— Pisa, 23 October 2014 —

Quantum quenches, Entanglement and the transverse field Ising chain from excited states

Leda Bucciantini

UNIVERSITÀ DI PISA

INFN Istituto Nazionale di Fisica Nucleare

mainly based on

LB, Kormos, Calabrese 1401.7250; Kormos, LB, Calabrese, 1406.5070

Outline of the talk

I. Introduction to non-equilibrium many-body quantum physics

II. State of the art: Relaxation, Light-Cone effect, Entanglement entropy

III. Extension to a quench in the transverse field Ising chain with a new ingredient: initial excited states

IV. Stationary and dynamical behaviour

V. Thermodynamic entropies of the stationary state

VI. Conclusions & Outlooks

I. Introduction

Long-Standing Questions

[Von Neumann '29; Birkhoff '30]

- Does an isolated quantum system reach a stationary state starting from an arbitrary initial state?
- If so, is there a way to economically describe the stationary state?
- How do correlation functions and observables depend on time?

What's the simplest way to drive a system out-of-equilibrium?

Sudden quantum quench $\quad H(\lambda) \xrightarrow[\text{Quench}]{t=0} H(\lambda')$

The Quench paradigm

- prepare a many-body quantum system in an eigenstate $|\psi_0\rangle$ of a pre-quenched hamiltonian H
- from $t = 0$ let it evolve unitarily with a different post-quenched time-independent hamiltonian H'

$$|\psi(t)\rangle = e^{-iH't}|\psi_0\rangle, \qquad [H, H'] \neq 0$$

Initial state is NOT an eigenstate
nor a finite superposition of eigenstates of H'

Evolution from an out-of-equilibrium state $|\psi_0\rangle$

II. State-of-the-art:

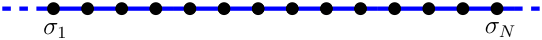
1. Relaxation

Can the whole system attain stationary behaviour?

Initial pure state + unitary evolution → it will be in a pure state $\forall t$

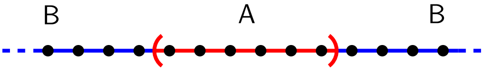
Global observables (i.e. the whole system) can never relax

As an example, a spin-chain



$\langle\psi(t)|\sigma_1 \cdots \sigma_N|\psi(t)\rangle$: persistent oscillations, quantum recurrence

What about local observables?

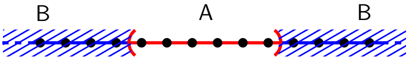


First taking B infinite, then $t \to \infty$ a finite subsystem A can relax!

Only local observables relax!

Physical picture: B acts like a “thermal” bath on A
No time averaging involved!

Density matrix:
$\rho_{\mathsf{A}\cup\mathsf{B}}(t) = e^{-iH't}|\psi_0\rangle\langle\psi_0|e^{iH't}$



Reduced Density Matrix of A:
$\rho_{\mathsf{A}}(t) \equiv \mathrm{Tr}_{\mathsf{B}}\left[\rho_{\mathsf{A}\cup\mathsf{B}}(t)\right]$

$$\lim_{t\to\infty}\lim_{N\to\infty}\rho_{\mathsf{A}}(t) = \lim_{N\to\infty}\mathrm{Tr}_{\mathsf{B}}\left[\rho^{\mathrm{mixed}}_{\mathsf{A}\cup\mathsf{B}}\right]$$

- ρ_{A} stationary and allows for an ensemble description (mixed state)
- determines all local correlation functions

Which is the statistical ensemble for $\rho^{\text{mixed}}_{\mathsf{A}\cup\mathsf{B}}$?

Non Integrable Systems

$$\rho^{\text{Gibbs}}_{\mathsf{A}\cup\mathsf{B}} = \frac{e^{-H/T_{\text{eff}}}}{Z_{\text{Gibbs}}}$$

Thermal ensemble
only one integral of motion E
few info on the whole Initial state

[Deutsch '91; Srednicki '95]

Integrable Systems

$$\rho^{\text{GGE}}_{\mathsf{A}\cup\mathsf{B}} = \frac{e^{-\sum_m \beta_m I_m}}{Z_{\text{GGE}}}$$

Non thermal ensemble
complete set of local commuting
integrals of motions I_m
$I_m = \sum_{j=1}^{N} O_{j,j+1,\cdots,j+m}$,
$\mathcal{O}(m)$-support, m finite
full info on the whole Initial state

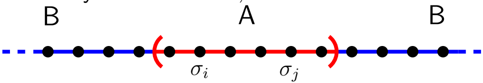
[Rigol et al '07; Eisert; Cramer...]

- based on many theoretical, experimental and numerical outcomes
  [Rigol, Muramatsu, Olshanii; Cazalilla; Calabrese, Cardy; Fioretto, Mussardo; Caux, Mossel...]
- not quite the end of the story [De Nardis et al '14, Kormos et al '14, Andrei et al '14]

Main test: exact solution of the full dynamics (free theories, TFIC, XY...)

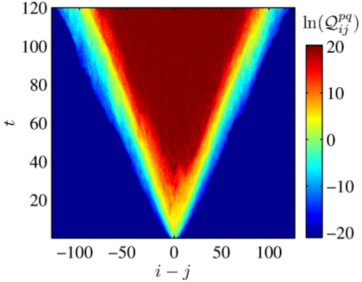
2. Light-cone spread

Do we really need $L \to \infty, t \to \infty$ to have relaxation?



Not really, as an example, the thermalization of $\langle \sigma_i \sigma_j \rangle$ occurs after $t \sim \frac{|i-j|}{2v_{max}}$.

In non relativistic quantum systems with finite-range interactions and a finite local Hilbert space: $\exists$ finite group velocity v_{max}, with exponentially small effects outside an effective light cone



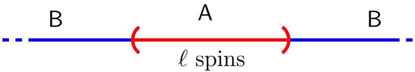
[Cheneau et al '12]

Is this a general feature? YES $\to$ Lieb-Robinson Bound! **[Lieb, Robinson '72]**

3. Entanglement entropy

A pure quantum state of a bipartite system
is not necessarily a pure state of each subsystem separately.

$$S_{\mathsf{A}} = -\mathrm{Tr}[\rho_{\mathsf{A}} \ln \rho_{\mathsf{A}}]$$



Entanglement entropy is a measure of how much a configuration of the subsystem A depends on one of B.

- product state: $|\psi\rangle = |\phi\rangle_A \otimes |\phi\rangle_B$: $S_A = 0$
- maximally entangled state: $|\psi\rangle = \frac{1}{\sqrt{D}} \sum_l |\phi_l\rangle_A \otimes |\phi_l\rangle_B$,

In an entangled state the state of A is not a vector but a density matrix.

Example: take a qubit in a singlet state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_A \otimes |\downarrow\rangle_B - |\downarrow\rangle_A \otimes |\uparrow\rangle_B), \qquad \rho_A = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

Entanglement in a quantum coherent system is responsible for appearance of entropy, hence for thermalization process!

III. What we did

So far, the focus has been put on initial states that are ground states of local hamiltonians

Objective

Study the time evolution and stationary limit of local observables after a quench [1 & 2-point functions, entanglement entropy ...]

Starting from an initial excited state

In the Transverse field Ising chain
[solvable but non-trivial as free theories]

So far, the focus has been put on initial states that are ground states of local hamiltonians

Objective

Study the time evolution and stationary limit of local observables after a quench [1 & 2-point functions, entanglement entropy ...]

Starting from an initial excited state

In the Transverse field Ising chain
[solvable but non-trivial as free theories]

So far, the focus has been put on initial states that are ground states of local hamiltonians

Objective

Study the time evolution and stationary limit of local observables after a quench [1 & 2-point functions, entanglement entropy ...]

Starting from an initial excited state

In the Transverse field Ising chain
[solvable but non-trivial as free theories]

So far, the focus has been put on initial states that are ground states of local hamiltonians

Objective

Study the time evolution and stationary limit of local observables after a quench [1 & 2-point functions, entanglement entropy ...]

Starting from an initial excited state

In the Transverse field Ising chain
[solvable but non-trivial as free theories]

So far, the focus has been put on initial states that are ground states of local hamiltonians

Objective

Study the time evolution and stationary limit of local observables after a quench [1 & 2-point functions, entanglement entropy ...]

Starting from an initial excited state
Let's discuss first this point

In the Transverse field Ising chain
[solvable but non-trivial as free theories]

Why should we focus on excited states?

- Radically different behaviour of entanglement entropy for excited states:

ground states:

  - massive non degenerate GS: $S_{GS} \simeq \partial l$ [Bombelli '88;Srendicki '93]
  - critical conformal theories: $S_{GS} \simeq \frac{c}{3} \log(l) + c'_1$ [Calabrese Cardy]

higly-excited states (# excitations $\simeq$ N)

$$S_{\text{exc}} \simeq l + \mathcal{O}(\log l)$$ [Alba, Fagotti, Calabrese, '09; Sierra, ...]

insensitive to the criticality of the ground states

- Look for universal behaviour
- Room for new effects

Quenched Transverse field Ising chain

$$H(h) = -\frac{1}{2}\sum_{j=1}^{N}[\sigma_j^x\sigma_{j+1}^x + h\sigma_j^z] \quad + \quad \text{PBC}$$

$\langle 0|\sigma_j^x|0\rangle \neq 0$ — $h_c = 1$ — $\langle 0|\sigma_j^x|0\rangle = 0$ $\longrightarrow h$

$|0\rangle$: ground state of $H(h)$

From interacting spins σ_i to free spinless fermions b_k

$$H(h) = \sum_k \epsilon_h(k)\big(b_k^\dagger b_k - \frac{1}{2}\big) \qquad \epsilon_h^2(k) = 1 + h^2 - 2h\cos\frac{2\pi k}{N}$$

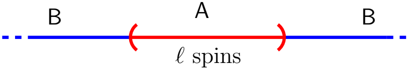
Interaction quench $h \to h'$

Initial state: $\quad |\psi_0\rangle = |m_k\rangle \equiv \prod_k (b_k^\dagger)^{m_k}|0\rangle$

- excited state of pre-quenched hamiltonian $H(h)$
- Z_2-invariant: $\langle\psi_0|\sigma_j^x|\psi_0\rangle = 0$
- m_k: fermionic initial occupation number of k-mode

IV. Stationary and dynamical behaviour

Local relaxation in the TFIC from excited states



"A" is a block of ℓ contiguous spins

$$\rho_A(t) = \mathrm{Tr}_B\big(|\psi_0(t)\rangle\langle\psi_0(t)|\big) \qquad |\psi_0(t)\rangle = e^{-iH(h')t}|\psi_0\rangle$$

Result: GGE works even for excited states!

$$\rho_{\mathrm{GGE},A} = \rho_A(\infty)$$

Idea:

Free systems → Wick's thm → just need to prove it for propagators!

- exactly solvable dynamics
- ensemble averages $\qquad \rho_{\mathrm{GGE,A}} = \dfrac{e^{-\sum_k \lambda_k n_k}}{Z}$

n_k: post-quench conserved fermionic occupation number operators

Local conserved charges from excited states

$$\langle I_n^+\rangle = \int_{-\pi}^{+\pi}\frac{dk}{4\pi}\cos(nk)\epsilon_k\left[1+m_k^S\cos\Delta_k\right] \qquad m_k^S \equiv m_{-k}+m_k-1$$

$$\langle I_n^-\rangle = -\int_{-\pi}^{+\pi}\frac{dk}{4\pi}\sin[(n+1)k]m_k^A \qquad m_k^A \equiv m_{-k}-m_k$$

Two classes of IS

- $m_k^A = 0$: Only $\langle I_n^+\rangle \neq 0$ (GS belongs to this class!)
- $m_k^A \neq 0$: Both $\langle I_n^+\rangle$ and $\langle I_n^-\rangle \neq 0$

Result: Doubling of non zero VEVs local conserved charges wrt ground state

Does it alter the asymptotic time dependence of correlations?

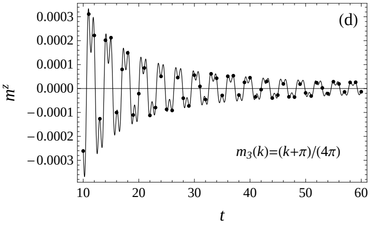
- transverse magnetization
- longitudinal two-point function

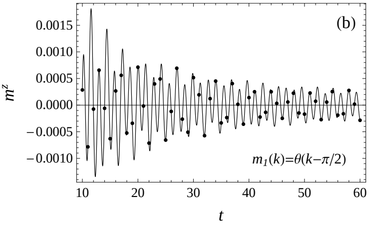
Transverse magnetization

$$m^z(t) = \underbrace{\int_{-\pi}^{\pi} \frac{dk}{4\pi} e^{i\theta_k} m_k^S \cos\Delta_k}_{\text{stationary part}} - i \underbrace{\int_{-\pi}^{\pi} \frac{dk}{4\pi} e^{i\theta_k} m_k^S \sin\Delta_k \cos(2\epsilon_k t)}_{\text{time-dependent}}$$

Asymptotic behaviour: stationary phase approximation

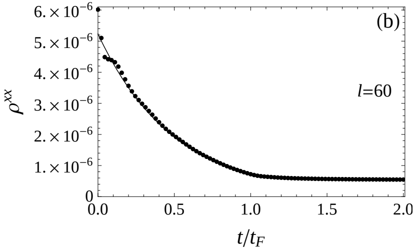
$m(k)$ analytic	$m(k)$ non-analytic
$m^z(t) \simeq t^{-\frac{3}{2}} + \mathcal{O}(t^{-\frac{2n+1}{2}})$	$m^z(t) \simeq t^{-1} + \mathcal{O}(t^{-\frac{2n+1}{2}})$
AS GROUND STATE	NOVELTY!





Longitudinal spin-spin function

$$\rho^{xx}(\ell, t) \equiv \langle \Psi_0(t) | \sigma_n^x \sigma_{\ell+n}^x | \Psi_0(t) \rangle$$



$$m(k) = \frac{k^2}{(2\pi)^2}$$

$\ell = 60$

$h = 1/3, \quad h' = 2/3$

$t_F = \ell/(2v_{\max})$

$v_{\max} = \min[h, 1]$

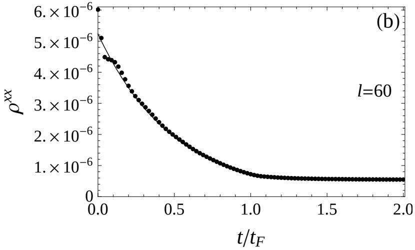
Results

- Emergent light-cone spreading of correlations (as for GS)
- Common behaviour $\forall m_k$ analyzed (double-stepfunction, linear, quadratic)...

...EXCEPT ONE!

Longitudinal spin-spin function

$$\rho^{xx}(\ell, t) \equiv \langle \Psi_0(t) | \sigma_n^x \sigma_{\ell+n}^x | \Psi_0(t) \rangle$$



$$m(k) = \frac{k^2}{(2\pi)^2}$$
$$\ell = 60$$
$$h = 1/3, \quad h' = 2/3$$
$$t_F = \ell/(2v_{\max})$$
$$v_{\max} = \min[h, 1]$$

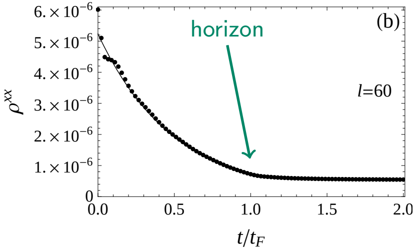
Results

- Emergent light-cone spreading of correlations (as for GS)
- Common behaviour $\forall m_k$ analyzed (double-stepfunction, linear, quadratic)...

...EXCEPT ONE!

Longitudinal spin-spin function

$$\rho^{xx}(\ell, t) \equiv \langle \Psi_0(t) | \sigma^x_n \sigma^x_{\ell+n} | \Psi_0(t) \rangle$$



$$m(k) = \frac{k^2}{(2\pi)^2}$$

$\ell = 60$

$h = 1/3, \quad h' = 2/3$

$t_F = \ell/(2v_{\max})$

$v_{\max} = \min[h, 1]$

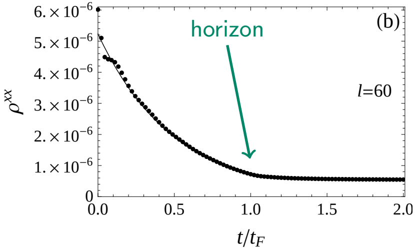
Results

- Emergent light-cone spreading of correlations (as for GS)
- Common behaviour $\forall m_k$ analyzed (double-stepfunction, linear, quadratic)...

...EXCEPT ONE!

Longitudinal spin-spin function

$$\rho^{xx}(\ell, t) \equiv \langle \Psi_0(t) | \sigma_n^x \sigma_{\ell+n}^x | \Psi_0(t) \rangle$$



$$m(k) = \frac{k^2}{(2\pi)^2}$$

$\ell = 60$

$h = 1/3, \quad h' = 2/3$

$t_F = \ell/(2v_{\max})$

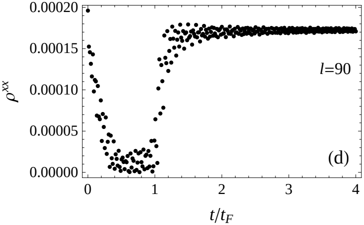
$v_{\max} = \min[h, 1]$

Results

- Emergent light-cone spreading of correlations (as for GS)
- Common behaviour $\forall m_k$ analyzed (double-stepfunction, linear, quadratic)...

...EXCEPT ONE!

The anomalous state: $m_k = \theta(k - \frac{\pi}{2})$



Still open problems

- Is it related to $\boxed{\langle I_n^- \rangle \neq 0}$?
- But other $m_k^A \neq 0$ display usual light-cone effect...

V. Thermodynamic entropies of the stationary state

The system will always be globally in a zero entropy state.

Can we define the entropy for the stationary state reached a quantum quench?

Diagonal ensemble

$$\rho_{\mathsf{D}} = \sum_j |\langle j|\psi_0\rangle|^2 |j\rangle\langle j|$$

[Polkovnikov '11]

- captures the long time averaged expectation value of all observables

but

- knows everything of the initial state!

Subsystem's stationary ensemble

$$\rho_{\mathsf{GGE}} = \frac{e^{-\sum_k \lambda_k I_k}}{Z}$$

- represents the long-time limit of only local observables
- its entropy coincides with the stationary value of the entanglement entropy

Diagonal entropy

$$S_{\mathsf{D}} = -\mathsf{Tr}[\rho_{\mathsf{D}} \ln \rho_{\mathsf{D}}]$$

GGE entropy

$$S_{\mathsf{GGE}} = -\mathsf{Tr}[\rho_{\mathsf{GGE}} \ln \rho_{\mathsf{GGE}}]$$

What is the relation between these entropies?

Initial Ground states

$$S_{\mathsf{GGE}} = 2S_{\mathsf{D}}$$

- verified in some integrable systems **[Gurarie '13, Calabrese '14]**
- consequence of the inequivalence of ρ_{GGE} and ρ_{D}

Initial Excited states

Many different microstates sharing the same macroscopical distribution of excitations in the thermodynamic limit

$$\left.\begin{array}{c} m_{1\,k} \\ m_{2\,k} \\ \cdots \\ \cdots \end{array}\right\} \xrightarrow[N \to \infty]{} m(p)$$

In the $N \to \infty$, averages over microstates need to be introduced.

What are the consequences on S_{GGE} and S_{D}?

Finite Systems: $m_k = \{0, 1\}$

$$S_{\mathsf{D}} = \sum_{k>0} \left[m_k m_{-k} + (1 - m_k)(1 - m_{-k}) \right] s_k$$

$$S_{\mathsf{GGE}} = \sum_{k} \left[m_k m_{-k} + (1 - m_k)(1 - m_{-k}) \right] s_k$$

only the modes with $m_k = m_{-k}$ contribute to the entropy!

$S_{\mathsf{GGE}} = 2S_{\mathsf{D}}$ even for excited states!

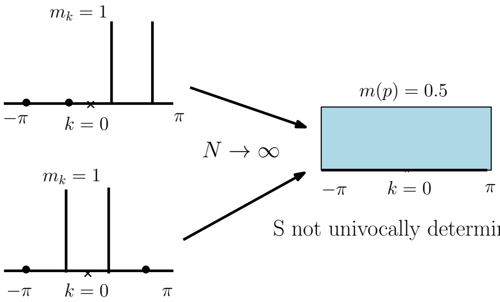
From N finite to the thermodynamic limit

$m_k \neq m_{-k} \quad \forall k$

$S_{GGE} = 0$

$m_k = m_{-k} \quad \forall k$

$S_{GGE} = \sum_k s_k$

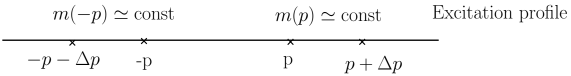


S not univocally determined by m(p)!

The way we take the average matters!

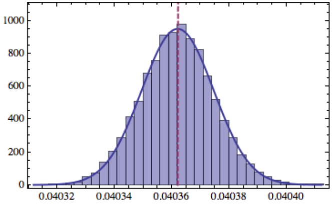
$$S^{TD} = \overline{-\mathrm{Tr}\rho \ln \rho} \qquad S^{ST} = -\mathrm{Tr}\overline{\rho} \ln \overline{\rho}$$

S^{TD}



$$S(\{m_k\}) \to \overline{S}, \qquad \frac{\sigma}{\overline{S}} \to \simeq \frac{1}{\sqrt{N}}$$

$$\frac{S^{TD}_{\mathsf{GGE}}}{N} = -\int_{-\pi}^{\pi} \frac{dp}{2\pi} \{m(p)m(-p) + [1 - m(p)]\,[1 - m(-p)]\,\} s(p)$$



- 10^4 randomly generated $\{m_k\}$ states satisfying $m(p) = e^{-p/2}$
- $h_0 = 7$, $h = 2$
- Sharply peaked around $\overline{S}$!

S^{ST}

$$S^{ST}_{\mathsf{GGE}} = N \int_{-\pi}^{\pi} \frac{dp}{2\pi} H[m(-p) - m(p) + (m(p) + m(-p) - 1) \cos \Delta(p)]$$

agrees with the stationary limit of entanglement entropy!

V. Conclusions and Outlook

We have considered quenches from excited states

Validity of GGE

Horizon effect for multipoint correlation functions

Still open problems

Non-trivial dependence for m_k^A?

Excitations in truly interacting models?

Thank you for your attention