"Supersymmetry, Supergravity, Superstrings"
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Solution-generating Symmetries of String Theory

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based on work with:

J. de Boer, R. Dijkgraaf and H. Ooguri (hep-th/0111210),
W. Abou Salem and J. Fröhlich (unpublished),
M. Gaberdiel (hep-th/0411067),
I. Brunner and M. Gaberdiel (in progress)

• String theory has given support to the hypothesis of supersymmetric unification, and has enriched our 'arsenal' of theoretical possibilities and ideas. It has, however, only had one successful semi-quantitative prediction [$\log M_{GUT} \simeq \log(M_{Planck}/g_{GUT})$] , and several striking failures (most notably the value of Λ_{cosm}). Fortunately the LHC is there to (hopefully) point the way about how/whether string theory may fit (in) our low-dimensional world.

• On the more formal side, and despite beautiful progress over the years, we still have no answer to the question 'What is String Theory?' (or any other non-perturbative quantum gravity). This question is in principle not unrelated to the one above. But LHC data may prove of little immediate help in this respect, in which case we must rely on the continuing obstination of (part of) the theoretical community, who find this question fascinating by itself.

- In the informal spirit of this workshop, I want to discuss some ideas that are speculative, but I believe interesting. My own published work on this is not new. I will, however, try to put it in broader context, and also to discuss some more recent results by other authors, and some of my own ongoing work.

- A way to introduce these ideas is as follows: the classical (super)gravity equations admit a set of solution-generating 'symmetries' . These include U dualities, which are relics of spontaneously-broken gauge symmetries, such as large reparametrizations of tori . They include, however, also transformations which change the physical properties of the solution. The simplest example is the transformation of scale, e.g. for 11D supergravity

$$g_{\mu\nu} \to \lambda^2 g_{\mu\nu} \ , \quad A_{\mu\nu\rho} \to \lambda^3 A_{\mu\nu\rho} \ , \quad \mathcal{L} \to \lambda^2 \mathcal{L} \ .$$

This breaks in general to a Z^+ group, because of the quantization of flux, but survives (in many cases) all other quantum corrections.

- A more intriguing example are the Ehlers-Geroch transformations of pure Einstein gravity, and its extensions to effective supergravities. Assuming e.g. the existence of a Killing vector field ξ^μ, one may define on-shell the complex scalar $z = \omega + i|\xi|^2$, where

$$\xi_\mu \xi^\mu = |\xi|^2 \ , \quad \epsilon^{\mu\nu\rho\sigma}\xi_\nu \nabla_\rho \xi_\sigma = D^\mu_{(h)}\omega \ , \quad \text{and} \quad h_{\mu\nu} = |\xi|^2 (g_{\mu\nu} - \xi_\mu \xi_\nu / |\xi|^2) \ .$$

The remaining equations are then invariant under SL(2,R) transformations of z . This symmetry of the equations becomes infinite - dimensional if there are two Killing vectors. It can be extended easily to effective string-theory actions.

Johnson+Myers, Sen, Bakas, ...

- The question that I want to ask is whether one can define such transformations <u>for the exact string equations of motion</u>. I will argue that there exist candidate generators at least in the case of open-string field theory: the conformal defects. I will speculate in the end on a possible extension to closed-string theory.

• Solutions of open-string theory can be thought of as boundary states that annihilate the (non-anomalous) super-Virasoro algebra:

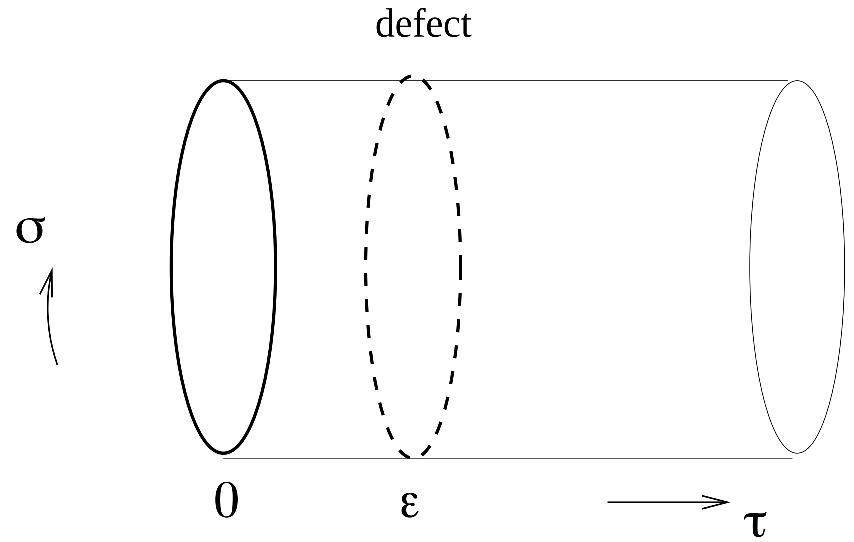
$$(L_N - \bar{L}_{-N})\,|\mathcal{B}\gg \;=\; (G_r - i\bar{G}_{-r})\,|\mathcal{B}\gg \;=\; 0 \qquad \text{for all } N, r \ .$$

In the open channel, this is the condition that no energy and supercharge flow to/from the worldsheet boundary. This must be supplemented by the condition of tadpole cancellation for certain RR fields that correspond to top-forms in the non-compact spacetime. The latter requires in general the introduction of an additional crosscap state $|\mathcal{C}\gg$. Let us for now ignore this complication, and focus on (space non-filling) D-branes, which do not couple to RR top forms.

• Let $\mathcal{O}$ be now an operator acting (formally) on the space of boundary states. For this to be a symmetry of the equations, we must demand:

$$[L_N - \bar{L}_{-N}\ ,\mathcal{O}] \;=\; [G_r - i\bar{G}_{-r}\ ,\mathcal{O}] \;=\; 0 \ .$$

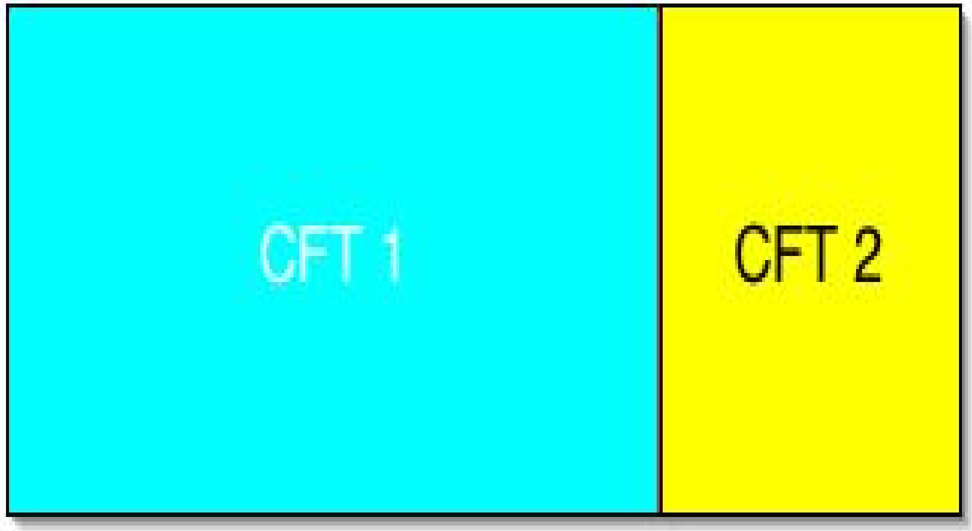
This is precisely the condition satisfied by conformal defects.



We further request that the defect operator be of the form

$$\mathcal{O} \;=\; \mathrm{tr}\, P \exp(-\int_0^{2\pi} d\sigma\, H_I) \;,$$

corresponding to an impurity which interacts locally with fields in the bulk. The trace is over the (finite-dimensional) space of states of the defect. Any given impurity Hamiltonian will flow, in principle, in the IR to a conformally-invariant defect.



Conformal defects were first considered in the condensed-matter literature, as impurities of quantum wires *(Fisher + Kane '92)* , or as lines of 'weak links' in the Ising model *(Affleck + Oshikawa '96)*. They are special cases of conformal interfaces, which obey:

$$(L_N^{(1)} - \bar{L}_{-N}^{(1)}) \times \mathcal{O} \; = \; \mathcal{O} \times (L_N^{(2)} - \bar{L}_{-N}^{(2)}) \ .$$

Their algebraic properties have been analyzed by many authors (see later).

They were introduced in string theory as the holographic duals of AdS2 branes *(CB, de Boer, Dijkgraaf + Ooguri '01; also Karch + Randall '01; CB '02).*

A special case, when CFT1 $\simeq$ CFT2, are topological defects, which obey the more stringent constraints:

$$[L_N, \mathcal{O}_{top}] \;=\; [\bar{L}_{-N}, \mathcal{O}_{top}] \;=\; 0 \qquad \text{for all} \quad N\ .$$

These can be deformed to arbitrary curves, as long as they do not encounter boundaries and/or bulk-operator insertions. To each $\mathcal{O}_{top}$ one can associate the orientation-reversed defect $\mathcal{O}'_{top}$. Furthermore topological defects have a fusion algebra *(Petkova + Zuber '00)*

$$\mathcal{O}^a_{top} \times \mathcal{O}^b_{top} = n^{ab}{}_c\, \mathcal{O}^c_{top}\ .$$

There is a subest of defects that obey $\mathcal{O}_{top} \times \mathcal{O}'_{top} = \mathbf{1}$, and which form a group of automorphisms of the operator algebra of the CFT (such as $\phi \to \phi + a$, or $\phi \to -\phi$ for a free scalar field). They generate real symmetry transformations of OSFT, such as translations and reflections of D-branes in a flat direction.

Fuchs, Fröhlich, Runkel + Schweigert '05

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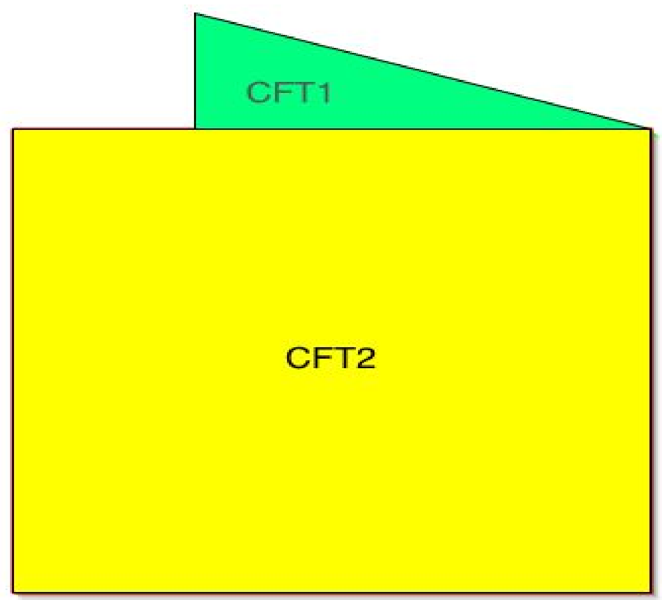
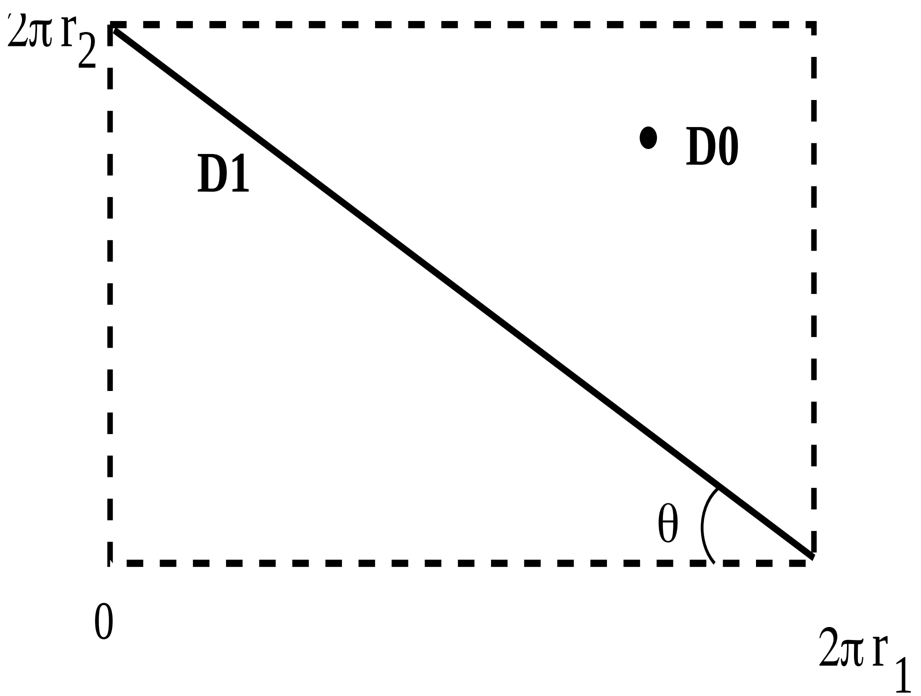
The proposal is that more general conformal defects can be used to generate Ehlers-Geroch type of transformations of open-string-field theory.

The challenge is to construct explicit examples, and to understand more generally whether/when such transformations are well-defined.

A very useful technical device for constructing conformal interfaces is the folding trick *(Affleck + Oshikawa '96)*. This maps interfaces onto boundary states of the tensor-product theory CFT1 $\otimes$ CFT2 (with left- and right-movers in CFT2 interchanged). The operation respects locality and reflection positivity (i.e. unitarity) of the underlying theories.



The trivial defect corresponds to a permutation brane, which in geometric language is the diagonally-embedded, middle-dimensional brane. This is deformed continuously when the volume (or other continuous moduli) are changed in one of the two CFTs. The deformed brane is not a topological defect, except for special values of the deformation parameter.



• For defects respecting locality and unitarity, we do not need to check the Cardy, or other consistency conditions of the transformed branes. This is not the case for the crosscap state, which obeys:

$$(L_N - (-)^N \bar{L}_{-N})|\mathcal{C} \gg = 0 \ .$$

Although topological defects respect this condition, they violate in general the requirement that in the closed channel of the Klein bottle multiplicities must be ± 1 (thus projecting out part of the closed-string spectrum).

• This additional requirement is obeyed by defects that act as Z_2 automorphisms of the operator algebra, and (possibly) by interfaces that correspond to allowed deformations of the orientifold theory. Other conformal defects, especially those carrying Chan-Paton charges, cannot be made to act on the crosscap state.

• Perturbations of the (trivial) middle-dimensional brane are described in the sigma-model approach by the following general (power-counting renormalizable) interactions:

$$\mathcal{O} \;=\; \mathrm{tr}\,\mathrm{P}\,\exp\left(i\int_0^{2\pi} d\sigma \left[A_M(X)\,\partial_\sigma X^M + \frac{1}{2\pi\alpha'}\,Y_M(X)\,\partial_\tau X^M\right]\right) \;.$$

Here the X^M are the worldsheet fields defined in a target-space $\mathcal{M}$, and A and Y are gauge and coordinate fields on the D-brane. These can be matrix-valued so as to account for Chan-Paton factors. When pulled-back to a boundary, half of these fields are (at least classically) set to zero .

• As a straightforward corollary, note that the conformal-invariance conditions for a defect line can be derived perturbatively from a (matrix-valued) Dirac-Born-Infeld action for the diagonally-embedded D-brane in $\mathcal{M}\times\mathcal{M}$.

- The discussion up to here has set the general framework, but as usual it is instructive to work out examples ! The simplest, but quite rich situation is that of free (bosonic or fermionic) world-sheet fields. Here I will consider instead the case of WZW models.

- Symmetric D-branes of WZW models are by now very well understood, both from the algebraic and the geometric viewpoint.

Cardy; Bianchi, Pradisi, Sagnotti + Stanev

Alekseev, Schomerus+Recknagel; CB, Douglas + Schweigert; ...

The RG flows between them describe the (partial) screening of magnetic impurities by the electron gas in a metal (Kondo problem).

Affleck + Ludwig '91

We will now revisit this problem, focussing on the role of the conformal defects.

♠ The existence of conformal defects in WZW models can be inferred from a semiclassical argument, similar to the one used by Witten '84 to infer conformal invariance in the bulk. One starts with the classical currents

$$J(x^+) = -i\kappa\,(\partial_+ g)g^{-1} \quad \text{and} \quad \bar{J}(x^-) = i\kappa\, g^{-1}\partial_- g\ ,$$

where $x^\pm = \tau \pm \sigma$, the $g(x^+, x^-)$ take values in a Lie group G, and $\kappa = \psi^2 k/2$ with ψ the length of long roots and k the level of the current algebra. The currents generate the left and right symmetry transformations

$$g \to u(x^+)^{-1}\, g\, \bar{u}(x^-)\ ,$$

under which they themselves transform in the same way as the components of a 2D gauge field:

$$J \to u^{-1} J u + i\kappa\, u^{-1}\partial_+ u \qquad \text{and} \qquad \bar{J} \to \bar{u}^{-1}\bar{J}\bar{u} + i\kappa\, \bar{u}^{-1}\partial_- \bar{u}\ .$$

Thus the following 'Wilson loops' [with t^a the Lie algebra generators in the representation R] will be invariant under all the symmetry transformations:

$$\mathcal{O}_{\mathsf{chir}}(\lambda; R) = \mathsf{Tr}_R \, \mathsf{P} \exp \left(i\lambda \oint_C dx^+ \, J^a t^a \right) \ ,$$

if we choose $\boxed{\lambda = \lambda^* \equiv -1/\kappa}$. Note that $(A_+, A_-) = \lambda(J^a t^a, 0)$ is a flat connection for any value of λ, so that $\mathcal{O}_{\mathsf{chir}}(\lambda; R)$ is always topological at the classical level. But this does not, a priori, survive renormalisation, which introduces (through dimensional transmutation) a length scale. For $\lambda = \lambda^*$ on the other hand, we have seen that

$$\{J^a_n \ , \ \mathcal{O}_{\mathsf{chir}}(\lambda^*; R)\} = \{\bar{J}^a_n \ , \ \mathcal{O}_{\mathsf{chir}}(\lambda^*; R)\} = 0 \ .$$

If these relations survive quantization, then the above special Wilson loops will describe topological defect lines. [In the classical theory these measure the monodromy of a solution.]

A similar semiclassical argument helps us identify also a class of conformal (but not topological) defects, by considering the more general 'Wilson lines'

$$\mathcal{O}(\lambda, \bar{\lambda}; R) = \mathrm{Tr}_R \, \mathrm{P} \exp \left(i \int_0^{2\pi} d\sigma \, (\lambda \, J^a \, - \, \bar{\lambda} \, \bar{J}^a) \, t^a \right) \, .$$

For $\boxed{\lambda = \bar{\lambda} = \lambda^*/2}$ these are invariant under (vector-like) transformations, i.e. transformations with $u(x) = \bar{u}(-x)$. It follows that

$$\left\{ J_n^a + \bar{J}_{-n}^a \, , \, \mathcal{O}\left(\frac{\lambda^*}{2}, \frac{\lambda^*}{2}; R \right) \right\} \; = \; 0 \; .$$

If these relations survive quantization, they would imply that $\mathcal{O}(\lambda^*/2, \lambda^*/2; R)$ are (G-symmetric) conformal defects. As we shall see, they correspond to unstable fixed points of the RG flow.

In order to construct the quantum defects we start with the formal expression

$$\mathcal{O}_{\mathsf{chir}}(\lambda; R) = \sum_{N=0}^{\infty} (i\lambda)^N \, \mathcal{O}^{(N)}(R) \ ,$$

where

$$\mathcal{O}^{(N)}(R) = \mathsf{Tr}_R \left(t^{a_1} \cdots t^{a_N} \right) \left(\prod_{i=1}^{N} \int_0^{2\pi} d\sigma_i \right) \theta_{\sigma_1 > \cdots > \sigma_N} \, J^{a_1}(\sigma_1) \cdots J^{a_N}(\sigma_N) \ .$$

Classically the order of the currents is irrelevant, but in the quantum theory there is an ambiguity due to the short-distance singularities of the OPE.

- To guide the choice, we insist that the following two symmetries be preserved: (i) the path can start at any point σ_0 on the circle, and (ii) the result is invariant if the loop orientation is reversed, and R is traded for its conjugate representation. These symmetries can be preserved by the following (non-unique) regularization prescription:

$$\mathcal{O}^{(N)}_{\text{reg}}(R) = \text{Tr}_R\,(t^{a_1} \quad \cdots t^{a_N}) \left(\prod_{i=1}^{N} \int_0^{2\pi} d\sigma_i \right) \theta_{\sigma_1 > \cdots > \sigma_N} \times$$
$$\times \frac{1}{2N} \left(J^{a_1}_{\text{reg}}(\sigma_1) \cdots J^{a_N}_{\text{reg}}(\sigma_N) \;+ \text{cyclic} \;+ \text{reversal} \right) \;,$$

where

$$J^a_{\text{reg}}(\sigma) = \sum_{n \in \mathbf{Z}} J^a_n \, e^{-in\sigma - |n|s/2} \quad .$$

Note (i) that since the bare currents at non-coincident points commute, the choice of ordering is part of the regularisation prescription and (ii) that the prescription guarantees that $\mathcal{O}^{(N)}_{\text{reg}}(R)$ commutes with the generator $L_0 - \bar{L}_0$. Thus, even without being topological, it can be transported to the boundary of the half-cylinder freely.

Plugging the mode expansion (with $\tilde{J}^a_n \equiv J^a_n\, e^{-|n|s/2}$) and performing explicitly the integrals leads to the following expressions for the first few values of N :

$$\mathcal{O}^{(2)}_{\rm reg}(R) = 2\pi^2\ {\rm Tr}_R(t^a t^b)\ J^a_0 J^b_0\ ,$$

$$\mathcal{O}^{(3)}_{\rm reg}(R) = \frac{2\pi^2}{3}\ {\rm Tr}_R(t^a t^b t^c) \left[\frac{\pi}{3}\ J^a_0 J^b_0 J^c_0 + \sum_{n\neq 0} \frac{i}{n} \tilde{J}^a_{-n} \tilde{J}^b_n J^c_0\ + {\rm cyclic} + {\rm reversal} \right]\ ,$$

$$\begin{aligned}\mathcal{O}^{(4)}_{\rm reg}(R) \ &= \ \frac{\pi^2}{2}\ {\rm Tr}_R(t^a t^b t^c t^d) \left[\frac{\pi^2}{6}\ J^a_0 J^b_0 J^c_0 J^d_0 + \sum_{n\neq 0} \frac{i\pi}{n} \tilde{J}^a_{-n} \tilde{J}^b_n J^c_0 J^d_0 \right. \\ &\quad + \sum_{n\neq 0} \frac{1}{n^2} \left(\tilde{J}^a_{-n} \tilde{J}^b_n J^c_0 J^d_0 - \tilde{J}^a_{-n} J^b_0 \tilde{J}^c_n J^d_0 \right) + \sum_{\substack{m,l,n\neq 0 \\ m+n+l=0}} \frac{1}{ml}\, \tilde{J}^a_m \tilde{J}^b_n \tilde{J}^c_l J^d_0 \\ &\quad \left. -\frac{1}{2} \sum_{m,n\neq 0} \frac{1}{mn} \tilde{J}^a_{-n} \tilde{J}^b_n \tilde{J}^c_{-m} \tilde{J}^d_m\ +\ {\rm cyclic} + {\rm reversal} \right] .\end{aligned}$$

After normal ordering, *i.e.* moving all positive modes to the right of negative modes, we get :

$$\mathcal{O}^{(2)}_{\rm reg}(R) = 2\pi^2 I_R\, J^a_0 J^a_0\,,$$

$$\begin{aligned}\mathcal{O}^{(3)}_{\rm reg}(R) \;&=\; \frac{2\pi^3}{3}\, I^{(3)}_R\, d^{abc}\, J^a_0 J^b_0 J^c_0 \;+\; 4\pi^2 I_R\, f^{abc} \sum_{n>0} \frac{1}{n}\, J^a_{-n} J^b_0 J^c_n \;- \\ &- 4\pi^2 i\; I_R\, h^\vee\, \psi^2 \left[\sum_{n>0} \frac{1}{n} J^a_{-n} J^a_n \;-\; \frac{1}{2}\, J^a_0 J^a_0 \left(\sum_{n>0} \frac{e^{-ns}}{n} \right) \;+\; \frac{\kappa}{6}\, \mathrm{dim}(g) \;\left(\sum_{n>0} e^{-ns} \right) \right],\end{aligned}$$

$$\begin{aligned}\mathcal{O}^{(4)}_{\rm reg}(R) \;&=\; :\mathcal{O}^{(4)}_{\rm reg}(R): -\, 2\pi^2\, I_R\, h^\vee\, \psi^2\, \kappa \left[\sum_{n>0} \frac{1}{n}\, J^a_{-n} J^a_n -\; J^a_0 J^a_0 \;\left(\sum_{n>0} \frac{e^{-ns}}{n} \right) + \right. \\ &\left. + \frac{\kappa}{4}\, \mathrm{dim}(G) \;\left(\sum_{n>0} e^{-ns} \right) \right] \;+\; \text{subleading}\,.\end{aligned}$$

where $I_R = C(R) \times \mathrm{dim}(R)/\mathrm{dim}(G)$, $\mathrm{Tr}_R\,(t^a t^b t^c) = \frac{i}{2}\, f^{abc}\, I_R + \frac{1}{2}\, d^{abc}\, I^{(3)}_R$, and h^V is the dual Coxeter number.

• We can absorb all divergences at this order with the help of the two local counterterms (a mass and coupling-constant renromalization):

$$\int_0^{2\pi} d\sigma(\Delta m + i\Delta\lambda\, J^a t^a) \ .$$

These are the only relevant operators consistent with the global G symmetry of the problem. The explicit form of these renormalizations is:

$$\Delta m = \pi\, C(R)\, h^\vee\, \psi^2 \left(\frac{1}{3}\kappa\lambda^3 + \frac{1}{4}\kappa^2\lambda^4 + \text{subleading}\right) \times \frac{1}{s} \ ,$$

$$\lambda_{\text{eff}} = \lambda + \frac{1}{2}(\lambda^2 + \kappa\lambda^3)\,\xi + \frac{1}{4}\lambda^3\xi^2 + O(\lambda^4) \ ,$$

where $\xi = h^\vee\psi^2 \log s$. Note that λ_{eff} is independent of the representation R. Note also that s is the ratio of the (only) two length scales in the problem: the short-distance cutoff and the circumference L of the cylinder.

The β-function of the chiral defect thus reads:

$$\beta(\lambda_{\text{eff}}) = -\frac{d\lambda_{\text{eff}}}{d\log s} = -\frac{1}{2}h^\vee\psi^2\left(\lambda_{\text{eff}}^2 + \kappa\lambda_{\text{eff}}^3 + O(\lambda_{\text{eff}}^4)\right) .$$

It is asymptotically free, and has an infrared fixed point at the critical value

$$\lambda^* = -\frac{1}{\kappa} + O\left(\frac{1}{\kappa^2}\right) .$$

This is perturbatively-small for large κ. If one brings this defect to a D0-brane boundary, one recovers the RG flow of the Kondo problem (derived in a non-conventional way).

The basic point, however, is that this is a universal RG flow, for any arbitrary UV Hamiltonian.

NB1: A similar universality of flows has been pointed out in the minimal models *(Graham + Watts '03)*. It should occur whenever there is a (non-invertible) topological defect obtained by flowing from a multiple of $\mathbf{1}$.

NB2: The topological defects constructed here are central elements of the envelopping current algebra. Their existence to all orders has been proved recently by *Alekseev + Monnier '07*.

NB3: The fusion of these topological defects, among themselves and with the (Cardy) boundary states, is the same as the fusion of primary fields. For instance, in the case $G = SU(2)$:

$$j \otimes j' = |j - j'| \oplus \cdots \oplus \text{max}(j + j', k - j - j') \ .$$

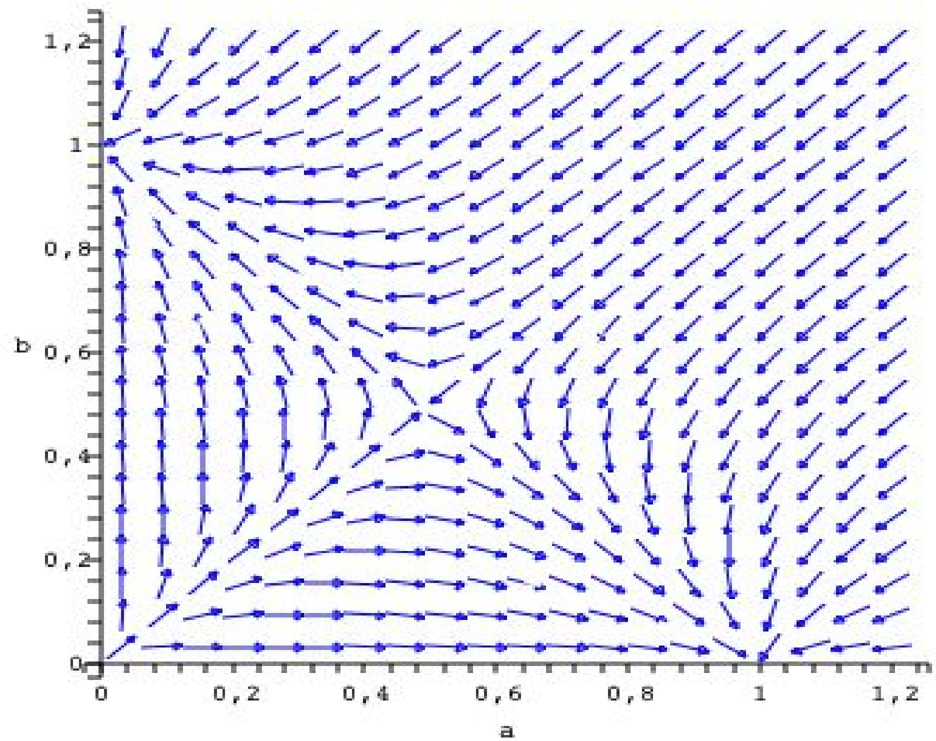
This can be verified explicitly above, and follows also from more formal arguments, e.g. from TFT in 3D *(Fröhlich, Fuchs, Runkel, Schweigert)* .

More general, non-topological defects, are neither purely reflective nor purely transmissive. Non-trivial examples have been constructed by Quella, Fredenhagen, Schomerus, Graham, Watts, and others. Their fusion is (exponentially) singular, and resembles the singular OPEs of bulk fields.

According to the semiclassicla argument, WZW models also have conformal but not topological defects. The leading-order renormalizations in the general case reads:

$$\begin{aligned}\lambda_{\mathrm{eff}} &= \lambda + \frac{1}{2}\xi\left(\lambda^2 + \kappa(\lambda^3 + \lambda\,\bar{\lambda}^2)\right) + \cdots \\ \bar{\lambda}_{\mathrm{eff}} &= \bar{\lambda} + \frac{1}{2}\xi\left(\bar{\lambda}^2 + \kappa(\bar{\lambda}^3 + \bar{\lambda}\,\lambda^2)\right) + \cdots\ .\end{aligned}$$

These equations lead to the following flow diagram, showing the existence of an unstable symmetric fixed point :



Conclusions

- Conformal defects and interfaces have the potential to unify all symmetry tranformations of the OSFT equations of motion.

- They also have a plethora of condensed-matter physics applications.

- They deserve further study.