Stochastic processes from physics to finance:

A hierarchical-model approach to the financial market

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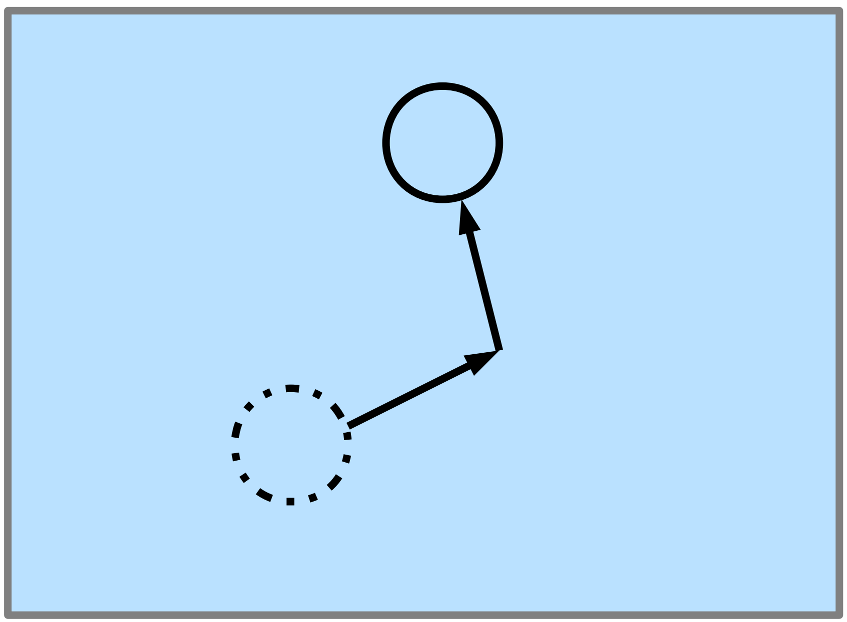
BROWNIAN MOTION

Pollen particle in water



Many independent collisions result in a **stochastic** force

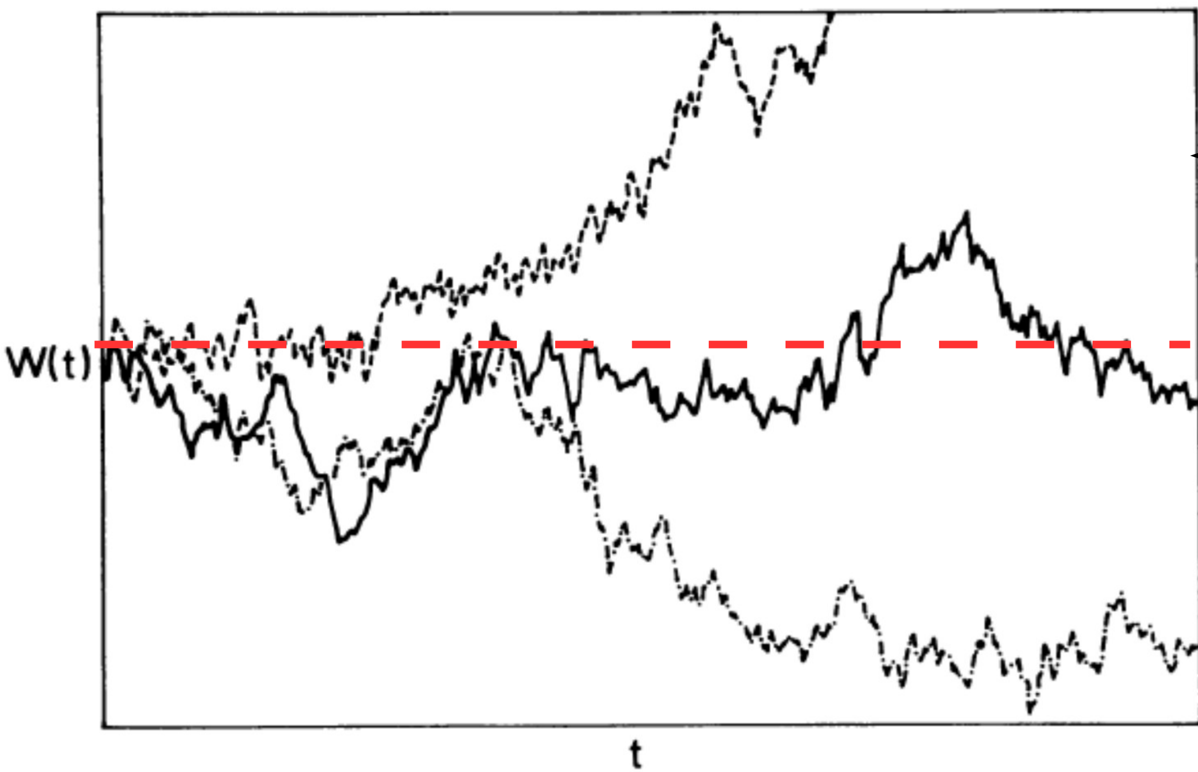
Each increment depends only on the actual position of the particle => MARKOVIAN PROCESS

Continuous, independent increments

$$\Delta W_t \sim N(0, t)$$

Increments are **gaussian-distributed**

BROWNIAN MOTION



$$\langle \Delta W_t \rangle = 0$$
$$\langle \Delta W_t^2 \rangle = t$$

More generally

$$dS_t = \boxed{\sigma} S_t dW_t$$

Changes the variance

FLUCTUATIONS

C. W. Gardiner – Handbook of Stochastic Methods for Physics, Chemistry and Natural Sciences

GEOMETRIC BROWNIAN MOTION

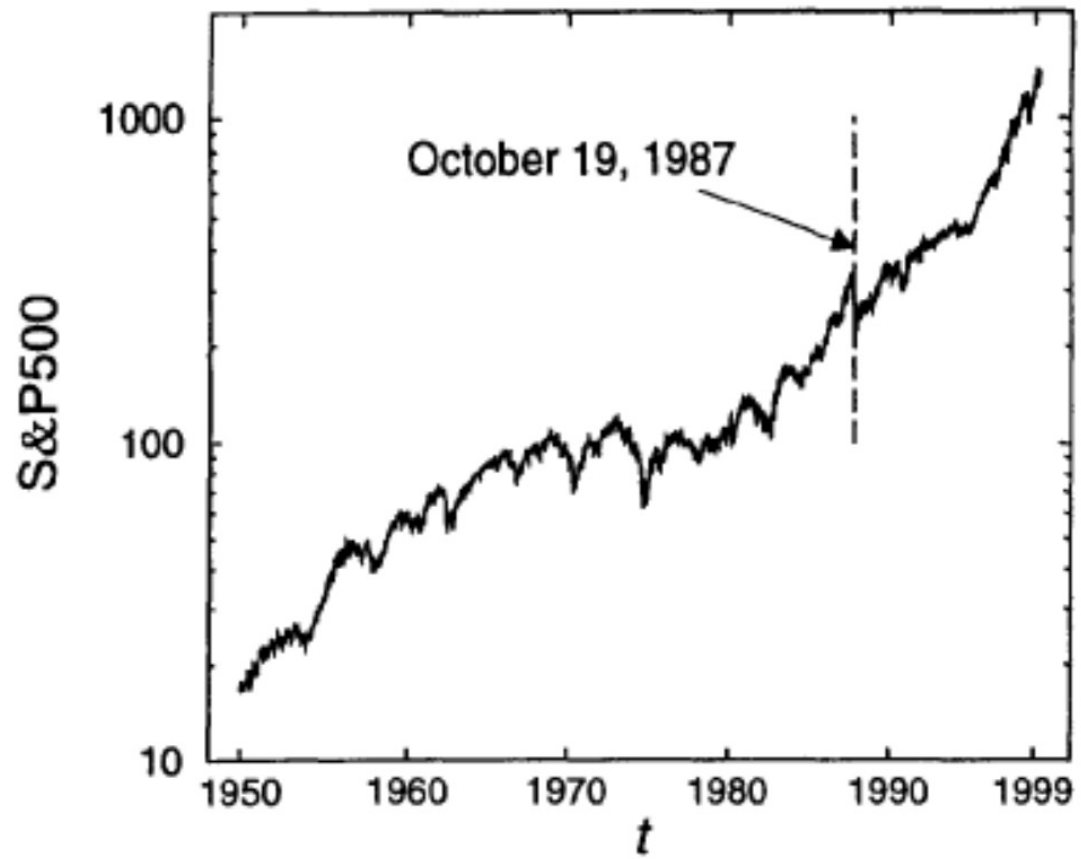
$$dS_t = \boxed{\mu S_t} + \boxed{\sigma S_t dW_t}$$

DETERMINISTIC

STOCHASTIC

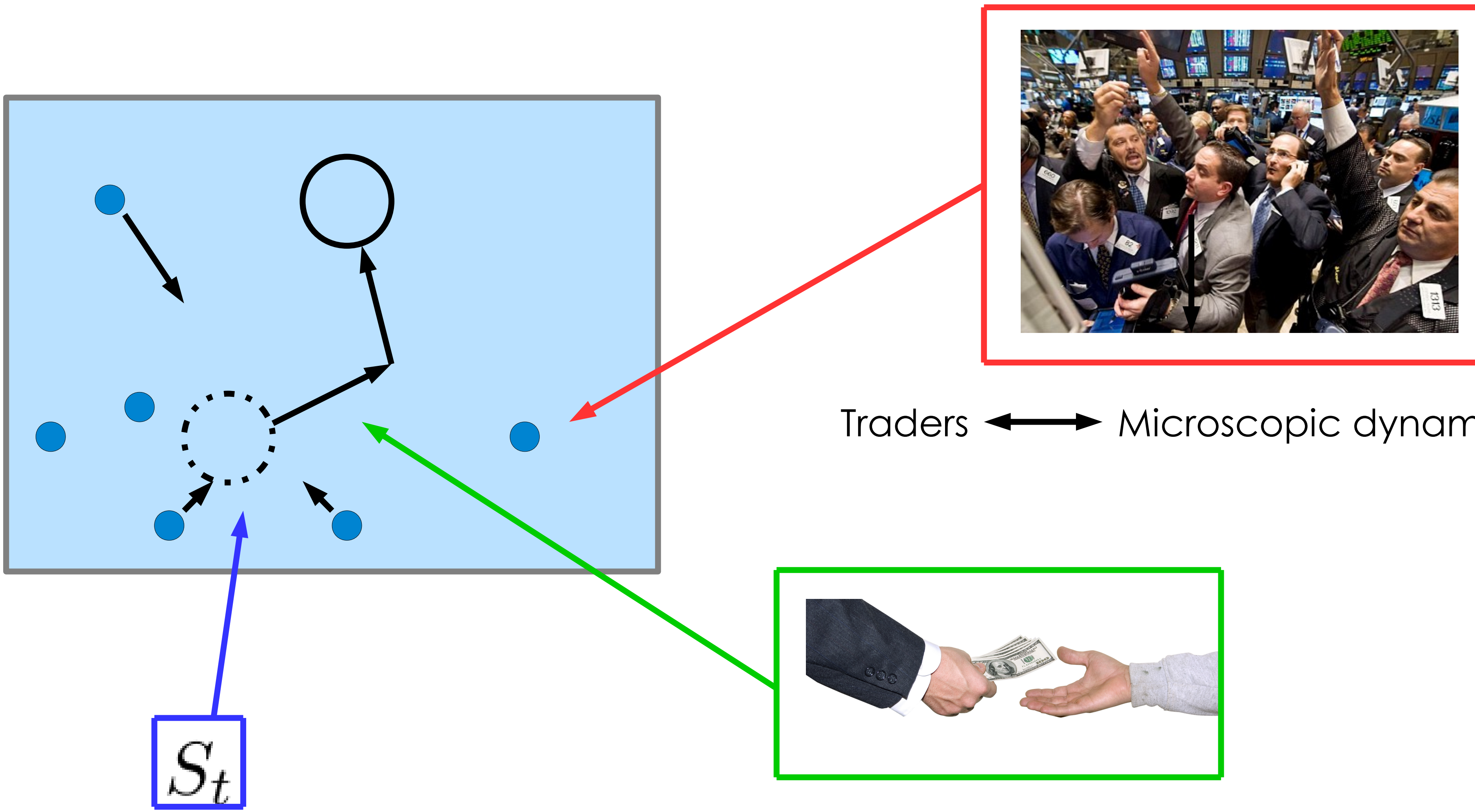
MARKOVIAN

Prices in the financial market



W. P. J. Bashnagel – Stochastic Processes from Physics to Finance

WHY DOES IT WORK?



Traders ⟷ Microscopic dynamics

Transactions ⟷ Collisions

Spot price ⟷ Pollen

A NOVEL APPROACH TO THE PROBLEM

- **Mesoscopic** and **macroscopic** models are used (order book, prices)
- They can be compared to **data**

BUT

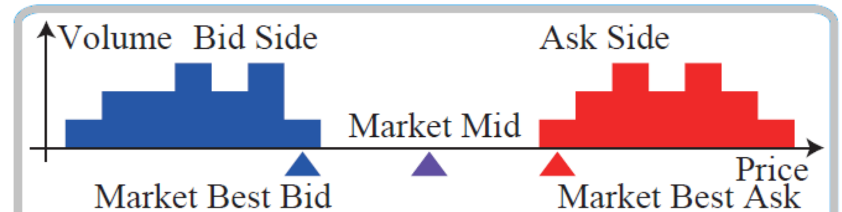
- **Theoretical microscopic** models only
- Not validated by **empirical analysis**

- **SOLUTION**: try a **hierarchical** approach from **microscopic** to **meso** and **macroscopic**, based on **empirical findings**

STRUCTURE OF THE HIERARCHICAL APPROACH

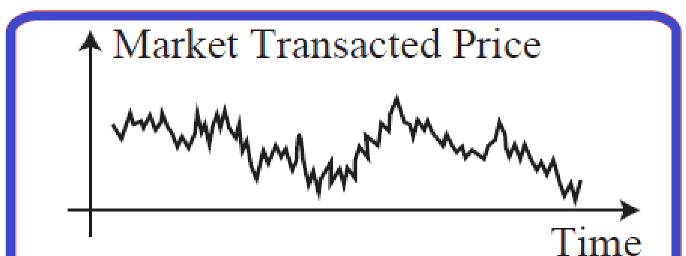
(a) Micro: traders' dynamics

Model the traders' dynamics on **direct observation**



(b) Meso: order-book dynamics

Build up the resulting **kinetic theory**



(c) Macro: price dynamics

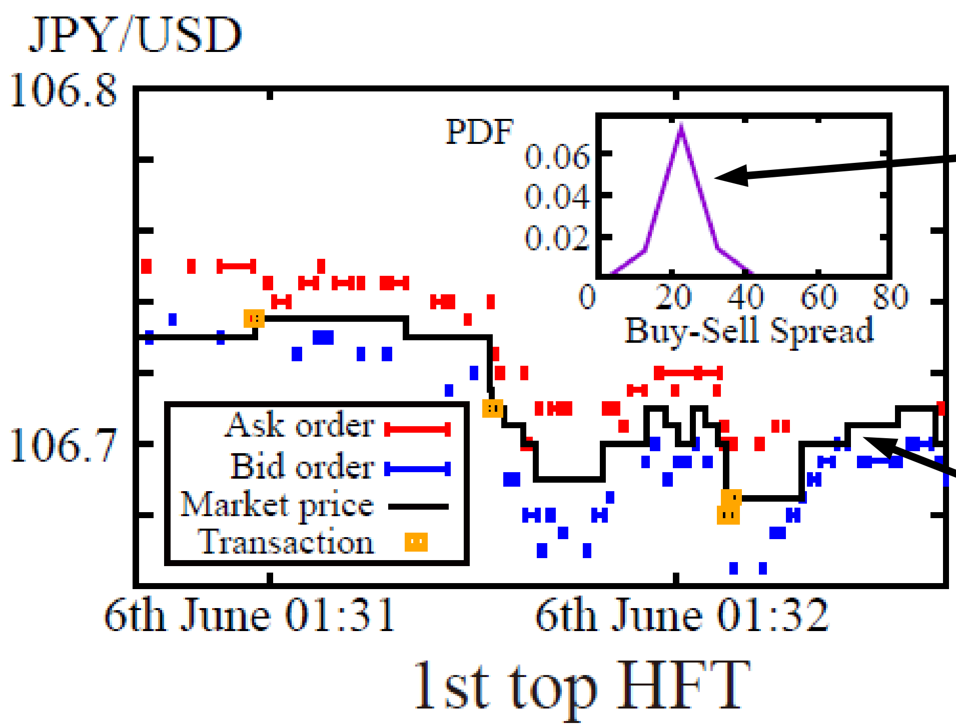
Compare to the **macroscopic** dynamics

Kanazawa et al, 2018

OBSERVATION OF THE TRADERS' DYNAMICS

- **High frequency traders** (HFT, more than 500 submissions a day) trading between the U.S. dollar(USD) and the Japanese Yen (JPY) on a **Foreign Exchange** (FX) were observed for one week. Currency unity =0.001 yen ≡ tpip

- **Microscopic trajectories** are identified by the **bid** and **ask** quoted prices for each trader



Specific to each trader, **constant**

TREND FOLLOWING: reaction of the trader to the market midprice

Kanazawa et al, 2018

TREND-FOLLOWING COMPONENT

- The microscopic trajectories have a **trend-following** component related to the **market price**
- Recast into the appropriate variables

Market price: price at which a **transaction** occurs

$$p(T) \longrightarrow \Delta p(T) = p(T) - p(T-1)$$

Best **ask/bid** price for trader **i**

$$a_i \;\; b_i \longrightarrow$$

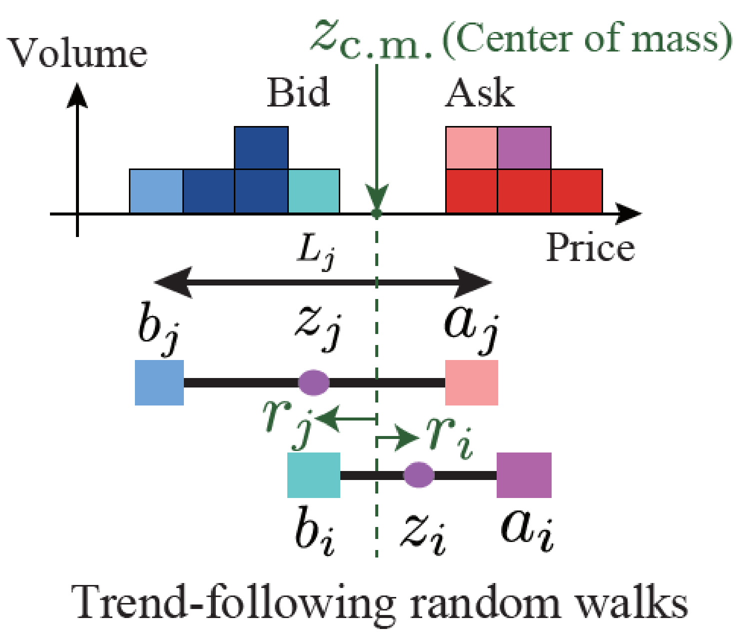
$$z_i = \frac{a_i + b_i}{2}$$

$$\Delta z_i(T) = z_i(T) - z_i(T-1)$$

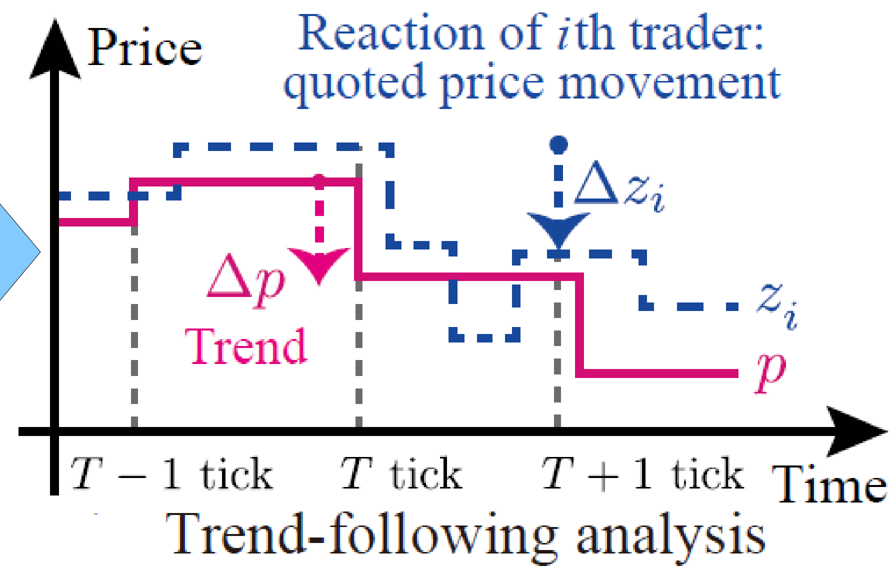
$$L_i = a_i - b_i$$ **Bid/ask spread**

TREND-FOLLOWING COMPONENT

The variables of the problem



Reaction of the traders to the market price



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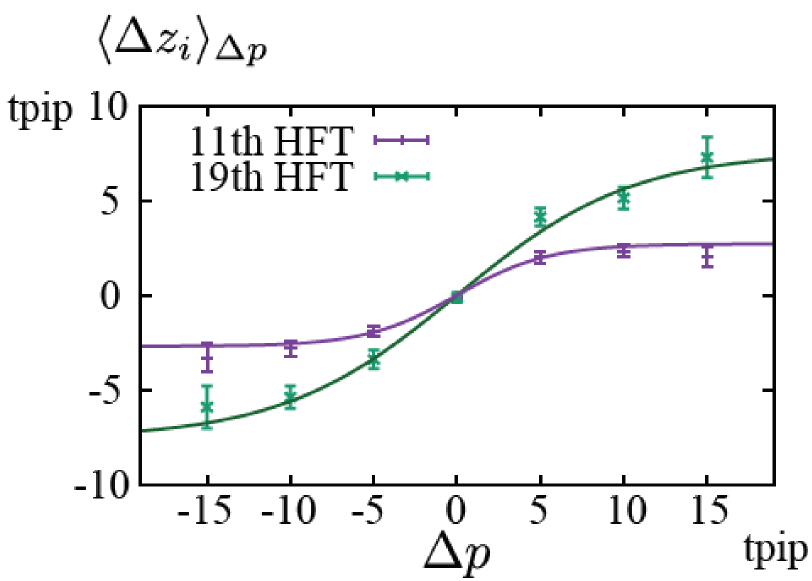
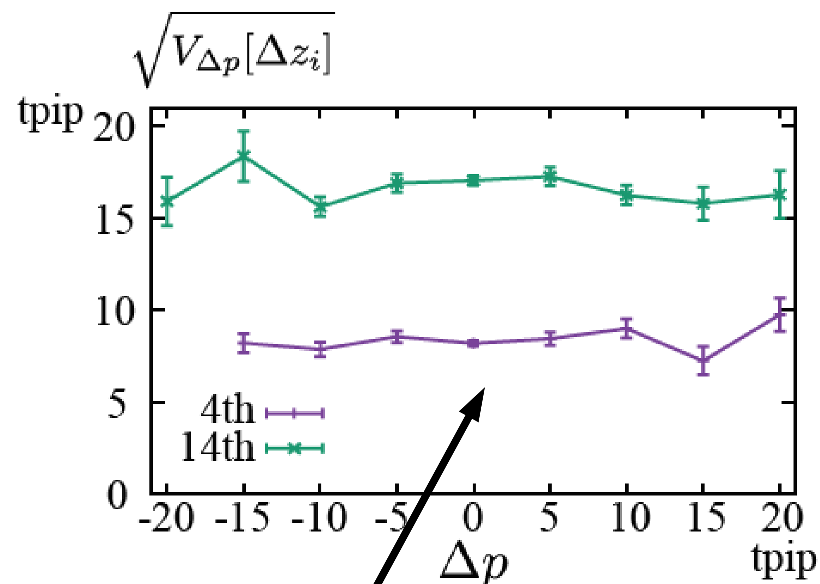
BUILDING UP THE MODEL: TREND-FOLLOWING AND NOISE COMPONENTS

- Consider the distribution of the **quoted midprice variations, conditional** to the **price movements**





- Formulate a **hypothesis**

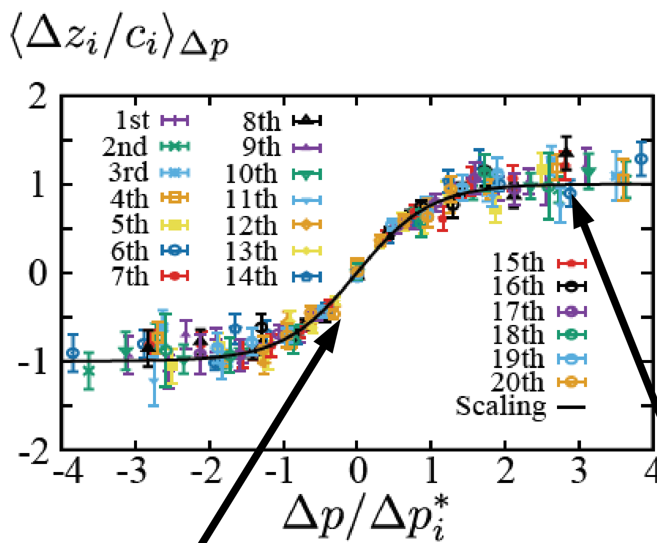
Independent of the price movements

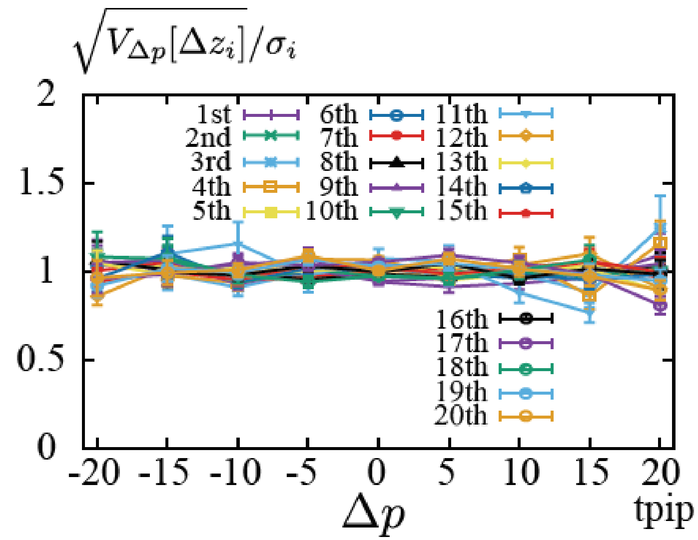
$$\langle \Delta z_i \rangle_{\Delta p} \approx c_i \tanh \frac{\Delta p}{\Delta p^*} \qquad V_{\Delta p}\left[\Delta z_i\right] \approx \sigma_i^2$$

Kanazawa et al, 2018

THE MICROSCOPIC MODEL: VALIDATION AND SCALING PROPERTY

- **Scaled averaged movements and noise** collapse onto the same curve





Almost **linear**

Saturates

Strength of trend following

- $$\frac{dz_i(t)}{dt} = c \tanh \frac{\Delta p(t)}{\Delta p^*} + \sigma \eta_i^R(t)$$

Gaussian noise

Kanazawa et al, 2018

FROM **MICRO** TO **MESO**: BOLTZMANN-LIKE EQUATION FOR FINANCE

- Introduce a term accounting for the "**jumps**" (requotations after transaction)

$$\frac{dz_i(t)}{dt} = c\tanh\frac{\Delta p(t)}{\Delta p^*} + \sigma\eta_i^R(t) + \eta_i^T(t)$$

- Introduce the **center of mass** and **relative coordinates** $z_{c.m.},\ r_i = z_i - z_{c.m.}$

- Consider $N \to \infty$, then this choice of coordinates **decouples the drift** and the **random motion**

- Assume a continuous distribution for the ask-bid spread $L \sim \rho(L)$

- From the dynamics of r_i derive a **master equation** for the conditional (on the spread) probability $\phi_L(r)$

- Suppose **molecular chaos** for the two-body correlation $\phi_{LL'}(rr') \approx \phi_L(r)\phi_{L'}(r')$

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FROM **MICRO** TO **MESO**: BOLTZMANN-LIKE EQUATION FOR FINANCE

$$\frac{\partial \phi_L(r)}{\partial t} = \frac{\sigma^2}{2}\frac{\partial^2 \phi_L(r)}{\partial r^2} + N \sum_{s=\pm 1} \int dL' \rho(L') \left[\tilde{J}^s_{LL'}(r + sL/2) - \tilde{J}^s_{LL'}(r) \right]$$

Mean field description

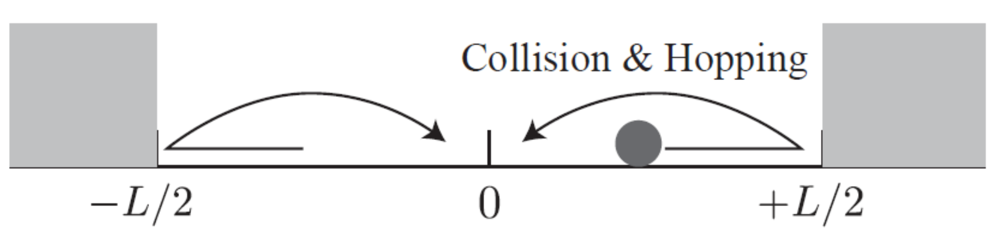
Diffusion

Collision. J: collision probability per unit time for bidder (asker), s=+1 (-1)

$$\tilde{J}^s_{LL'}(r) = \frac{\sigma^2}{2} |\tilde{\partial}_{rr'}| \{\phi_L(r)\phi_{L'}(r')\}\Big|_{r-r'=s(L+L')/2}$$

- Analytical **steady solution** for infinitely-many traders $\psi_L(r)$

$$\psi_L(r) \equiv \lim_{t\to\infty} \lim_{N\to\infty} \phi_L(r;t) = \frac{4}{L^2} \max\left\{\frac{L}{2} - |r|, 0\right\}$$



Brownian motion confined by hopping barriers!

Kanazawa et al, 2018 Giulia Pagano

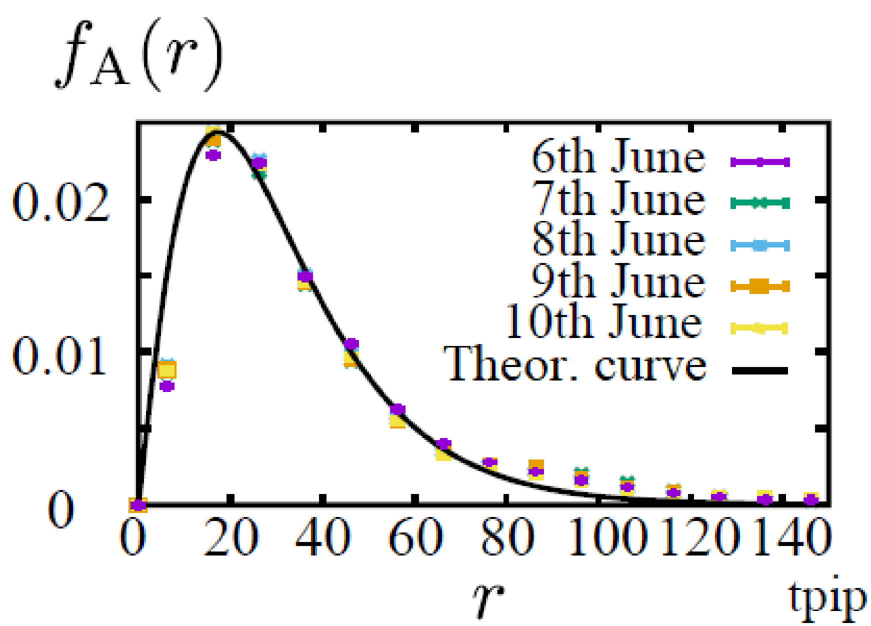
MESOSCOPIC VALIDATION

- **Order-book ask profile** induced by the microscopic dynamics:

$$f_A(r) = \int dL \rho(L) \phi_L \left(r - \frac{L}{2} \right) \quad + \quad \rho(L) = \frac{L^3}{6L^{*4}} e^{-L/L^*}$$

empirically validated on a daily basis. L^* = 15.5 ± 0.2 tpip

$$f_A(r) = \frac{4e^{-\frac{3r}{2L^*}}}{3L^*} \left[\left(2 + \frac{r}{L^*} \right) \sinh \frac{r}{2L^*} - \frac{re^{-\frac{r}{2L^*}}}{2L^*} \right]$$



Without any fitting parameter!

Kanazawa et al, 2018

FROM MICRO TO MACRO: LANGEVIN-LIKE EQUATION FOR FINANCE

- Introduce the **price movement at one tick precision**

$$\Delta p(T+1) \equiv p(T+1) - p(T) = z_i + \Delta z_{ij} - p(T)$$

and use **Ito** on the microscopic model

$$\Delta p(T+1) = c\ \tau(T) \tanh \frac{\Delta p(T)}{\Delta p^*} + \zeta(T)$$

$\tau(T)$: Time between transactions

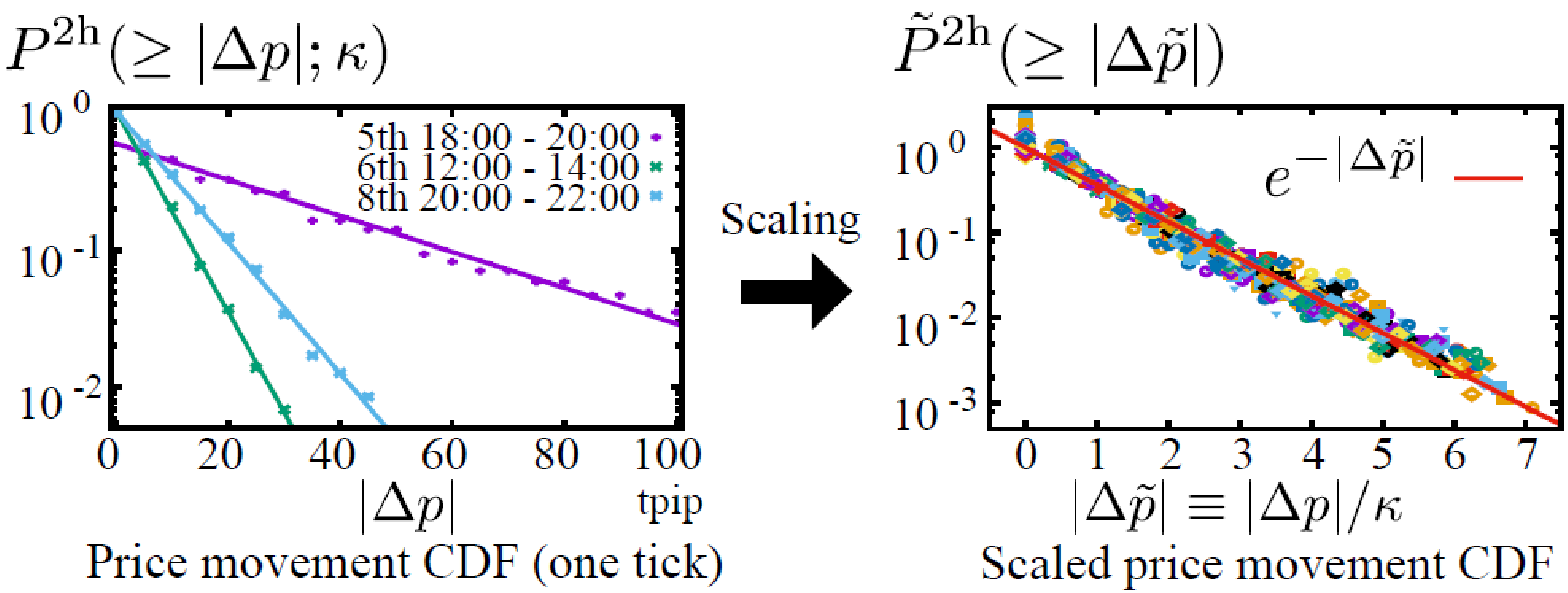
$\zeta(T)$: Random noise at T-th tick

- From the **statistics of the time intervals** (assumed to be **Poissonian**): $P^{2\mathrm{h}}(\geq |\Delta p|; \kappa) \approx e^{-|\Delta p|/\kappa} \quad (|\Delta p| \to \infty)$

- The **2h-decay length k fluctuates**, hence a 2h-segmeted distribution

MACROSCOPIC VALIDATION

$$P^{2\mathrm{h}}(\geq |\Delta p|;\kappa) \approx e^{-|\Delta p|/\kappa} \quad (|\Delta p| \to \infty)$$



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CONCLUSIONS

- **Prices** in the financial market can be modelled through **stochastic processes**

- **Models** are usually built by observing the **mesoscopic** or **macroscopic dynamics**

- Nevertheless, the **observation** of the **microscopic dynamics** of traders enables to build a **microscopic model** based on **empirical data** and the related **kinetic theory**

- Such model is able to **predic** the **mesoscopic** (order book) and **macroscopic** (prices) dynamics and can be **validated on observations**

REFERENCES

[1] K. Kanazawa, T. Sueshige, H. Takayasu and M. Takayasu - Derivation of the Boltzmann Equation for Financial Brownian Motion: Direct Observation of the Collective Motion of High-Frequency Traders , Phys. Rev. Lett. 120, 138301

[2] C. W. Gardiner – Handbook of Stochastic Methods for Physics, Chemistry and Natural Sciences

[3] W. P. J. Bashnagel – Stochastic Processes from Physics to Finance

[4] R. N. Mantegna and H. E. Stanley – An Introduction to Econophysics. Correlations and Complexity in Finance