Phase space techniques: the Wigner function and its applications

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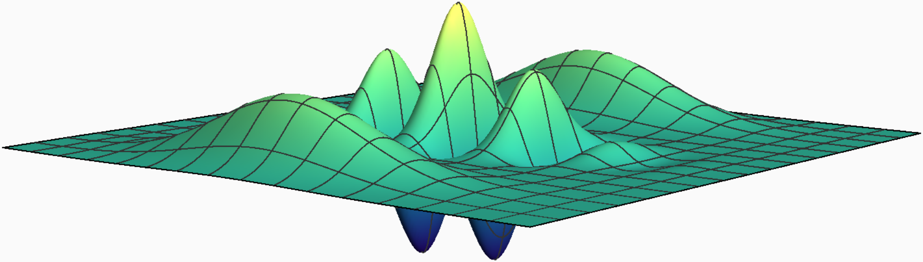

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Phase space in Quantum Mechanics: the Wigner function

A jump start of the Wigner function

The quantum transition of a particle from x' to x'' is described by the density matrix element $\langle x' \mid \hat{\rho} \mid x'' \rangle$, or equivalently by the *Wigner function*:

Definition of Wigner function

$$W(x, p) = \int_{-\infty}^{+\infty} \frac{\mathrm{d}\xi}{2\pi} e^{-ip\xi} \left\langle x + \frac{\xi}{2} \middle| \hat{\rho} \middle| x - \frac{\xi}{2} \right\rangle .$$

The variables $x \equiv (x' + x'')/2$ and p (conjugate to $\xi \equiv (x'' - x')/2$) span the phase space in which the $W(x, p)$ lives.

Wigner function of a pure state

$$W(x, p) = \int_{-\infty}^{+\infty} \frac{\mathrm{d}\xi}{2\pi} e^{-ip\xi} \psi^*(x - \xi/2)\psi(x + \xi/2) .$$

(Some) Properties of the Wigner function

Marginals

Marginalizing in x and p respectively provides momentum and position distributions:

$$W(x) \equiv \int \mathrm{d}x\, W(x,p) = \langle x \mid \hat{\rho} \mid x \rangle\,, \quad W(p) \equiv \int \mathrm{d}p\, W(x,p) = \langle p \mid \hat{\rho} \mid p \rangle\,.$$

Trace-product rule

$$\mathrm{Tr}(\hat{\rho}_1 \hat{\rho}_2) = 2\pi \int \mathrm{d}x\, \mathrm{d}p\, W_{\hat{\rho}_1}(x,p) W_{\hat{\rho}_2}(x,p)\,.$$

Two important corollaries of the trace-product rule:

- Cannot squeeze a state to a phase space domain smaller than $2\pi\hbar$:

$$2\pi\hbar \leq \frac{1}{\int \mathrm{d}x\, \mathrm{d}p\, W^2(x,p)}\,.$$

- The Wigner function is real and can take on negative values.

A (quasi)probability distribution

The Wigner function efficiently bridges QM and Statistical Physics. It highlights striking similarities...

Averages

In QM, $\langle A \rangle \equiv \mathrm{Tr}(\hat{A}\hat{\rho})$. Wigner transforming $\hat{A} \to A(x,p)$ and using the trace product rule, W plays the role of a probability distribution:

$$\left\langle \hat{A} \right\rangle = \int \mathrm{d}x\, \mathrm{d}p\, A(x,p) W(x,p)\,.$$

Time evolution

The operator equation $\mathrm{d}\hat{\rho}/\,\mathrm{d}t = -i[\hat{H},\hat{\rho}]$, with $\hat{H} = \hat{p}^2/(2m) + U(\hat{x})$, becomes a c-number equation for $W(x,p)$:

$$L \cdot W(x,p,t) = \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell}(\hbar/2)^{2\ell+1}}{(2\ell+1)!} \frac{\mathrm{d}^{2\ell+1}}{\mathrm{d}x^{2\ell+1}} U(x) \frac{\partial^{2\ell+1}}{\partial p^{2\ell+1}} W(x,p,t)\,,$$

where L is the Liouville operator associated to $\hat{H}$.

A (quasi)probability distribution

The Wigner function efficiently bridges QM and Statistical Physics. It highlights striking similarities but also important differences:

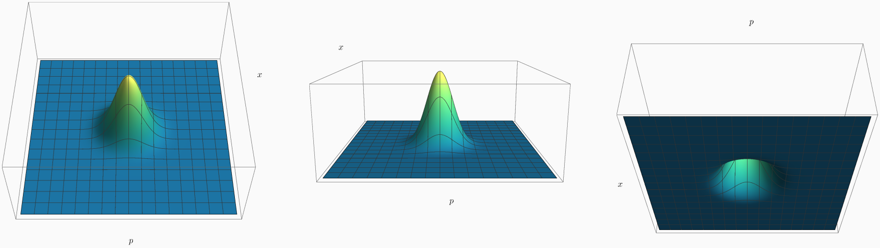
- Quantum interference effects can make W negative.
- W does not satisfy the Lioville equation $LW(x, p) = 0$ because of quantum corrections.

These non-classicalities are absent in important situations:

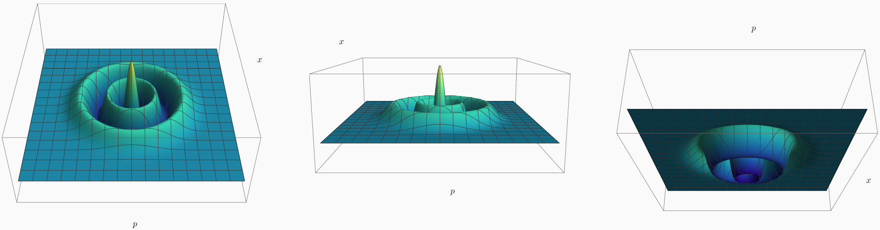
- For pure states, W is positive only if they are Gaussian (Hudson-Piquet theorem). E.g.: coherent and squeezed states.
- If U is at most quadratic in $\hat{x}$, the time evolution of W is entirely classical.

A gallery of Wigner functions: number states

Vacuum state

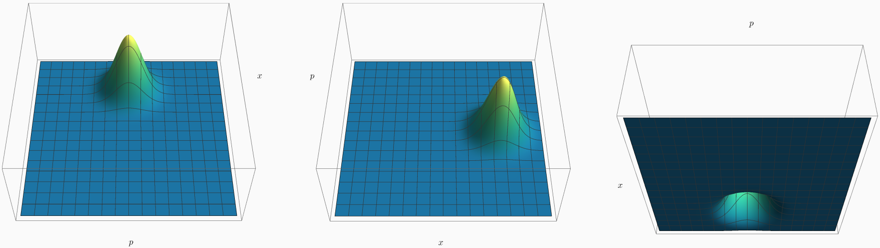


Number state ($n = 4$)

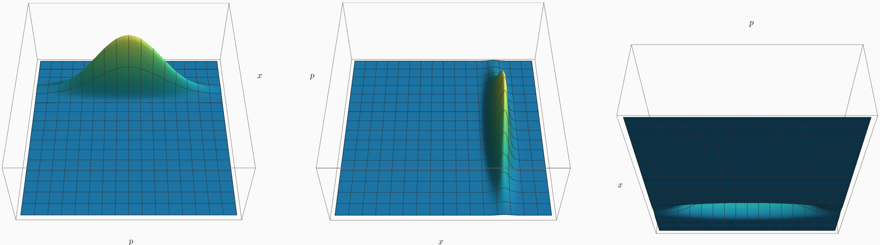


A gallery of Wigner functions: coherent and squeezed states

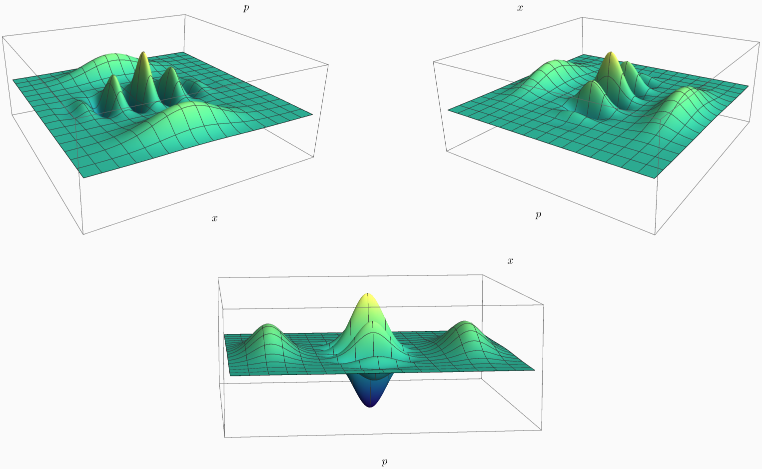
Coherent state



Squeezed state



Schrödinger cat state: $|\psi\rangle = \mathcal{N}/\sqrt{2}\left(|\alpha e^{i\varphi}\rangle + |\alpha e^{-i\varphi}\rangle\right)$



Applications

Quantum state reconstruction: the theoretical side

The Wigner function encodes all the physical properties of a quantum system.

Q: Is it possible to reconstruct it via a set of appropriate distributions?

A: Yes.

It is a two-step procedure:

1. Compute the Wigner function $W_{|X_\theta\rangle}$ of the eigenstate $|X_\theta\rangle$ of the quadrature operator $\hat{X}_\theta \equiv \cos\theta\,\hat{x} + \sin\theta\,\hat{p}$.
2. Use the trace-product rule and a Radon transformation to get $W_{\hat{\rho}}$ in terms of $W(X_\theta) \equiv \mathrm{Tr}(|X_\theta\rangle\langle X_\theta|\,\hat{\rho})$:

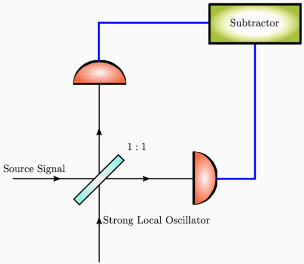
$$W_{\hat{\rho}}(x,p) = \frac{1}{4\pi^2}\int \mathrm{d}t|t|\int_{-\pi/2}^{\pi/2}\mathrm{d}\theta\int \mathrm{d}X_\theta\, e^{it(X_\theta - x\cos\theta - p\sin\theta)}W(X_\theta)\,.$$

Quantum state reconstruction: the experimental side

The Wigner function encodes all the physical properties of a quantum system.

Q: Is it possible to experimentally reconstruct it via a set of appropriate measurements?

A: Yes: in the Quantum Optics jargon, it is called homodyne tomography.



It is three-step procedure:

1. Inject the desired light state and a strongly classical coherent light $|\alpha e^{i\theta}\rangle$ in a homodyne detector measuring $W(n_{21}) \simeq W(X_\theta)/(\sqrt{2}|\alpha|)$.
2. Vary θ in N steps from $-\pi/2$ to $\pi/2$, and obtain a sequence of distributions $\{W(X_{\theta_1}), \ldots, W(X_{\theta_N})\}$.
3. Use this sampling of $W(X_\theta)$ to numerically gain $W_{\hat{\rho}}$ from the Radon transform above.

QHD: Generalities

- Quantum kinetic theory encodes a large amount of information in the quantum Boltzmann equation. Extracting it, though, is very painful.
- One needs simplified models containing only the necessary physical information: QHD.

QHD models are constructed by taking moments of the quantum Boltzmann equation.

Advantages One can recover physically relevant quantities:

$$n(x,t) = \int \mathrm{d}v\, W(x,v,t)\,, \quad u(x,t) = n^{-1} \int \mathrm{d}v\, vW(x,v,t)\,,$$
$$p(x,t) = m\left(\int \mathrm{d}v\, v^2 W(x,v,t) - n(x,t)u^2(x,t)\right), \quad \ldots$$

Drawbacks One has to provide a closure approximation to truncate the resulting hierarchy.

QHD: A fluid model for a quantum electron gas

A gas of non-relativistic, collisionless quantum electrons can be efficiently described by a single-particle QHD model. The first two moments at $\mathcal{O}(\hbar)$ of the quantum Boltzmann equation read

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(nu) = 0\,, \qquad \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \frac{e}{m}\frac{\partial \phi}{\partial x} - \frac{1}{mn}\frac{\partial p}{\partial x}\,.$$

- ϕ is the electrostatic potential, satisfying the Poisson equation

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{e}{\epsilon_0}\left(\int \mathrm{d}v\, W(x,v,t) - n_0\right).$$

- p is the pressure, composed by a classical and a quantum contribution, p_c and p_q. It can be related to n by making an "equal amplitude approximation" [3] (closure approximation).

Dispersion relations

This QHD model matches the Wigner-Poisson prediction for the dispersion relation of linear plasma perturbations. Same result, less effort.

Deformation quantization: basics

Classical		Quantum
(x, p)		$(\hat{x}, \hat{p})$
Γ	canonical $\longrightarrow$ quantization	$\mathcal{H}$
$\{\cdot, \cdot\}$		$[\cdot, \cdot]$

Moyal $\star$ product

The Wigner-Weyl correspondence induces the Moyal product:

$$f(x,p) \star_M g(x,p)$$
$$\equiv f\left(x + \frac{i\hbar}{2}\vec{\partial}_x, p - \frac{i\hbar}{2}\vec{\partial}_p\right) g(x,p)\,.$$

Deformation quantization

Similar to canonical quantization, but “gentler”:

- No operators, only *c*-number functions.
- Operator non-commutativity encoded in a $\star$ product:

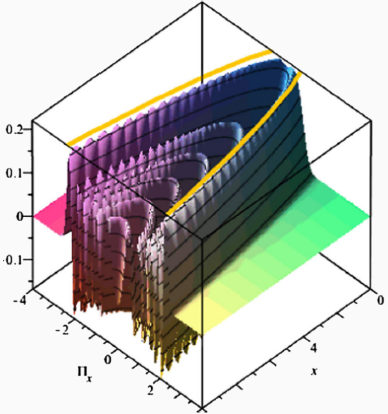
$$f \star g \equiv \sum_{n=0}^{\infty} (i\hbar)^n C_n(f, g)\,,$$
$$f \cdot g \equiv C_0\,,$$
$$\{f, g\} \equiv C_1(f, g) - C_1(g, f)\,.$$

- $\hbar$ is a deformation parameter:

$$[f, g]_\star \equiv f \star g - g \star f$$
$$= i\hbar\{f, g\} + \mathcal{O}(\hbar^2)\,.$$

Deformation quantization: FRW cosmology

- Quantizing cosmological models is easier because one can resort to symmetry reduction. At the practical level, this means simpler constraint equations.
- In the deformation quantization scheme, [4] solved the Hamiltonian constraint $H(x,p) \star_M W(x,p) = 0$ in a radiation-dust filled Universe:



- The most probable solutions have a $\tilde{\Omega}_r$ different to the classical one:

$$\tilde{\Omega}_r = \Omega_r - (\Omega_m/2)^{2/3} a_n \,,$$

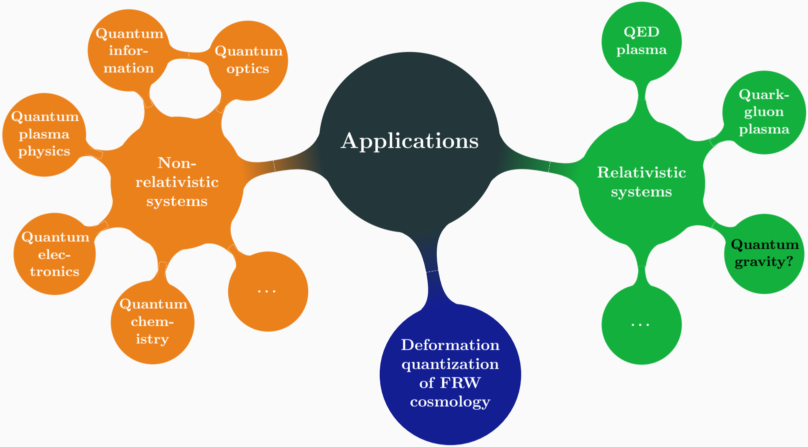
  a_n such that $\mathrm{dAi}(-\xi)/\,\mathrm{d}\xi|_{\xi=a_n} = 0$.
- Good match with classical results for large scale factor: quantum effects observable only for small x.
- $x = 0$ is no more a singular point.

Conclusions

Conclusions

Main advantages of the phase space Wigner function approach

- More concrete description of states and c-number equations.
- Strong connection between quantum mechanics and kinetic theory.



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