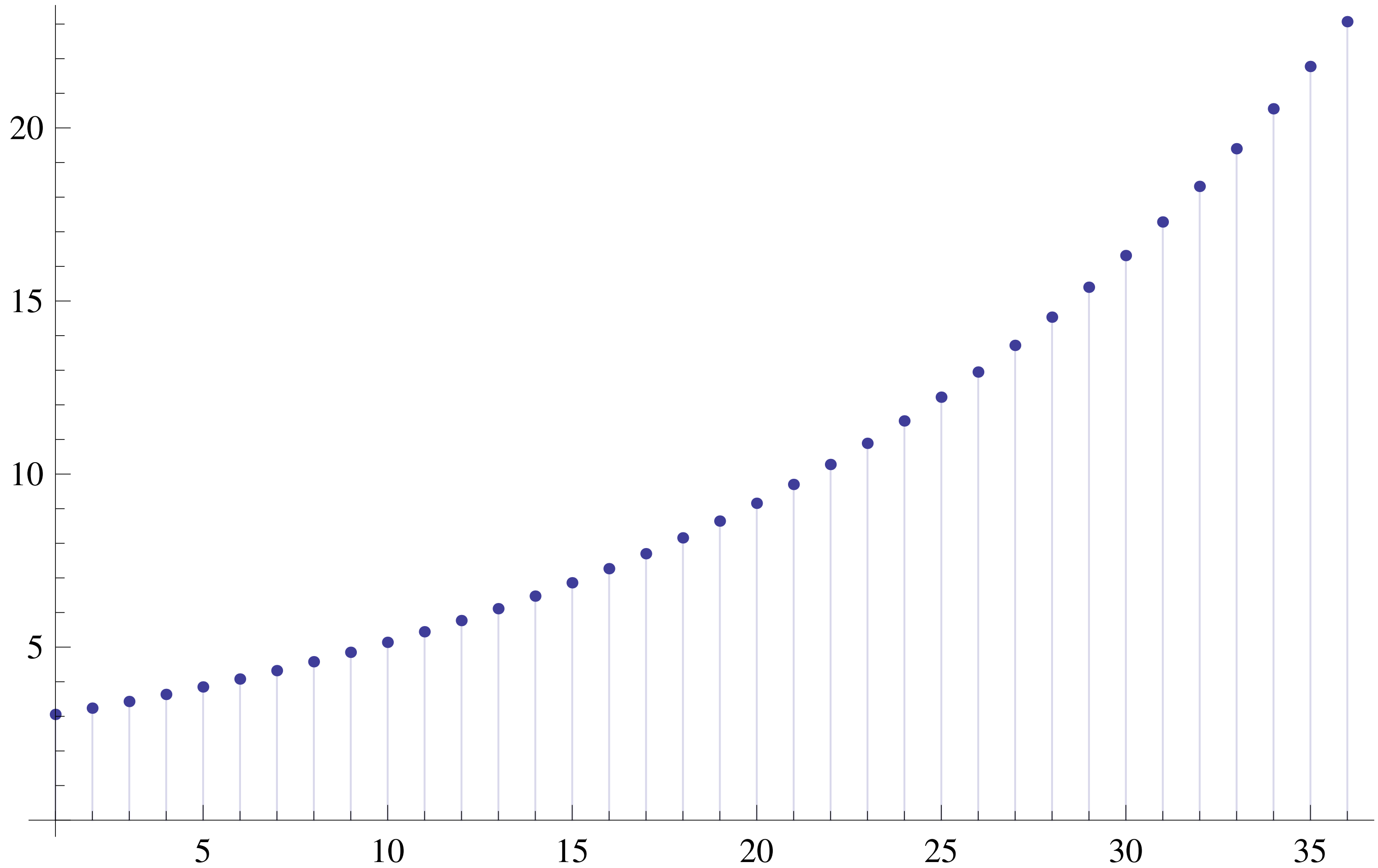
The sphere ?

Spectrum of sphere $= \{\sqrt{j(j+1)} \mid j \geq 0\}$ looks like this for small values





High frequencies of sphere

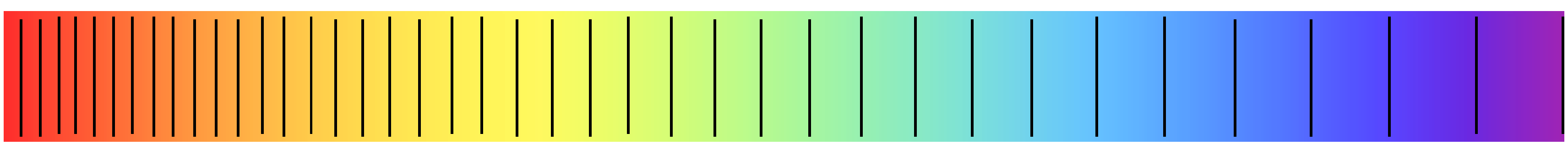


Musical Shape has Dimension $= 0$

The quantum sphere S_q^2

Poddles, Dabrowski, Sitarz, Landi, Wagner, Brain...

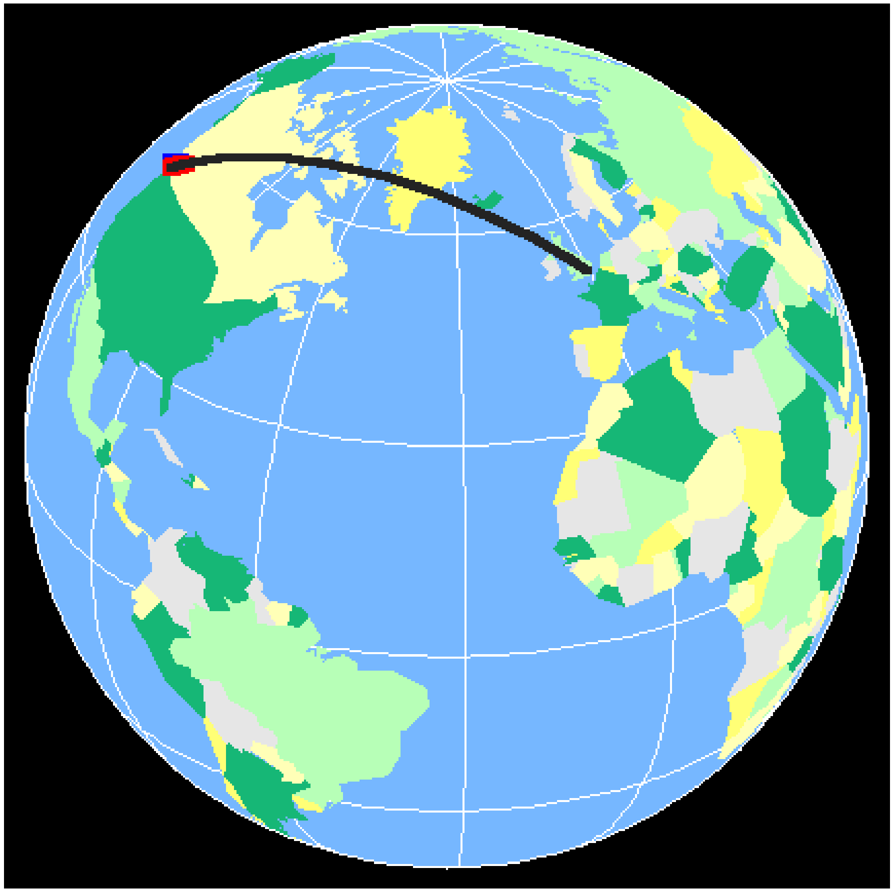
$$\{\frac{q^j - q^{-j}}{q - q^{-1}} \mid j \in \mathbb{N}\} \text{ with multiplicity } O(j)$$



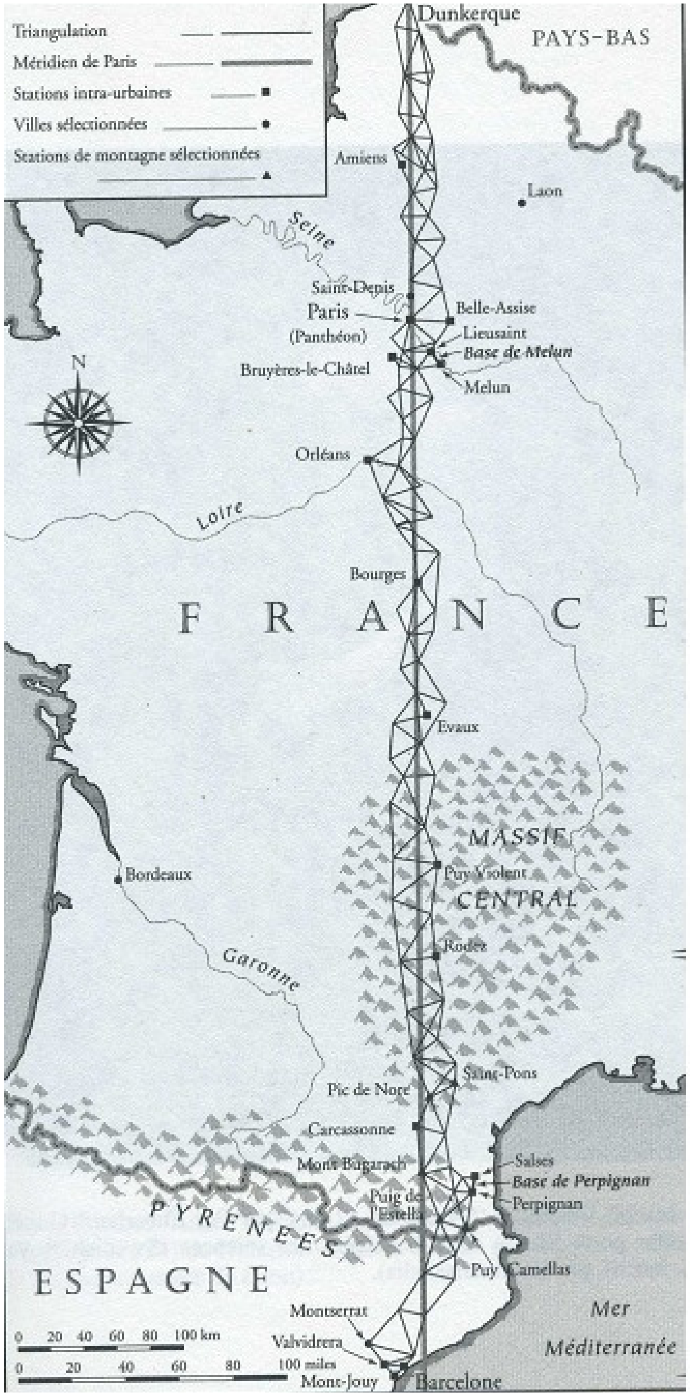
L. Dabrowski, A. Sitarz, *D*irac operator on the standard Podles' quantum sphere. Noncommutative geometry and quantum groups (Warsaw, 2001), 49–58, Banach Center Publ., 61, Polish Acad. Sci., Warsaw, 2003.

L. Dabrowski, F. D'Andrea, G. Landi, E.Wagner, *D*irac operators on all Podles quantum spheres J. Noncomm. Geom. 1 (2007) 213–239 arXiv :math/0606480

S. Brain, G. Landi, *T*he 3D Spin geometry of the quantum 2-sphere Rev. Math. Phys. 22 (2010) 963–993 arXiv :1003.2150



$$d(A,B) = \text{Inf} \int_\gamma \sqrt{g_{\mu\,\nu}\, dx^\mu\, dx^\nu}$$



J-B. J. DELAMBRE

P. F. A. MECHAIN

1792--1799

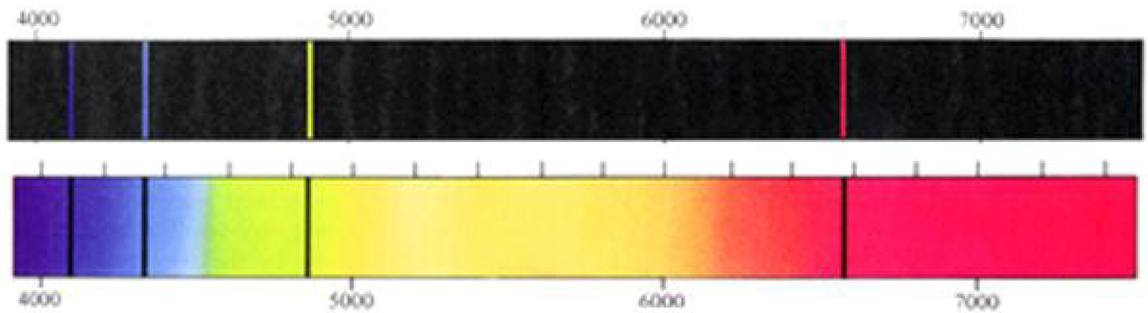
DUNKERQUE--BARCELONE

Dirac's square root of the Laplacian

Spectral triples

$$(\mathcal{A}, \mathcal{H}, D)\,, \qquad ds = D^{-1}\,,$$

$$d(A,B) = \text{Sup}\,\{|f(A) - f(B)|\,;\, f \in \mathcal{A}\,,\ \|[D,f]\| \leq 1\,\}$$



Meter $\rightarrow$ Wave length (Krypton (1967) spectrum of 86Kr then Caesium (1984) hyperfine levels of C133)

Gauge transfos = Inn($\mathcal{A}$)

Let us consider the simplest example

$$\mathcal{A} = C^\infty(M, M_n(\mathbb{C})) = C^\infty(M) \otimes M_n(\mathbb{C})$$

Algebra of $n \times n$ matrices of smooth functions on manifold M.

The group Inn($\mathcal{A}$) of inner automorphisms is locally isomorphic to the group $\mathcal{G}$ of smooth maps from M to the small gauge group $SU(n)$

$$1 \to \mathrm{Inn}(\mathcal{A}) \to \mathrm{Aut}(\mathcal{A}) \to \mathrm{Out}(\mathcal{A}) \to 1$$

becomes identical to

$$1 \to \mathrm{Map}(M, G) \to \mathcal{G} \to \mathrm{Diff}(M) \to 1.$$

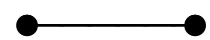
Einstein–Yang-Mills

We have shown that the study of pure gravity on this space yields Einstein gravity on M minimally coupled with Yang-Mills theory for the gauge group $SU(n)$. The Yang-Mills gauge potential appears as the inner part of the metric, in the same way as the group of gauge transformations (for the gauge group $SU(n)$) appears as the group of inner diffeomorphisms.

Geometry from the Quantum

The goal is to reconcile Quantum Mechanics and General Relativity by showing that the latter naturally arises from a higher degree version of the Heisenberg commutation relations.

We have discovered a geometric analogue of the Heisenberg commutation relations $[p,q] = i\hbar$. The role of the momentum p is played by the Dirac operator. It plays the role of a measuring rod and at an intuitive level it represents the inverse of the line element ds familiar in Riemannian geometry

$$ds =$$ 

Position variables

The role of the position variable q was the most difficult to uncover. The answer that we discovered is to encode the analogue of the position variable q in the same way as the Dirac operator encodes the components of the momenta, just using the Feynman slash.

Feynman Slash

To be more precise we let $Y \in \mathcal{A} \otimes C_\kappa$ be of the Feynman slashed form $Y = Y^a \Gamma_a$, and fulfill the equations

$$Y^2 = \kappa, \qquad Y^* = \kappa Y \tag{1}$$

Here $\kappa = \pm 1$ and $C_\kappa \subset M_s(\mathbb{C})$, $s = 2^{n/2}$, is the Clifford algebra on $n+1$ gamma matrices Γ_a, $0 \leq a \leq n$

$$\Gamma_a \in C_\kappa, \quad \left\{\Gamma^a, \Gamma^b\right\} = 2\kappa\, \delta^{ab}, \; (\Gamma^a)^* = \kappa \Gamma^a$$

Higher Heisenberg equation

The one-sided higher analogue of the Heisenberg commutation relations is then (up to a normalization factor $\frac{1}{2^{n/2}n!}$)

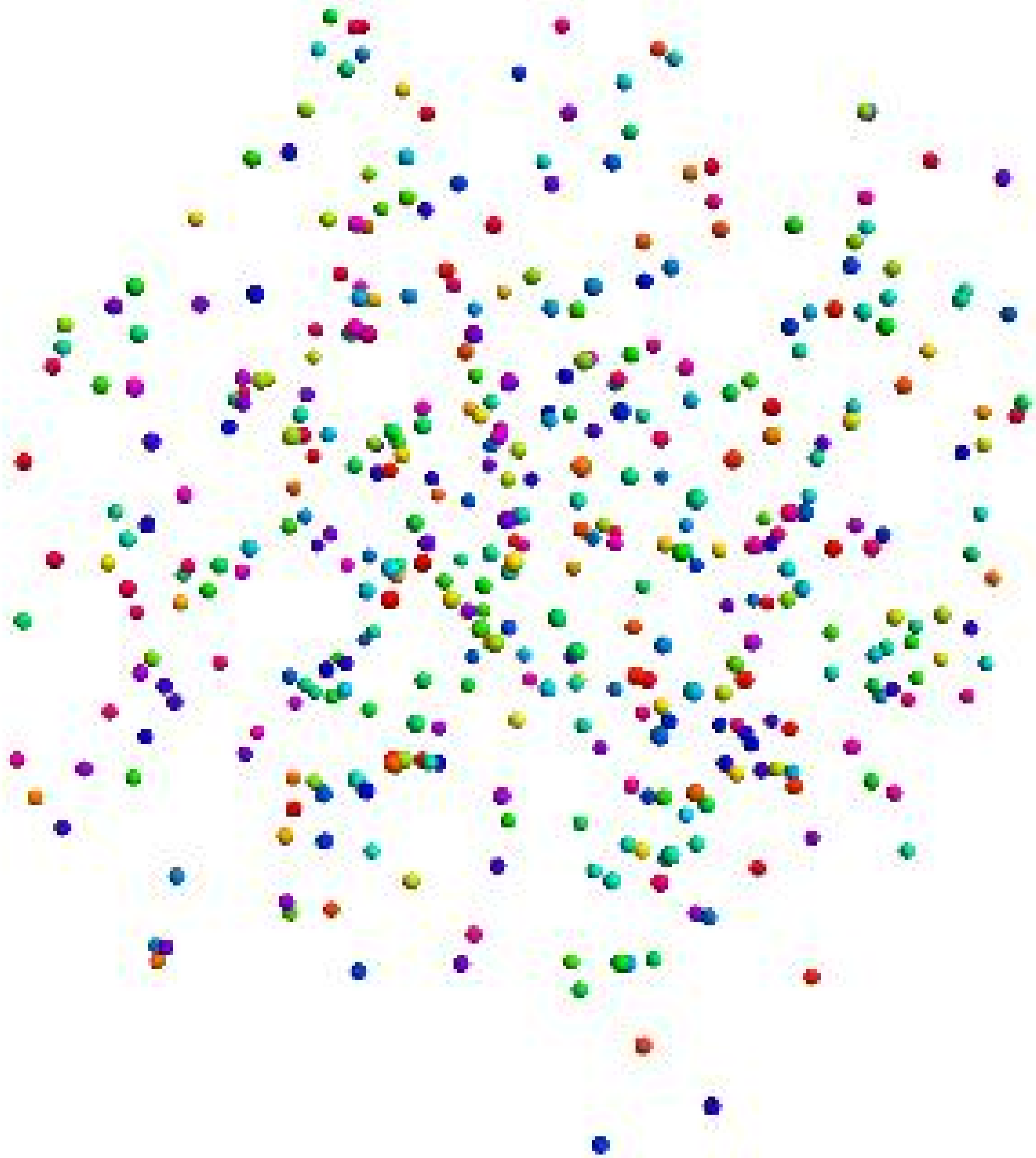
$$\langle Y\,[D,Y]\cdots[D,Y]\rangle = \sqrt{\kappa}\,\gamma \quad (n \text{ terms } [D,Y]) \qquad (2)$$

Volume is quantized

For even n, equation (2), together with the hypothesis that the eigenvalues of D grow as in dimension n, imply that the volume, expressed as the leading term in the Weyl asymptotic formula for counting eigenvalues of the operator D, is "quantized" by being equal to the index pairing of the operator D with the K-theory class of $\mathcal{A}$ defined by the projection $e = (1 + \sqrt{\kappa}\, Y)/2$.

Theorem 1 : bubbles

Let M be a spin Riemannian manifold of even dimension n and $(\mathcal{A}, \mathcal{H}, D)$ the associated spectral triple. Then a solution of the one-sided equation exists if and only if M breaks as the disjoint sum of spheres of unit volume. On each of these irreducible components the unit volume condition is the only constraint on the Riemannian metric which is otherwise arbitrary for each component.



Two kinds of quanta

It would seem at this point that only disconnected geometries fit in this framework but this is ignoring an essential piece of structure of the NCG framework, which allows one to refine (2). It is the real structure J, an antilinear isometry in the Hilbert space H which is the algebraic counterpart of charge conjugation.

Two sided equation

This leads to refine the quantization condition by taking J into account as the two-sided equation

$$\langle Z\,[D,Z]\cdots[D,Z]\rangle = \gamma \quad Z = 2EJEJ^{-1} - 1, \tag{3}$$

where E is the spectral projection for $\{1,i\} \subset \mathbb{C}$ of the double slash $Y = Y_+ \oplus Y_- \in C^\infty(M, C_+ \oplus C_-)$.

Geometry $\Longrightarrow$ Standard Model !

It turns out that in dimension 4

$$C_{+} = M_2(\mathbb{H}), \quad C_{-} = M_4(\mathbb{C})$$

which give the algebraic constituents of the Standard Model exactly in the form of our joint work with Ali Chamseddine ! ! ! !

The two maps $Y_\pm : M \to S^n$

One now gets two maps $Y_\pm : M \to S^n$ while, for $n = 2, 4$, (3) becomes,

$$\det\left(e^a_\mu\right) = \Omega_+ + \Omega_-, \tag{4}$$

with $\Omega_\pm$ the Jacobian of $Y_\pm$ (the pullback of the volume form of the sphere).

In the next theorem the algebraic relations between $Y_\pm$, D, J, $C_\pm$, γ are assumed to hold.

Theorem 2 : Large

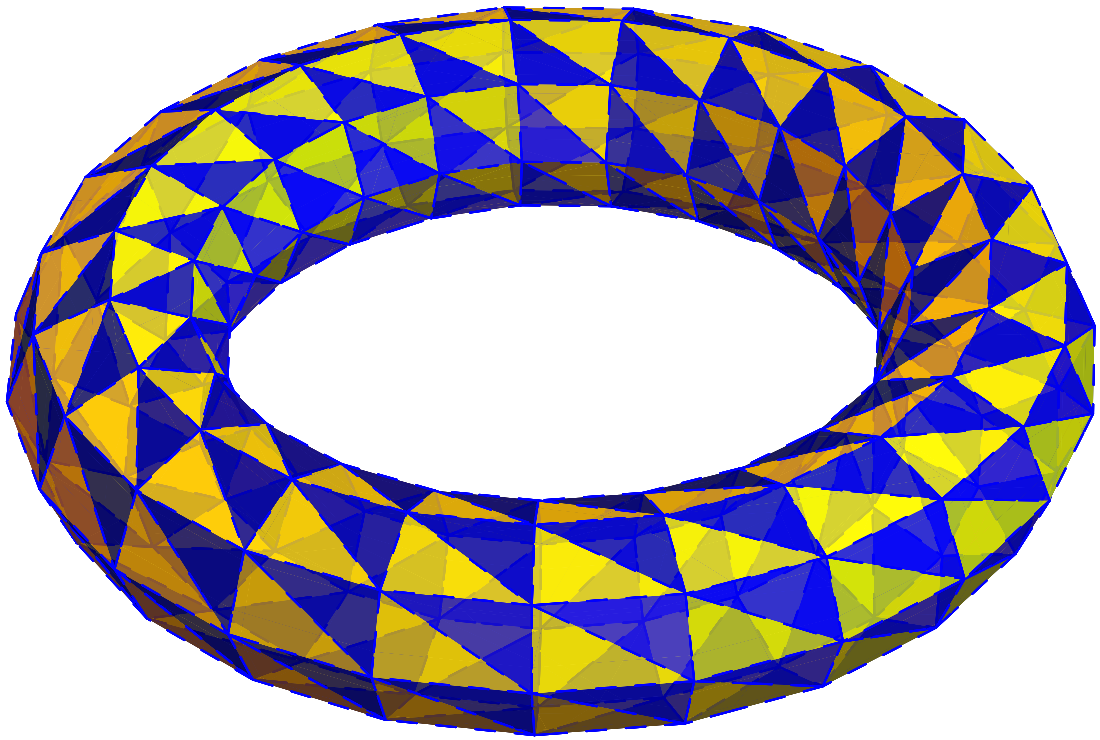
Let $n = 2$ or $n = 4$.

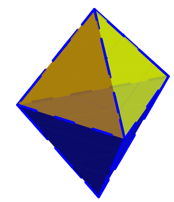
(i) In any operator representation of the two sided equation (3) in which the spectrum of D grows as in dimension n the volume (the leading term of the Weyl asymptotic formula) is quantized.

(ii) Let M be a compact oriented spin Riemannian manifold of dimension n. Then a solution of (4) exists if and only if the volume of M is quantized to belong to the invariant $q_M \subset \mathbb{Z}$ defined as the subset of $\mathbb{Z}$

$$q_M = \{\deg(\phi_+) + \deg(\phi_-) \mid \phi_\pm : M \to S^n,$$
$$|\phi_+|(x) + |\phi_-|(x) \neq 0, \forall x \in M,$$

where deg is the topological degree of the smooth maps and $|\phi|(x)$ is the Jacobian of ϕ at $x \in M$.





The invariant q_M

The invariant q_M makes sense in any dimension. For $n = 2, 3$, and any M, it contains all sufficiently large integers. The case $n = 4$ is more difficult but for our purposes it will suffice to know that q_M contains arbitrarily large numbers in the two relevant cases $M = S^4$ and $M = S^3 \times S^1$.

A first shot at QG

Three "variables", in a fixed Hilbert space with fixed representation of $C_\pm$, γ, J :

$$(D, Y_+, Y_-)$$

$$\langle Z\,[D, Z] \cdots [D, Z] \rangle = \gamma$$

where $Z = 2EJEJ^{-1} - 1$ and E is the spectral projection for $\{1, i\} \subset \mathbb{C}$ of $Y = Y_+ \oplus Y_-$.

Whitney strong embedding

Let us explain why it is natural from the point of view of differential geometry also, to consider the two sets of Γ-matrices and then take the operators $Y_\pm$ as being the correct variables for a first shot at a theory of quantum gravity. The first question which comes in this respect is if, given a compact 4-dimensional manifold M one can find a map $(Y_+, Y_-) : M \to S^4 \times S^4$ which embeds M as a submanifold of $S^4 \times S^4$.

Reconstruction of M

A : It is true that the joint spectrum of the Y_+^a and Y_-^b is of dimension 4 while one has 8 variables.

B : It is it true that the non-commutative integral

$$\int\!\!\!\!\!- \gamma \,\langle Y\,[D,Y]^n\rangle$$

remains quantized.

Spectral action

The bothering cosmological leading term of the spectral action is now quantized and thus it no longer appears in the variation of the spectral action which now reproduces the Einstein equations coupled with matter. The geometry appears from the joint spectrum of the $Y_{\pm}$ and is a 4-dimensional immersed submanifold in the 8-dimensional product $S^4 \times S^4$. One has the strong Whitney embedding theorem : $M^4 \subset \mathbb{R}^4 \times \mathbb{R}^4 \subset S^4 \times S^4$.

Standard Model	Spectral Action
Higgs Boson	Inner metric$^{(0,1)}$
Gauge bosons	Inner metric$^{(1,0)}$
Fermion masses u, ν	Dirac$^{(0,1)}$ in $\uparrow$
CKM matrix Masses down	Dirac$^{(0,1)}$ in $(\downarrow 3)$
Lepton mixing Masses leptons e	Dirac$^{(0,1)}$ in $(\downarrow 1)$

Standard Model	Spectral Action
Majorana mass matrix	Dirac$^{(0,1)}$ on $E_R \oplus J_F E_R$
Gauge couplings	Fixed at unification
Higgs scattering parameter	Fixed at unification
Tadpole constant	$-\mu_0^2\,\lvert\mathbf{H}\rvert^2$

Reduction to SM gauge group

We showed that requiring that these two copies of M stay a finite distance apart reduces the symmetries from the group $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(4)$ of inner automorphisms of the even part of the algebra to the symmetries $U(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3)$ of the Standard Model. This reduction of the gauge symmetry occurs because of the order one condition

$$[[D, a], b^0] = 0 \,, \quad \forall\, a, b \in \mathcal{A}$$

Spectral Model

Let M be a Riemannian spin 4-manifold and F the finite noncommutative geometry of KO-dimension 6 described above. Let $M \times F$ be endowed with the product metric.

1. The unimodular subgroup of the unitary group acting by the adjoint representation $\mathrm{Ad}(u)$ in $\mathcal{H}$ is the group of gauge transformations of SM.
2. The unimodular inner fluctuations of the metric give the gauge bosons of SM.
3. The full standard model (with neutrino mixing and seesaw mechanism) minimally coupled to

Einstein gravity is given in Euclidean form by the action functional

$$S = \mathrm{Tr}(f(D_A/\Lambda)) + \frac{1}{2}\langle J\,\tilde{\xi}, D_A\,\tilde{\xi}\rangle\,, \quad \tilde{\xi} \in \mathcal{H}_{cl}^{+},$$

where D_A is the Dirac operator with the unimodular inner fluctuations.

Standard Model

$$
\begin{aligned}
\mathcal{L}_{SM} = &-\frac{1}{2}\partial_\nu g^a_\mu \partial_\nu g^a_\mu - g_s f^{abc}\partial_\mu g^a_\nu g^b_\mu g^c_\nu - \frac{1}{4}g_s^2 f^{abc} f^{ade} g^b_\mu g^c_\nu g^d_\mu g^e_\nu \\
&-\partial_\nu W^+_\mu \partial_\nu W^-_\mu - M^2 W^+_\mu W^-_\mu - \frac{1}{2}\partial_\nu Z^0_\mu \partial_\nu Z^0_\mu - \frac{1}{2c_w^2} M^2 Z^0_\mu Z^0_\mu - \frac{1}{2}\partial_\mu A_\nu \partial_\mu A_\nu \\
&-igc_w(\partial_\nu Z^0_\mu (W^+_\mu W^-_\nu - W^+_\nu W^-_\mu) - Z^0_\nu (W^+_\mu \partial_\nu W^-_\mu - W^-_\mu \partial_\nu W^+_\mu) \\
&+Z^0_\mu (W^+_\nu \partial_\nu W^-_\mu - W^-_\nu \partial_\nu W^+_\mu)) - igs_w(\partial_\nu A_\mu (W^+_\mu W^-_\nu - W^+_\nu W^-_\mu) \\
&-A_\nu (W^+_\mu \partial_\nu W^-_\mu - W^-_\mu \partial_\nu W^+_\mu) + A_\mu (W^+_\nu \partial_\nu W^-_\mu - W^-_\nu \partial_\nu W^+_\mu)) \\
&-\frac{1}{2}g^2 W^+_\mu W^-_\mu W^+_\nu W^-_\nu + \frac{1}{2}g^2 W^+_\mu W^-_\nu W^+_\mu W^-_\nu \\
&+g^2 c_w^2 (Z^0_\mu W^+_\mu Z^0_\nu W^-_\nu - Z^0_\mu Z^0_\mu W^+_\nu W^-_\nu) + g^2 s_w^2 (A_\mu W^+_\mu A_\nu W^-_\nu - A_\mu A_\mu W^+_\nu W^-_\nu) \\
&+g^2 s_w c_w (A_\mu Z^0_\nu (W^+_\mu W^-_\nu - W^+_\nu W^-_\mu) - 2A_\mu Z^0_\mu W^+_\nu W^-_\nu) - \frac{1}{2}\partial_\mu H \partial_\mu H - \frac{1}{2}m_h^2 H^2 \\
&-\partial_\mu \phi^+ \partial_\mu \phi^- - M^2 \phi^+ \phi^- - \frac{1}{2}\partial_\mu \phi^0 \partial_\mu \phi^0 - \frac{1}{2c_w^2} M^2 \phi^0 \phi^0 \\
&-\beta_h \left(\frac{2M^2}{g^2} + \frac{2M}{g} H + \frac{1}{2}(H^2 + \phi^0\phi^0 + 2\phi^+\phi^-) \right) + \frac{2M^4}{g^2}\alpha_h \\
&-g\alpha_h M \left(H^3 + H\phi^0\phi^0 + 2H\phi^+\phi^- \right)
\end{aligned}
$$

$$-\frac{1}{8}g^2\alpha_h\left(H^4+(\phi^0)^4+4(\phi^+\phi^-)^2+4(\phi^0)^2\phi^+\phi^-+4H^2\phi^+\phi^-+2(\phi^0)^2H^2\right)$$

$$-gMW_\mu^+W_\mu^-H-\frac{1}{2}g\frac{M}{c_w^2}Z_\mu^0Z_\mu^0H$$

$$-\frac{1}{2}ig\left(W_\mu^+(\phi^0\partial_\mu\phi^--\phi^-\partial_\mu\phi^0)-W_\mu^-(\phi^0\partial_\mu\phi^+-\phi^+\partial_\mu\phi^0)\right)$$

$$+\frac{1}{2}g\left(W_\mu^+(H\partial_\mu\phi^--\phi^-\partial_\mu H)+W_\mu^-(H\partial_\mu\phi^+-\phi^+\partial_\mu H)\right)$$

$$+\frac{1}{2}g\frac{1}{c_w}(Z_\mu^0(H\partial_\mu\phi^0-\phi^0\partial_\mu H)-ig\frac{s_w^2}{c_w}MZ_\mu^0(W_\mu^+\phi^--W_\mu^-\phi^+)$$

$$+igs_wMA_\mu(W_\mu^+\phi^--W_\mu^-\phi^+)-ig\frac{1-2c_w^2}{2c_w}Z_\mu^0(\phi^+\partial_\mu\phi^--\phi^-\partial_\mu\phi^+)$$

$$+igs_wA_\mu(\phi^+\partial_\mu\phi^--\phi^-\partial_\mu\phi^+)-\frac{1}{4}g^2W_\mu^+W_\mu^-\left(H^2+(\phi^0)^2+2\phi^+\phi^-\right)$$

$$-\frac{1}{8}g^2\frac{1}{c_w^2}Z_\mu^0Z_\mu^0\left(H^2+(\phi^0)^2+2(2s_w^2-1)^2\phi^+\phi^-\right)$$

$$-\frac{1}{2}g^2\frac{s_w^2}{c_w}Z_\mu^0\phi^0(W_\mu^+\phi^-+W_\mu^-\phi^+)-\frac{1}{2}ig^2\frac{s_w^2}{c_w}Z_\mu^0H(W_\mu^+\phi^--W_\mu^-\phi^+)$$

$$+\frac{1}{2}g^2s_wA_\mu\phi^0(W_\mu^+\phi^-+W_\mu^-\phi^+)+\frac{1}{2}ig^2s_wA_\mu H(W_\mu^+\phi^--W_\mu^-\phi^+)$$

$$-g^2\frac{s_w}{c_w}(2c_w^2-1)Z_\mu^0A_\mu\phi^+\phi^--g^2s_w^2A_\mu A_\mu\phi^+\phi^-$$

$$+\frac{1}{2}ig_s\lambda^a_{ij}(\bar{q}^\sigma_i\gamma^\mu q^\sigma_j)g^a_\mu - \bar{e}^\lambda(\gamma\partial + m^\lambda_e)e^\lambda - \bar{\nu}^\lambda\gamma\partial\nu^\lambda - \bar{u}^\lambda_j(\gamma\partial + m^\lambda_u)u^\lambda_j$$

$$-\bar{d}^\lambda_j(\gamma\partial + m^\lambda_d)d^\lambda_j + igs_wA_\mu\left(-(\bar{e}^\lambda\gamma^\mu e^\lambda) + \frac{2}{3}(\bar{u}^\lambda_j\gamma^\mu u^\lambda_j) - \frac{1}{3}(\bar{d}^\lambda_j\gamma^\mu d^\lambda_j)\right)$$

$$+\frac{ig}{4c_w}Z^0_\mu\{(\bar{\nu}^\lambda\gamma^\mu(1+\gamma^5)\nu^\lambda) + (\bar{e}^\lambda\gamma^\mu(4s^2_w - 1 - \gamma^5)e^\lambda)$$

$$+(\bar{d}^\lambda_j\gamma^\mu(\frac{4}{3}s^2_w - 1 - \gamma^5)d^\lambda_j) + (\bar{u}^\lambda_j\gamma^\mu(1 - \frac{8}{3}s^2_w + \gamma^5)u^\lambda_j)\}$$

$$+\frac{ig}{2\sqrt{2}}W^+_\mu\left((\bar{\nu}^\lambda\gamma^\mu(1+\gamma^5)e^\lambda) + (\bar{u}^\lambda_j\gamma^\mu(1+\gamma^5)C_{\lambda\kappa}d^\kappa_j)\right)$$

$$+\frac{ig}{2\sqrt{2}}W^-_\mu\left((\bar{e}^\lambda\gamma^\mu(1+\gamma^5)\nu^\lambda) + (\bar{d}^\kappa_jC^\dagger_{\kappa\lambda}\gamma^\mu(1+\gamma^5)u^\lambda_j)\right)$$

$$+\frac{ig}{2\sqrt{2}}\frac{m^\lambda_e}{M}\left(-\phi^+(\bar{\nu}^\lambda(1-\gamma^5)e^\lambda) + \phi^-(\bar{e}^\lambda(1+\gamma^5)\nu^\lambda)\right)$$

$$-\frac{g}{2}\frac{m^\lambda_e}{M}\left(H(\bar{e}^\lambda e^\lambda) + i\phi^0(\bar{e}^\lambda\gamma^5e^\lambda)\right)$$

$$+\frac{ig}{2M\sqrt{2}}\phi^+\left(-m^\kappa_d(\bar{u}^\lambda_jC_{\lambda\kappa}(1-\gamma^5)d^\kappa_j) + m^\lambda_u(\bar{u}^\lambda_jC_{\lambda\kappa}(1+\gamma^5)d^\kappa_j\right)$$

$$+\frac{ig}{2M\sqrt{2}}\phi^-\left(m^\lambda_d(\bar{d}^\lambda_jC^\dagger_{\lambda\kappa}(1+\gamma^5)u^\kappa_j) - m^\kappa_u(\bar{d}^\lambda_jC^\dagger_{\lambda\kappa}(1-\gamma^5)u^\kappa_j\right)$$

$$-\frac{g}{2}\frac{m^\lambda_u}{M}H(\bar{u}^\lambda_ju^\lambda_j) - \frac{g}{2}\frac{m^\lambda_d}{M}H(\bar{d}^\lambda_jd^\lambda_j) + \frac{ig}{2}\frac{m^\lambda_u}{M}\phi^0(\bar{u}^\lambda_j\gamma^5u^\lambda_j) - \frac{ig}{2}\frac{m^\lambda_d}{M}\phi^0(\bar{d}^\lambda_j\gamma^5d^\lambda_j)$$

First interplay with experiment

Historically, the search to identify the structure of the noncommutative space followed the bottom-up approach where the known spectrum of the fermionic particles was used to determine the geometric data that defines the space.

This bottom-up approach involved an interesting interplay with experiments. While at first the experimental evidence of neutrino oscillations contradicted the first attempt, it was realized several years later in 2006 that the obstruction to get neutrino oscillations was naturally eliminated by dropping the equality between the metric dimension of space-time (which is equal to 4 as far as we know) and its KO-dimension which is only defined modulo 8. When the latter is set equal to 2 modulo 8 (using the freedom to adjust the geometry of the finite space encoding the fine structure of space-time) everything works fine, the neutrino oscillations are there as well as the see-saw mechanism which appears for free as an unexpected bonus. Incidentally, this also solved the fermionic doubling problem by allowing a simultaneous Weyl-Majorana condition on the fermions to halve the degrees of freedom.

Second interplay with experiment

The second interplay with experiments occurred a bit later when it became clear that the mass of the Brout-Englert-Higgs boson would not comply with the restriction (that $m_H \succeq 170$ Gev) imposed by the validity of the Standard Model up to the unification scale.

We showed that the inconsistency between the spectral Standard Model and the experimental value of the Higgs mass is resolved by the presence of a real scalar field strongly coupled to the Higgs field. This scalar field was already present in the spectral model and we wrongly neglected it in our previous computations.

It was shown recently by several authors, independently of the spectral approach, that such a strongly coupled scalar field stabilizes the Standard Model up to unification scale in spite of the low value of the Higgs mass. In our recent work, we show that the noncommutative neutral singlet modifies substantially the RG analysis, invalidates our previous prediction of Higgs mass in the range 160–180 Gev, and restores the consistency of the noncommutative geometric model with the low Higgs mass.

Lesson

One lesson which we learned on that occasion is that we have to take all the fields of the noncommutative spectral model seriously, without making assumptions not backed up by valid analysis, especially because of the almost uniqueness of the Standard Model (SM) in the noncommutative setting.