On cascading bubbles in Guinness beer

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Physics Department

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Presentation plan

1. Half pint of history
2. Are those bubbles really sinking?
3. What about that texture?
4. The last round

Presentation plan

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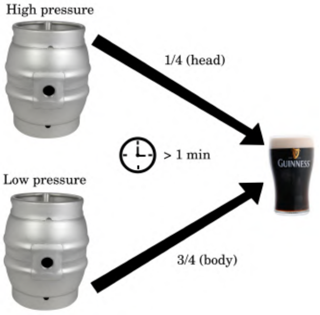
- **1901:** The first **Guinness research laboratory** is established under Oxford-educated chemist, Alexander Forbes-Watson.

The "draught problem"



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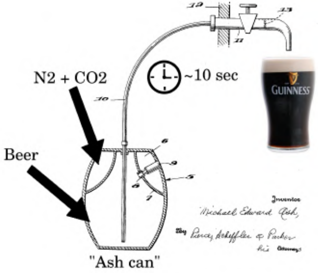

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In **1951**, the mathematician **Michael Edward Ash** joins the research laboratory.
In **1959** (200th anniversary), he completes the modern Guinness draught system, using a $CO_2 + N_2$ **solution** (instead of just CO_2).

A well-deserved pint

Presentation plan

What everybody asks:

Are those bubbles really sinking?

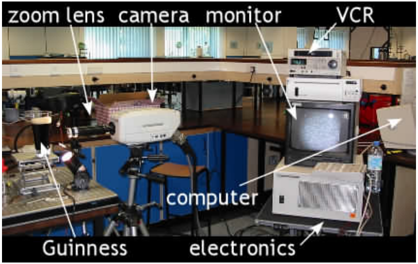
It is just an optical illusion?

If they really go down, why???

Did the Irish find a way to overcome Archimedes' law?

Is this the real life? Is this just fantasy?

In **2004**, Alexander and Zare[1] shoot the **first video proof**: bubbles really go down in Guinness, though only near the glass wall. Experimental set-up consists of an **high-speed digital camera** (up to 4500 FPS), data-acquiring hardware, and Guinness beer.



[1] https://web.stanford.edu/group/Zarelab/guinness/index.html

Caught in a *beer*slide, no escape from reality!

Bond number

$$B_0 = \frac{\rho_l g}{\sigma / d_b^2} \approx 0.002 \ll 1$$

Stokes' law

$$u_b = \frac{(\rho_l - \rho_b) g d_b^2}{18 \mu_l} \approx 3.96 mm/s$$

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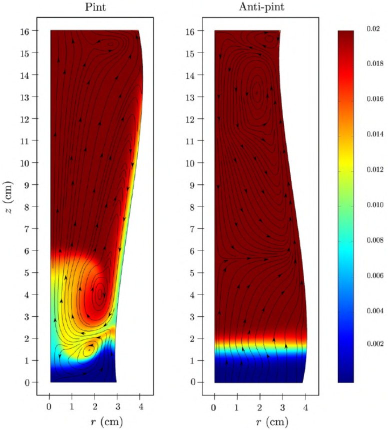

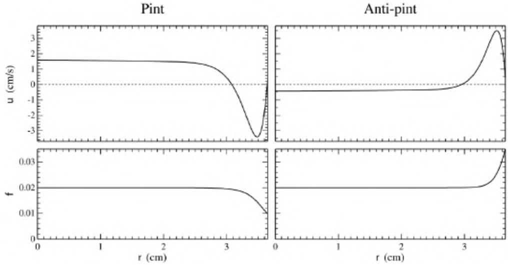

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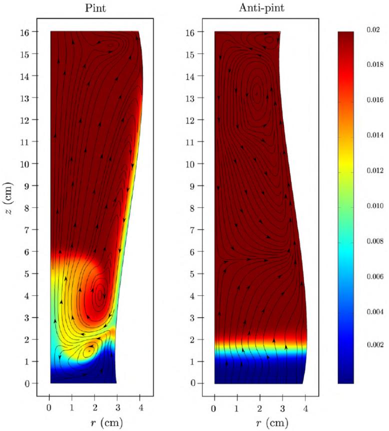

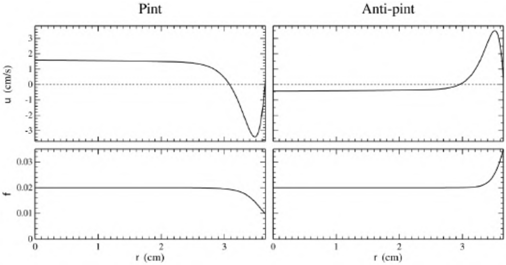

Near the walls, the **beer slides down faster than the bubbles rise** inside it!

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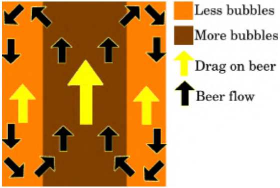
The simple mechanism: the Boycott effect[4]

Near wall variation of **bubble density** $\Longrightarrow$ Circulating **current**



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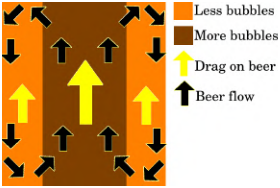

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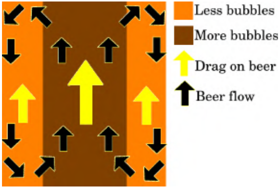

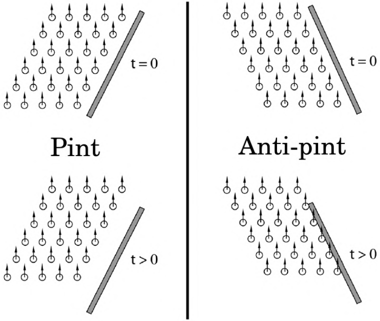

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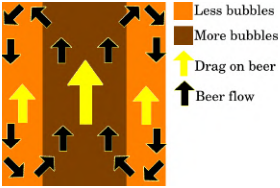

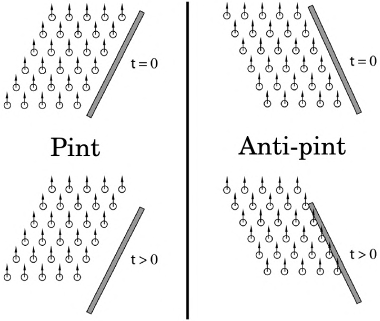

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Presentation plan

What someone asks:

Continuous two-phase analytical model[5]

Mass conservation and **momentum equation**:

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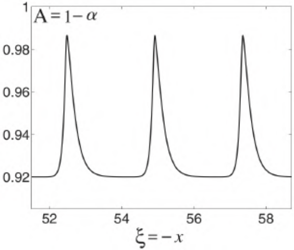
Model predictions

Numerical solution:



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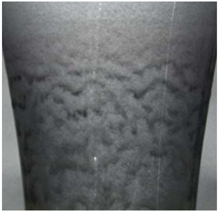

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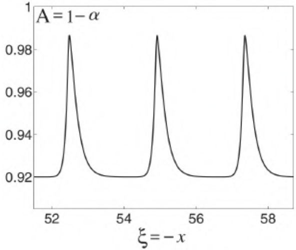

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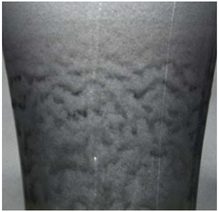

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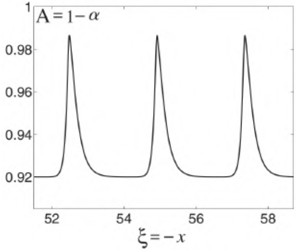

$$c_w \approx 21mm/s$$

$$0.92 \leq A \leq 0.98$$

$$\lambda \approx (0.082 - 0.095)mm$$

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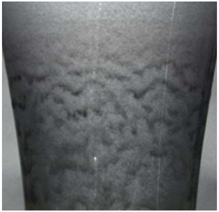

$$c_w \approx (20.9 - 21.8)mm/s$$

$$0.75 \leq A \leq 1.00$$

$$\lambda \approx 6mm$$

Qualitatively predictive, "extreme" value for A not reached, problem for λ selection.

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Well-controlled experiments[7]

Guinness: $d_b \approx 61\mu m;\ \alpha = 8\%$

Pseudo-Guinness: $d_b \approx 47\mu m;\ \alpha = (0.5-10)\%$

(tap water + glass hollow sphere)

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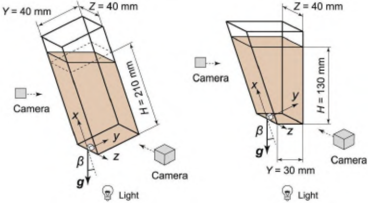

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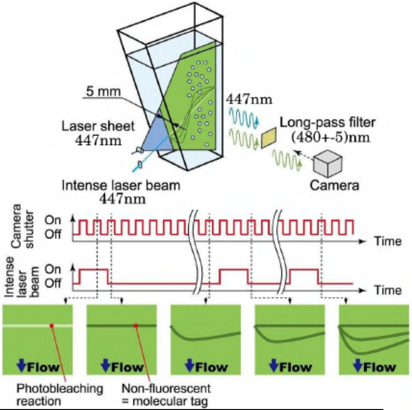

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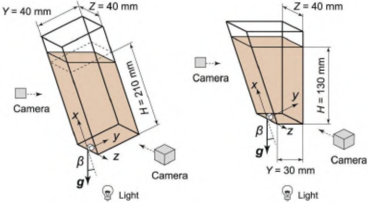

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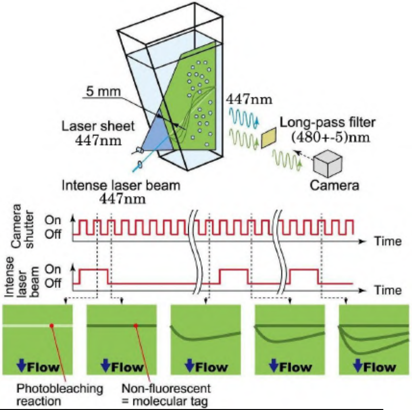

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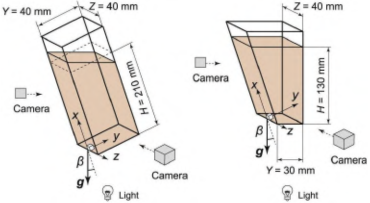
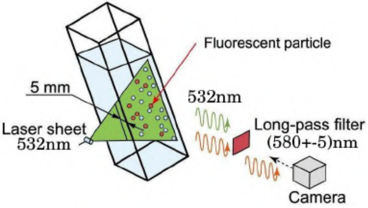


Particle Image Velocimetry:



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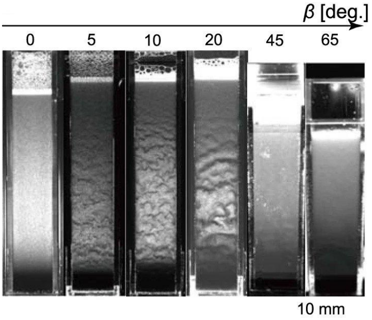
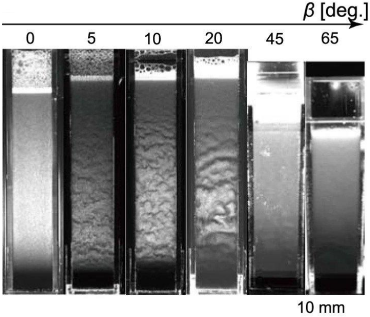


Texture for $\beta \geq 45$ may be absent due to the **finite length** of the container (the texture develops spatially).

Angle dependency



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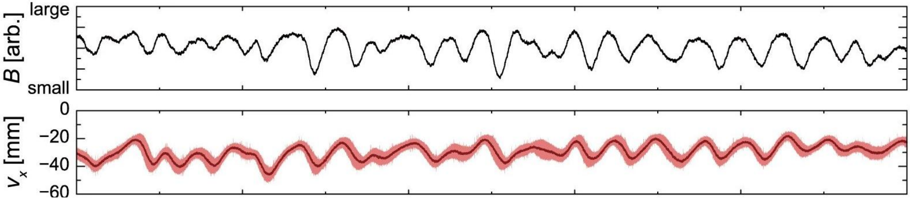
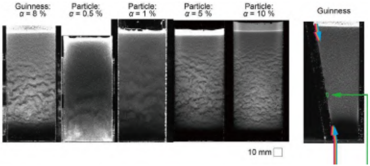


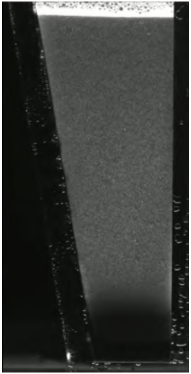
$\mathrm{Corr}(B, -v_x) \approx 0.8 \Rightarrow$

$\Rightarrow$ Bubbles move down quicker in low-α regions, and vice-versa $\Rightarrow$

$\Rightarrow$ **Fluid "blobs" cascade in the bubble-rich bulk**

Concentration dependency and the clear-fluid region





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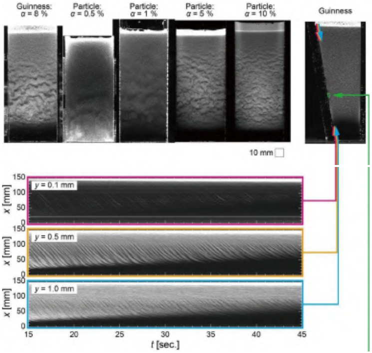

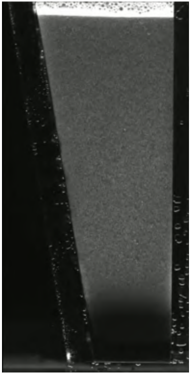

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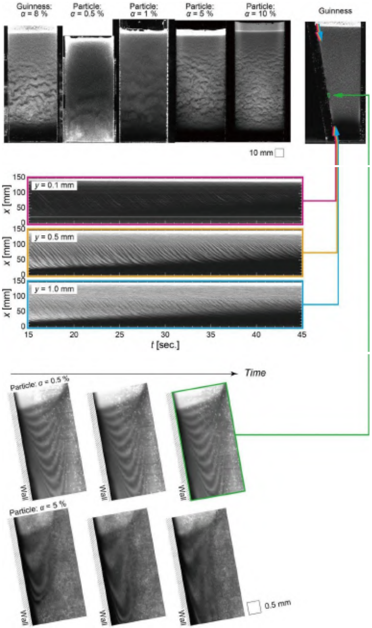
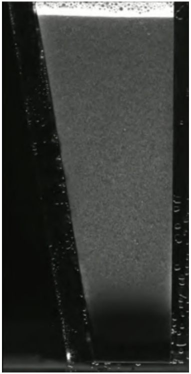
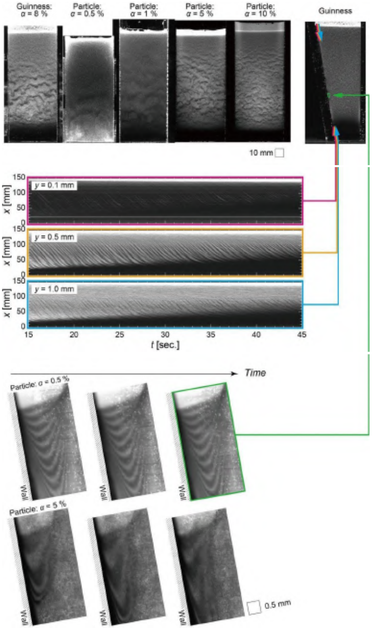
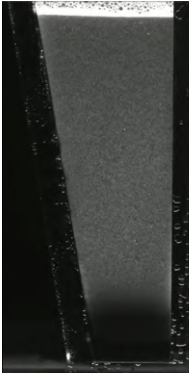
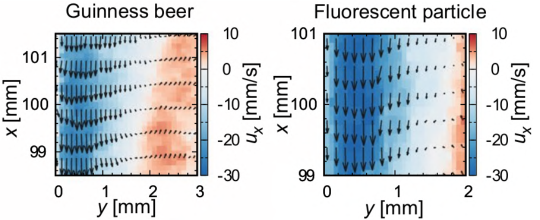


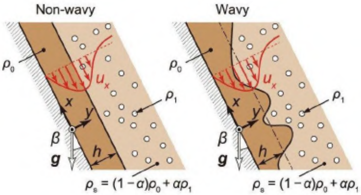



Guinness: σ = 8 %
Particle: σ = 0.5 %
Particle: σ = 1 %
Particle: σ = 5 %
Particle: σ = 10 %
Guinness
10 mm
y = 0.1 mm
y = 0.5 mm
y = 1.0 mm
x [mm]
150
100
50
0
15
20
25
30
35
40
45
t [sec.]
Time
Particle: σ = 0.5 %
Particle: σ = 5 %
Wall
0.5 mm

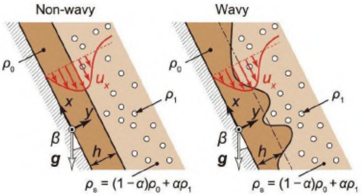


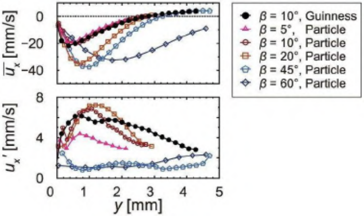

Guinness beer
Fluorescent particle
x [mm]
101
100
99
y [mm]
0
1
2
3
u_x [mm/s]
10
0
-10
-20
-30

Critical condition: the nature of the instability



[8] Needham and Merkin, Proc. R. Soc. Lond. A **394**,259-278 (1984).

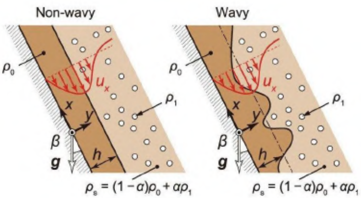


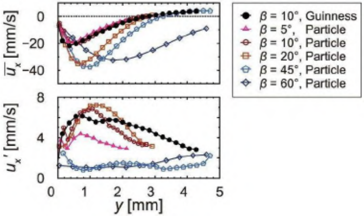


Texture formation is reflected in **velocity fluctuations**:

$$\frac{u'_{x,max}}{|\bar{u}_x|_{max}}$$

[8]Needham and Merkin, Proc. R. Soc. Lond. A **394**,259-278 (1984).

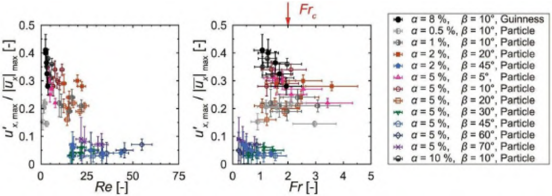




Texture formation is reflected in **velocity fluctuations**:

$$\frac{u'_{x,max}}{|\bar{u}_x|_{max}}$$

$$R_e = \frac{(1-\alpha)\rho_0 h |\bar{u}_x|_{max}}{(1+\frac{5}{2}\alpha)\mu}; \quad F_r = \frac{2|\bar{u}_x|_{max}}{3\sqrt{\frac{\rho_0-\rho_1}{\rho_0}\alpha h g \sin(\beta)}}$$



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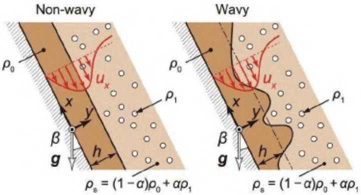
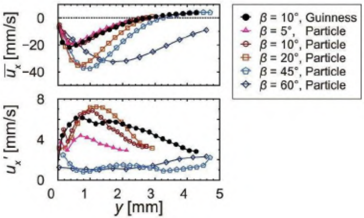
Critical condition: the nature of the instability





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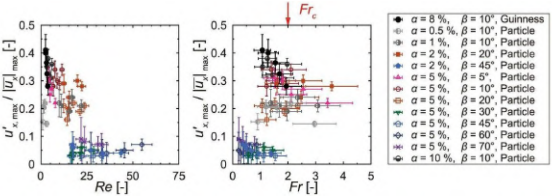

It is a **gravity current instability** (roll waves), not a shear instability (Kelvin-Helmholtz). Observed F_r threshold is lower than the theoretical one[8] due to model over-simplifications.

[8] Needham and Merkin, Proc. R. Soc. Lond. A **394**,259-278 (1984).

Presentation plan

Summary

- The nitrogen bubbles **really go downward**.
- The **shape** of the container determines the motion of the bubble via the **Boycott effect**.
- **Analytical models** can be built to reproduce qualitatively (and partially quantitatively) the observed texture.
- Both models and observations suggest **analogies with the roll waves**.
- Recent experiments reveal an almost **bubble-free film near the container wall**: there is a gravity current.
- The wavy texture originates from **gravity current instability**, rather than shear ones, like the roll waves.

Summary

- The nitrogen bubbles **really go downward**.
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- **Analytical models** can be built to reproduce qualitatively (and partially quantitatively) the observed texture.
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- Recent experiments reveal an almost **bubble-free film near the container wall**: there is a gravity current.
- The wavy texture originates from **gravity current instability**, rather than shear ones, like the roll waves.

- Experiments with $34\mu m$ and $75\mu m$ diameter bubbles still show the wavy cascade, but larger CO_2-like bubbles ($(300-500)\mu m$) show none of these features.
- Studies on two-phase fluids have important applications in industry (heat exchangers, reactor cooling, oil extraction, food processing, ...) and medicine.

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University of Pisa
Physics Department
Room 47

Eötvös and Bond numbers

Eötvös number: an adimensional number measuring the importance of buoyancy compared to surface tension (two phases involved).

$$E_o = \frac{\Delta\rho g L^2}{\sigma}$$

Bond number: an adimensional number measuring the importance of gravity compared to surface tension (one phase involved).

$$B_o = \frac{\rho g L^2}{\sigma}$$

Reynolds and Froude numbers

From **Navier-Stokes** equation:

$$\rho(\partial_t\vec{v} + \underbrace{\vec{v}\cdot\vec{\nabla}\vec{v}}_{inertia}) = -\vec{\nabla}P + \underbrace{\mu\nabla^2\vec{v}}_{viscosity} + \underbrace{\rho\vec{g}}_{gravity}$$

Scaling via $x = x'L$, $\vec{v} = \vec{v}'U$, $g = g'g_0$, $P = P'\rho U^2$, $\partial_t = \partial_{t'}U/L$, $\vec{\nabla} = \vec{\nabla}'/L$:

$$\partial_{t'}\vec{v}' + \vec{v}'\cdot\vec{\nabla}'\vec{v}' = -\vec{\nabla}'P' + \underbrace{\frac{\mu/\rho}{LU}}_{1/R_e}\nabla'^2\vec{v}' + \left(\underbrace{\frac{\sqrt{Lg_0}}{U}}_{1/F_r}\right)^2\vec{g}'$$

Reynolds number: importance of inertia over viscosity.

$$R_e = \frac{LU}{\mu/\rho}$$

Froude number: importance of inertia over gravity (or other external fields).

$$F_r = \frac{U}{\sqrt{Lg_0}}$$

Let us consider a sphere of radius a in a steady, viscous ($R_e \ll 1$), and incompressible flow, with asymptotic velocity U_0 along the polar axis: $0 = -\vec{\nabla} P + \mu \nabla^2 \vec{v}; \quad \vec{\nabla} \cdot \vec{v} = 0$
Thanks to axisymmetry, we can define the **Stokes stream function**[9] ψ in spherical coordinates as:

$$v_r = \frac{1}{r^2 \sin\theta} \partial_\theta \psi$$
$$v_\theta = \frac{1}{r \sin\theta} \partial_r \psi$$

By substitution in the N.-S. system one get

$$\psi = \frac{U_0}{2} r^2 \sin^2\theta \left(1 - \frac{3a}{2r} + \frac{a^3}{2r^3}\right)$$

$$v_r = U_0 \cos\theta \left(1 - \frac{3a}{2r} + \frac{a^3}{2r^3}\right); \; v_\theta = -U_0 \sin\theta \left(1 - \frac{3a}{4r} + \frac{a^3}{4r^3}\right)$$

$$P = -\frac{3a\mu U_0 \cos\theta}{2r^2} \text{ (by integration from the N.-S. radial eq.)}$$

[9] In the same spirit of the complex stream functions for 2D flows.

The drag along the polar axis caused by P is:

$$F_P = 2\pi a^2 \int_0^\pi P(a,\theta)\sin\theta\cos\theta d\theta = 2\pi a\mu U_0$$

The shear stress is given by $\tau_{r\theta} = -\mu[r\partial_r(v_\theta/r) + (\partial_\theta v_r)/r]$, from which the shear force is:

$$F_s = 2\pi a^2 \int_0^\pi \tau_{r\theta}(a,\theta)\sin^2\theta d\theta = 4\pi a\mu U_0$$

The total drag force is known as **Stokes' law**:

$$F_d = 6\pi a\mu U_0$$

So, the **terminal velocity** of a free sphere in a fluid is given by the condition $F_d = \Delta\rho\frac{4}{3}\pi a^3 g$, which gives:

$$v_S = \frac{2\Delta\rho g a^2}{9\mu}$$

Guinness and pseudo-Guinness parameters

	$T[C]$	$\rho_l[Kg/m^3]$	$\rho_g[Kg/m^3]$	$\mu_l[P_a s]$
Guinness$^\alpha$	5	1007	1.223	2×10^{-3}
Guinness$^\beta$	6	1007	1.223	2.06×10^{-3}
Guinness$^\gamma$	5	1006	1.223	$2.03 \times 10^{-3} - 2.21 \times 10^{-3}$
Pseudo-Guinness$^\gamma$	25	997	140	0.89×10^{-3}
	$\alpha[\%]$	$d_b[\mu m]$	$v_S[mm/s]$	
Guinness$^\alpha$	$23.3 - 26.6$	$94 - 122$	$2.40 - 4.06$	
Guinness$^\beta$	$2, 5, 10$	$90, 122^\alpha$	$2.14 - 3.94$	
Guinness$^\gamma$	8	$54 - 68$	$0.716 - 1.15$	
Pseudo-Guinness$^\gamma$	$0.5 - 10$	47	1.2	

$^\alpha$Robinson *et al.*, Phys. of Fluids **20**, 067101 (2008).
$^\beta$Belinov *et al.*, Am. J. of Phys. **81**, 88 (2013).
$^\gamma$Watamura *et al.*, Sci. Rep. **9**, 5718 (2019).

Some simulation details

Simplifications:

- all bubbles have the same size;
- the system is axisymmetric;
- liquid and gas phases are interpenetrating continua;
- $\rho_g \ll \rho_l$
- gas velocity relative to the fluid is given by the balance of pressure and viscous drag forces;
- $P_g = P_l$.

$\Rightarrow$ Bubbles have no inertia, no need for bubbles initial velocity, no need for bubbles momentum equation.

Border conditions are: null velocity for the fluid (no-slip on the wall), null normal velocity for the gas (no flux across the wall).

Model details

In Watamura *et al.*[10], they assume

$$C_{VM} = 1/2;\ H = 1/4,$$

values appropriate for isolated spherical bubbles[11]. Also $F_{lw} = 0$ (waves in a quiescent medium). Instead, for the interphases interaction:

$$F_{gi} = \frac{\mu_l D(\alpha)(v-u)}{a^2},$$

corresponding to laminar drag on a sphere. For $\alpha \ll 1$ we have Stokes drag: $D(\alpha) \approx 3\alpha$. For high α we have $D(\alpha) = 180\alpha^2/(1-\alpha)$, corresponding to the Carman-Kozeny law[12].

[10] Watamura *et al.*, Sci. Rep. **9**, 5718 (2019).
[11] Pauchon and Banerjee, Int. J. Multiphase Flow **14**, 253 (1988).
[12] Fowler, "Mathematical models in the applied Sciences", Cambridge University Press, Cambridge (1997).

The **base model** (without D_l, C_{VM}, pressure jump, eddy viscosity) is **ill-posed** as a Cauchy problem: the characteristics are complex. Some physics-based therms must be added.
The average nonlinear effects due to a non-uniform sectional profile can be added via the **profile coefficient** in the acceleration term: $u_t + D_l u u_x$. This makes possible to have real characteristics for $\rho_g \ll \rho_l$.
When a gas element accelerates in the liquid, it have to push away some of it. This action-reaction effect may be considered via addition of **virtual mass**: $\pm C_{VM} \alpha \rho_l (v_t + v v_x - u_t - u u_x)$.
The relative motion of the phases should also produce a **pressure jump** at the interface, which can be modeled as[13,14]: $\Delta P_i = -H \rho_l (v-u)^2$.

All these terms just post-pone the problem until a critical value of α (≈ 0.25). At this, point the ill-posedness is manifested as a **physical instability**, leading non-linearly to shock formation.

[13] Batchelor, J. Fluid Mech. **193**, 75 (1988).
[14] Stuhmiller, Int. J. Multiphase Flow **3**, 511 (1977).

Derivation of the model (2/2)

To regularize shocks, **diffusive terms** are needed. In turbulent flow, energy transport at small scales can be modeled as an effective viscosity, the **eddy viscosity**:

$$[\eta(1-\alpha)u_x]_x.$$

In laminar flow, eddy viscosity is hard to justify, but we can consider the dissipation due to bubbles deformation as another effective viscosity, a **bulk viscosity**. These deformations are caused by the interfacial pressure difference:

$$\Delta P_{gl} = \eta(\alpha_t + u_i \alpha_x),$$

where $\eta = 4\mu/(3\alpha)$, and $u_i = u$ is the average interfacial velocity (ΔP_{gl} deforms bubbles, i.e. it changes α). Using the conservation $-\alpha_t + [(1-\alpha)u]_x = 0$, we get:

$$\Delta P_{gl} = \eta(1-\alpha)u_x.$$

Thus, the added force is $\partial_x \Delta P_{gl}$, that has exactly the same form of the eddy viscosity.

Stability of the model (1/2)

Starting from the **basic model with** the only addition of D_l and **bulk/eddy viscosity**, eliminating P from the moment equations we get:

$$-\alpha_t + [(1-\alpha)u]_x = 0; \; u_t + D_l u u_x = \nu u_{xx} + \underbrace{\frac{1}{\rho_l}\left[\frac{F_{gi}}{\alpha(1-\alpha)} - \frac{F_{lw}}{1-\alpha}\right]}_{R(\alpha,u)}$$

From a steady solution (α^*, u^*), we **add a perturbation** $\sim e^{ikx+\sigma u}$, and by substitution $(d = D_l - 1)$:

$$\sigma = -iku^* \left(1 + \frac{1}{2}d\right) - \frac{1}{2}(\nu k^2 + |R_u|) \left\{ 1 \pm \underbrace{\left[\left(1 + \frac{ikdu^*}{|R_u| + \nu k^2}\right)^2 \frac{4ik(1-\alpha^*)R_\alpha}{(|R_u| + \nu k^2)^2}\right]^{1/2}}_{p+iq} \right\}$$

There is instability if $\Re(\sigma) > 0$, which in this case means $p > 1 \Rightarrow$

Stability of the model (2/2)

$\Rightarrow$ The **instability condition** reads:

$$\frac{(1-\alpha^*)|R_\alpha|}{|R_u| + \nu k^2} > du^*$$

We can say that:

- $d = 0 \Rightarrow$ always unstable;
- $\nu = 0 \Rightarrow$ for each k, there is a critical value for α^* (just D_l postpone the instability);
- $d, \nu > 0 \Rightarrow$ it is more difficult for high k modes to become unstable.

For $k \to +\infty$ we can expand σ:

$$\sigma \approx -\frac{bdu^+}{\nu^2 k^2} + \frac{b^2}{\nu^3 k^4};\ a = \nu + \frac{idu^*}{k} + \frac{|R_u|}{k^2};\ b = (1-\alpha^*)|R_\alpha|$$

So, in the end:

- the effective viscosity regularize the model, i.e. the growth rate goes to zero as $k \to +\infty$;
- $d > 0$ (and similar corrections) are still necessary to damp high k modes.