How do airplanes fly?

Manuela Sisti
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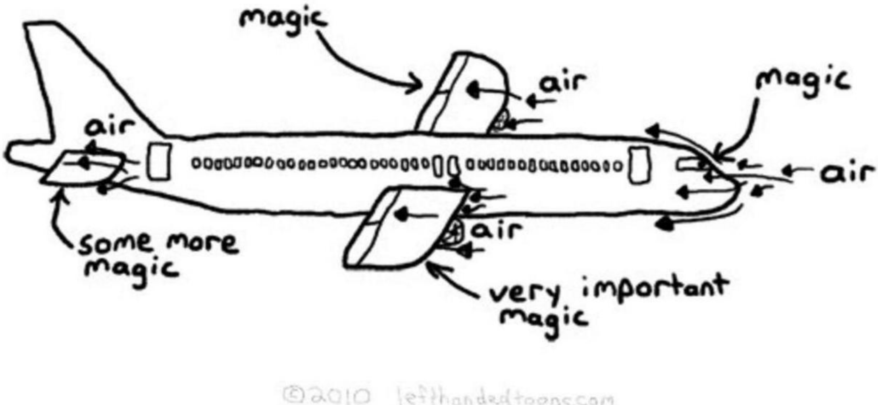
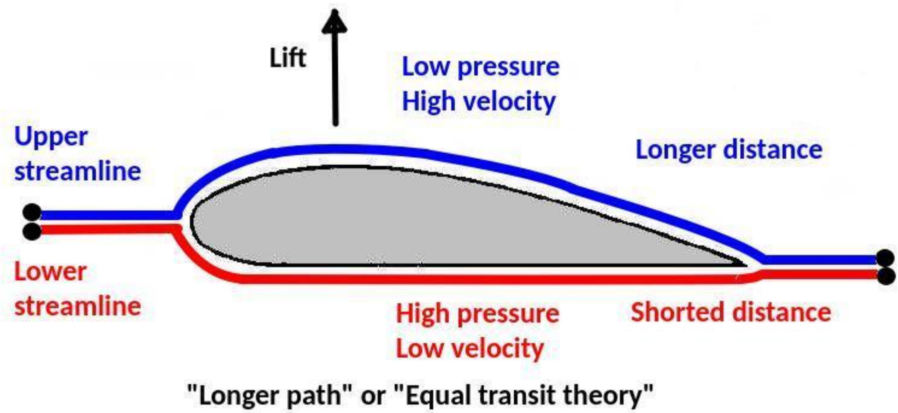


The first flight

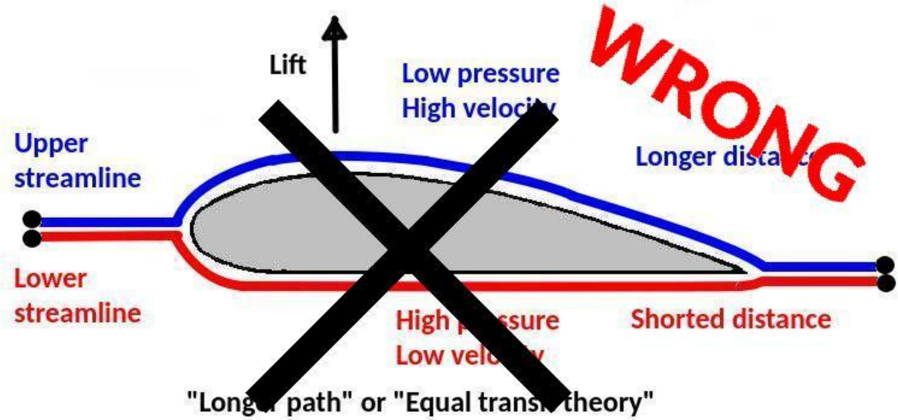
Figure: The first airplane flight, near Kitty Hawk, North Carolina (December 17th, 1903).

The “explanation” using Bernoulli’s theorem

$$\frac{v^2}{2} + \frac{p}{\rho} + gz = \text{constant} \quad (1)$$



The WRONG explanation using Bernoulli's theorem



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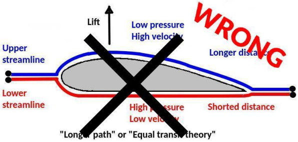

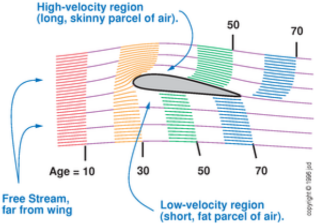

Figure: taken from "See How it Flies", NASA Research Article at `www.av8n.com/how/htm/spins.html`, Copyright 1996-2005 jsd; wind tunnel simulation with smoke injectors

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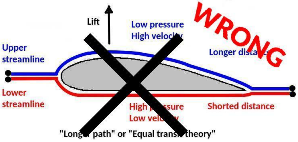

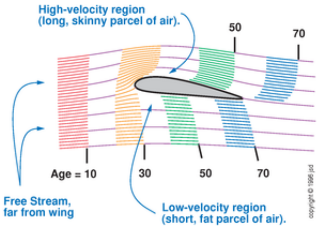

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Figure: Upside-down flight

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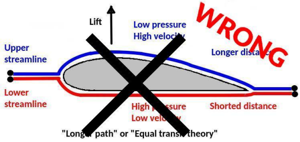

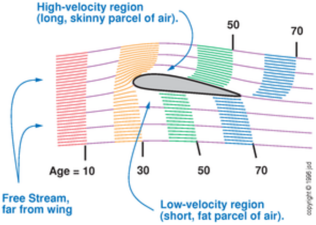

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Figure: Upside-down flight

What about viscosity?

A bit of definitions

2D case, setting $\mathbf{u} = (u, v, 0)$

$$\text{Irrotational flow} \quad (\nabla \wedge \mathbf{u} = 0) \rightarrow \mathbf{u} = \nabla \phi \tag{2}$$

$$\text{Incompressible flow} \quad (\nabla \cdot \mathbf{u} = 0) \rightarrow \mathbf{u} = \nabla \wedge \psi \tag{3}$$

$$\nabla^2 \phi = 0, \qquad \nabla^2 \psi = 0 \tag{4}$$

Thus, ϕ and ψ are harmonic functions!

$$\textbf{complex potential} \qquad w = \phi + i\psi \tag{5}$$

which is an analytic function of the complex variable $z = x + iy$

$$\tilde{u} = \frac{dw}{dz} = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x} = u - iv \tag{6}$$

2D flow past a cylinder

$$w(z) = U\left(z + \frac{a^2}{z}\right) \quad (7)$$

$$w(z) = U\left(z + \frac{a^2}{z}\right) - \frac{i\Gamma}{2\pi}\log z \quad (8)$$

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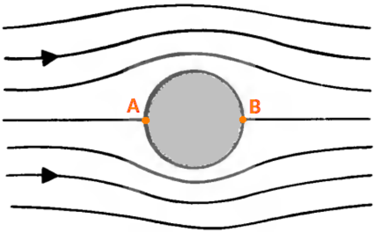

Figure: taken from "Elementary Fluid Dynamics", D. J. Acheson. Flow past a cylinder ($|z| = a$).

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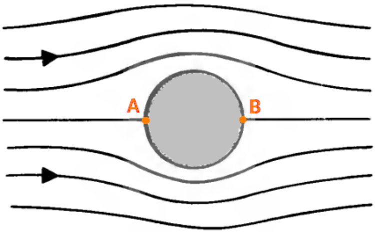

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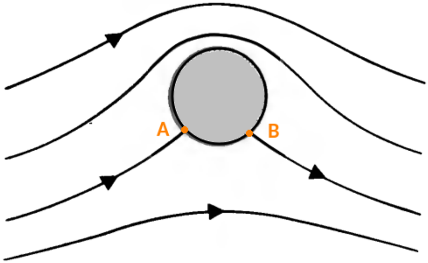

Figure: taken from "Elementary Fluid Dynamics", D. J. Acheson. Flow past a cylinder: superposition of irrotational + uniform flow and line vortex flow ($\Gamma < 0$).

2D flow past a cylinder

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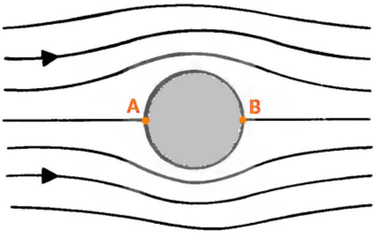

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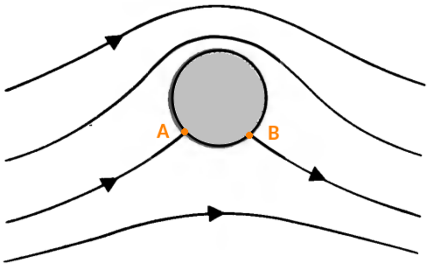


Figure: taken from "Elementary Fluid Dynamics", D. J. Acheson. Flow past a cylinder: superposition of irrotational + uniform flow and line vortex flow ($\Gamma < 0$).

Angle of attack: $$w(z) = U\left(ze^{-i\alpha} + \frac{a^2}{z}e^{i\alpha}\right) - \frac{i\Gamma}{2\pi}\log z \qquad (9)$$

Conformal mapping: circle into ellipse

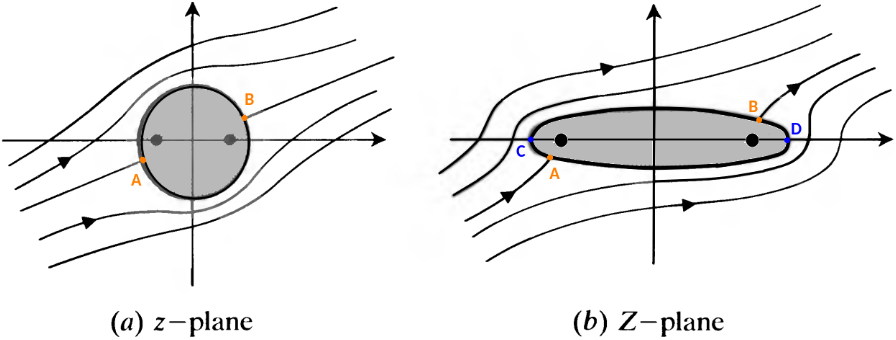


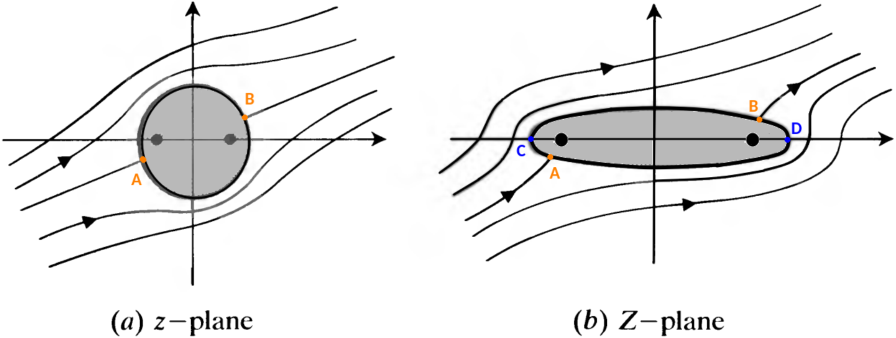
Figure: modified from "Elementary Fluid Dynamics", D. J. Acheson. (a) Flow past a cylinder ($|z| = a$), oncoming stream at α (attack angle); (b) flow past an elliptical wing by conformal mapping. No circulation ($\Gamma = 0$). In orange stagnation points.

$$\textbf{Joukowski transformation:} \qquad Z = z + \frac{c^2}{z} \tag{10}$$

with ($0 \leq c \leq a$), whose inverse is:

$$z = \frac{1}{2}Z + \left(\frac{1}{4}Z^2 - c^2\right) \tag{11}$$

Conformal mapping: circle into ellipse



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$$W(Z) = Ue^{-i\alpha}\left[\frac{1}{2}Z + \left(\frac{1}{4}Z^2 - c^2\right)^{1/2}\right] + Ue^{i\alpha}\frac{a^2}{c^2}\left[\frac{1}{2}Z + \left(\frac{1}{4}Z^2 - c^2\right)^{1/2}\right]$$

$$\xrightarrow{c=a} \tilde{U} = \frac{dW}{dZ} = \frac{dw}{dz}\frac{dz}{dZ} = \frac{U(e^{-i\alpha} - e^{i\alpha}\frac{a^2}{z^2})}{(1 - \frac{a^2}{z^2})} \quad (12)$$

More mathematical tricks

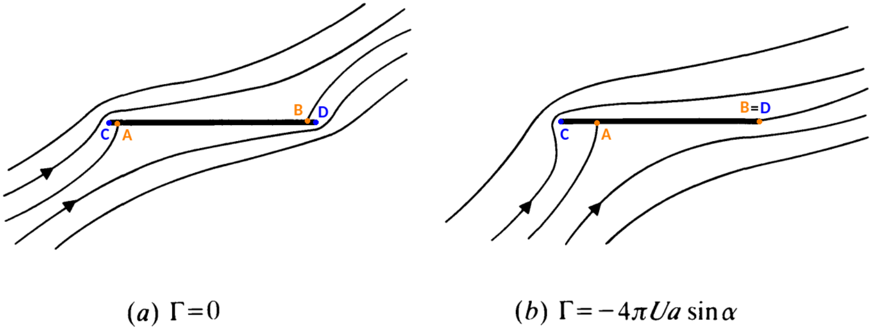
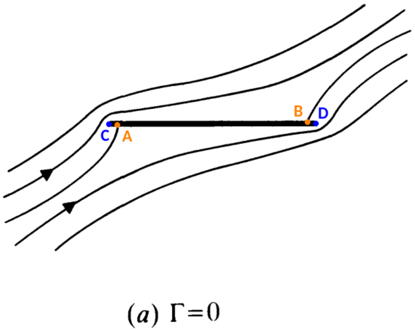
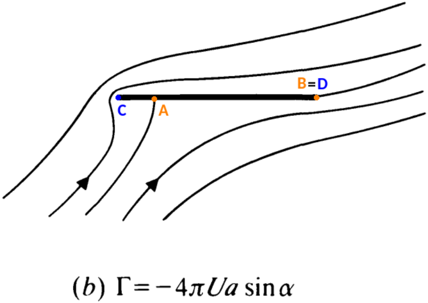


Figure: modified from "Elementary Fluid Dynamics", D. J. Acheson. Flow past a finite flat plate, the oncoming stream at angle α (attack angle); (a) case $\Gamma = 0$; (b) case $\Gamma = -4\pi Ua \sin\alpha$. In orange stagnation points. In blue singularity points.

$$W(Z) = Ue^{-i\alpha}\left[\frac{1}{2}Z + \left(\frac{1}{4}Z^2 - c^2\right)^{1/2}\right] + Ue^{i\alpha}\frac{a^2}{c^2}\left[\frac{1}{2}Z + \left(\frac{1}{4}Z^2 - c^2\right)^{1/2}\right] - \frac{i\Gamma}{2\pi}\log\left[\frac{1}{2}Z + \left(\frac{1}{4}Z^2 - c^2\right)^{1/2}\right] \quad (13)$$

More mathematical tricks





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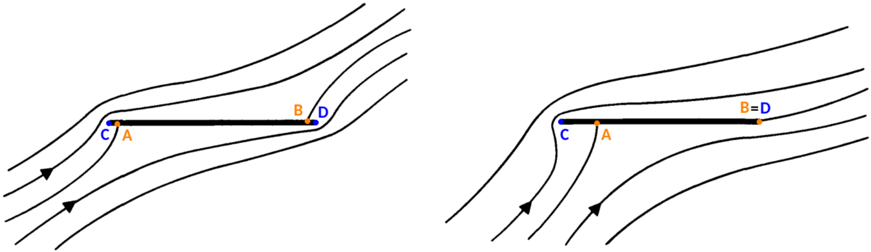
$$\tilde{U} = \frac{dW}{dZ} = \frac{dw}{dz}\frac{dz}{dZ} = \frac{U(e^{-i\alpha} - e^{i\alpha}\frac{a^2}{z^2}) - \frac{i\Gamma}{2\pi z}}{(1 - \frac{a^2}{z^2})} \tag{14}$$

More mathematical tricks



(a) $\Gamma = 0$　　　(b) $\Gamma = -4\pi Ua \sin\alpha$

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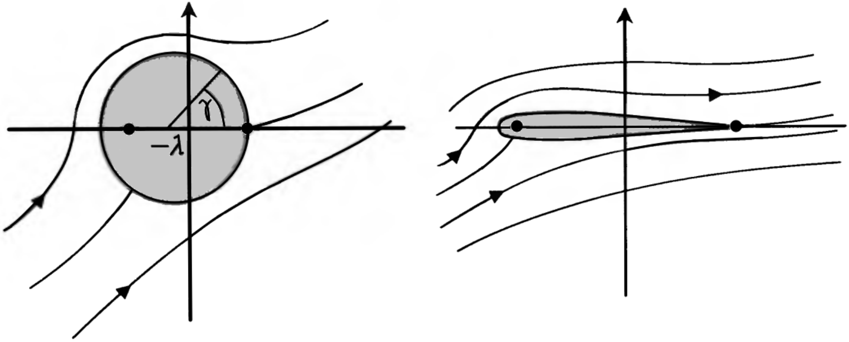
Kutta-Joukowski condition: $$\Gamma = -4\pi Ua \sin\alpha \quad (15)$$

More mathematical tricks, again!



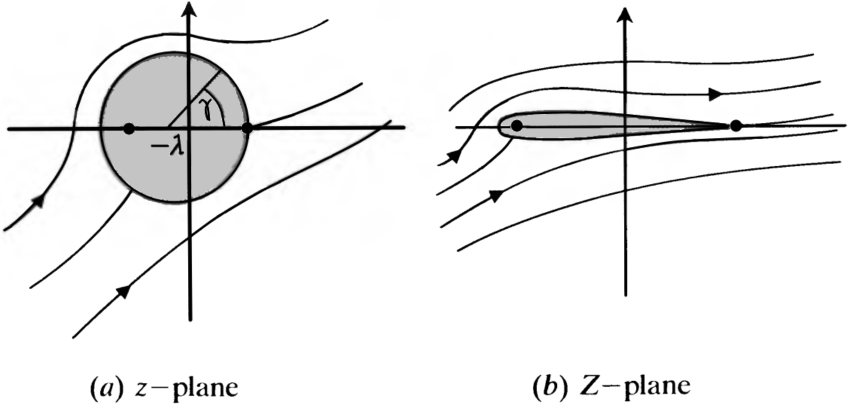
(*a*) *z*−plane (*b*) *Z*−plane

Figure: modified from “Elementary fluid dynamics”, D. J. Acheson. Flow past a symmetric Joukowski aerofoil by conformal mapping.

$$Z = -\lambda + (a+\lambda)e^{i\gamma} + \frac{a^2}{-\lambda + (a+\lambda)e^{i\gamma}} \tag{16}$$

$$w(z) = U\left[(z+\lambda)e^{-i\alpha} + \frac{(a+\lambda)^2}{(z+\lambda)}e^{i\alpha}\right] - \frac{i\Gamma}{2\pi}\log(z+\lambda) \tag{17}$$

More mathematical tricks, again!



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$$\frac{dW}{dZ} = \frac{U[e^{-i\alpha} - \frac{(a+\lambda)^2}{(z+\lambda)} e^{i\alpha}] - \frac{i\Gamma}{2\pi(z+\lambda)}}{(1 - \frac{a^2}{z^2})} \tag{18}$$

$$\Gamma = -4\pi U(a+\lambda)\sin\alpha \tag{19}$$

Why attack angle is so important?

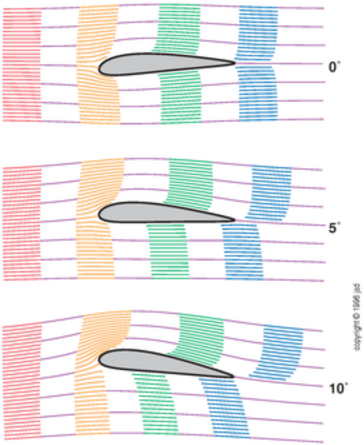

Figure: Figure taken from "See How it Flies", NASA Research Article at www.av8n.com/how/htm/spins.html. Wind tunnel simulation with smoke injectors. Flow for three different angle of attack: 0, 5, 10 degrees from top to bottom.

What about viscosity?

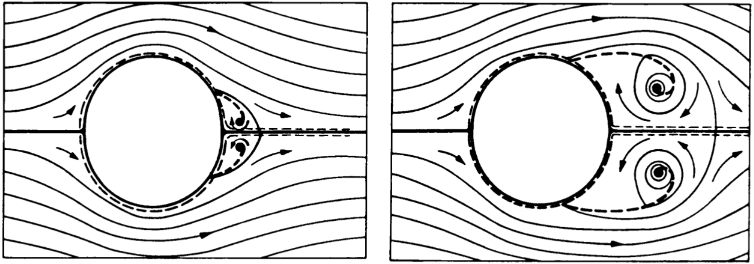

Figure: Figure taken from "Elementary Fluid Dynamics", D. J. Acheson. Flow relative to an impulsively moved circular cylinder at two different times (from Prandtl 1905). Dashed lines indicate layers of strong vorticity.

What about viscosity?

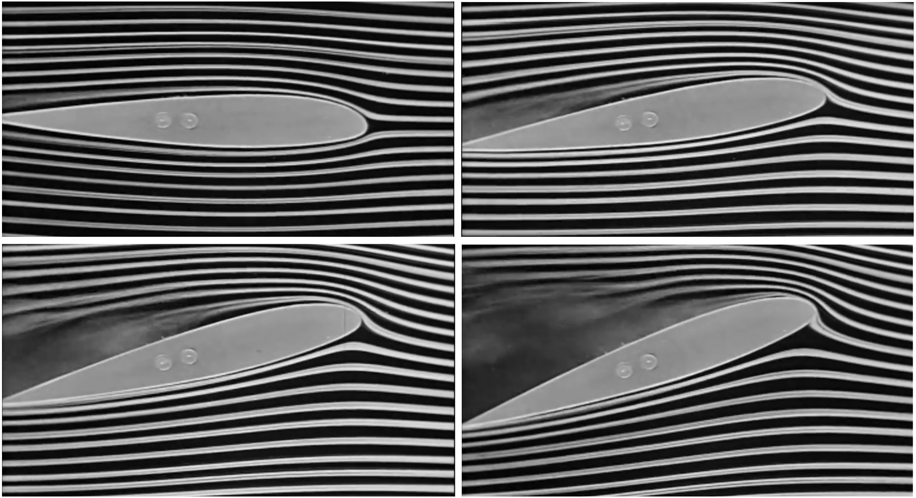

Figure: Snapshots from a 1930s test conducted at NASA Langley Research Center's 6 (https://www.youtube.com/watch?v=3_WgkVQWtno).

The generation of lift: the starting vortex

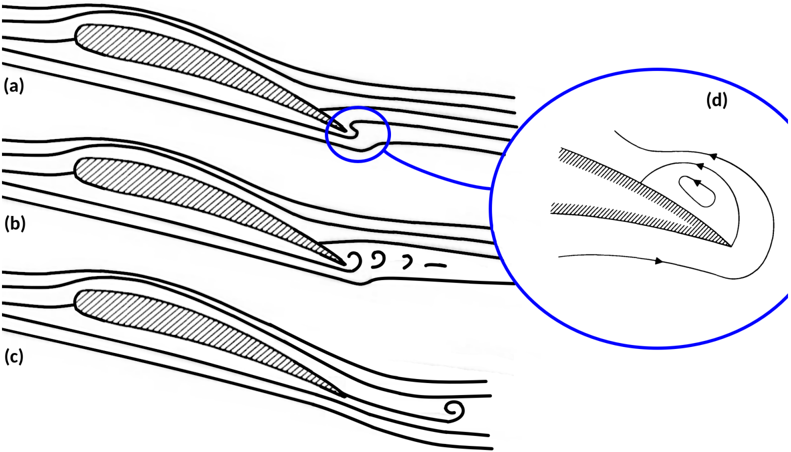

Figure: Figure modified from "Elementary fluid dynamics", D. J. Acheson. Starting vortex formation.

The generation of lift: the starting vortex

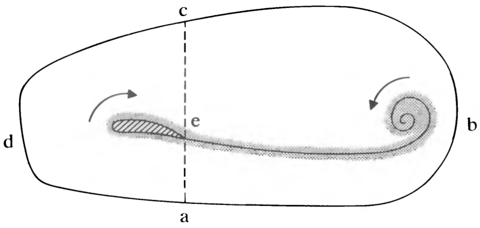

Figure: Figure taken from "Elementary Fluid Dynamics", D. J. Acheson. The generation of circulation by means of vortex shedding.

Theorem (Kelvin's circulation)

Let an inviscid, incompressible fluid of constant density be in motion in the presence of a conservative body force $g = -\nabla\chi$ per unit mass. Let $C(t)$ denote a closed circuit that consists of the same fluid particles as time proceeds. Then the circulation $\Gamma = \int_{C(t)} \mathbf{u} \cdot d\mathbf{x}$ round $C(t)$ is independent of time.

$$\int_{abcda} \mathbf{u} \cdot d\mathbf{x} = \int_{aecda} \mathbf{u} \cdot d\mathbf{x} + \int_{abcea} \mathbf{u} \cdot d\mathbf{x} = 0 \qquad (20)$$

The 3D case

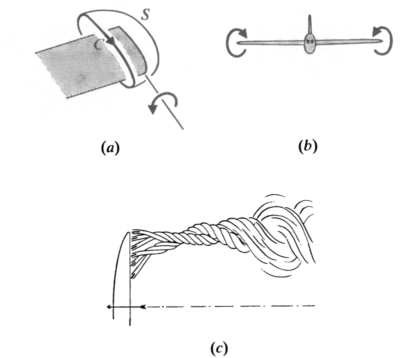

Figure: taken from "Elementary Fluid Dynamics", D. J. Acheson. Trailing vortices: (a) definition sketch for surface S; (b) view from some distance ahead the aircraft; (c) the original drawing from Lanchester's *Aerodynamics* (1907).

Figure: Photo showing wake turbulence. Credit: Ryoh Ishihara.

In conclusion...

Figure: dialogue from "Harry Potter and the Half-Blood Prince" by J. K. Rowling

- Angle of attack
- Viscosity

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Figure: dialogue from "Harry Potter and the Half-Blood Prince" by J. K. Rowling

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Thank you!

"Elementary fluid dynamics", D. J. Acheson, Oxford University Press (1990)

"See How it Flies", J. S. Denker, NASA Research Article at `www.av8n.com/how/htm/spins.html`, Copyright 1996-2005 jsd

Some important definitions

2D case, setting $\mathbf{u} = (u, v, 0)$

- Irrotational flow ($\nabla \wedge \mathbf{u} = 0$):

$$u = \frac{\partial \phi}{\partial x}, \qquad v = \frac{\partial \phi}{\partial y} \qquad \rightarrow \qquad \mathbf{u} = \nabla \phi \tag{21}$$

- Incompressible ($\nabla \cdot \mathbf{u} = 0$):

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x} \qquad \rightarrow \qquad \mathbf{u} = \nabla \wedge \psi \tag{22}$$

Using the definitions above:

$$\nabla^2 \phi = 0, \qquad \nabla^2 \psi = 0 \tag{23}$$

Thus, ϕ and ψ are harmonic functions!
(22) and (21) give the Cauchy-Rienmann equations of complex variable theory, thus we can define the "complex potential":

$$w = \phi + i\psi \tag{24}$$

which is an analytic function of the complex variable $z = x + iy$.

Some important definitions

If the function w is analytic, this implies that it is also differentiable, meaning that:

$$\frac{dw}{dz} = \lim_{dz \to 0} \frac{w(z + dz) - w(z)}{dz} \tag{25}$$

is finite and independent of the direction of dz, thus we can take $dz = dx$ and write

$$\tilde{u} = \frac{dw}{dz} = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x} = u - iv \tag{26}$$

The flow speed at any point is therefore

$$q = (u^2 + v^2)^{1/2} = \left|\frac{dw}{dz}\right| \tag{27}$$

Let's give some examples

- The complex potential for a uniform flow at an angle α to the x-axis is:

$$w = Uze^{-i\alpha} \quad \leftrightarrow \quad u = U\cos\alpha, \quad v = U\sin\alpha \tag{28}$$

- The complex potential for a line vortex at the origin is

$$w = -\frac{i\Gamma}{2\pi}\log z \quad \leftrightarrow \quad \mathbf{u} = \frac{\Gamma}{2\pi r}\mathbf{e}_\theta \tag{29}$$

where $\Gamma = \oint_C \mathbf{u} \cdot d\mathbf{x}$.

The method of images

The method of images is about getting flow which satisfies some boundary conditions.

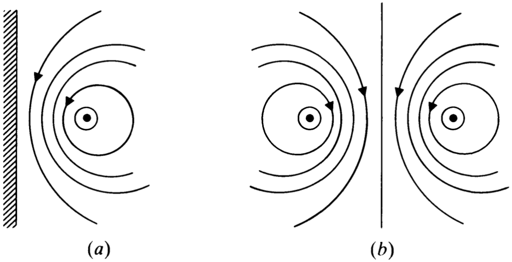

Figure: Figure taken from "Elementary Fluid Dynamics", D. J. Acheson. Sketch for the application of the method of images.

The complex potential is:

$$w = -\frac{i\Gamma}{2\pi}\log(z-d) + \frac{i\Gamma}{2\pi}\log(z+d) \qquad (30)$$

The method of images applied to the flow past a cylinder

The complex potential for the flow inside a circular cylinder $|z| = a$ due to a line vortex at $z = c < a$ is:

$$w = -\frac{i\Gamma}{2\pi}\log(z-c) + -\frac{i\Gamma}{2\pi}\log\left(z - \frac{a^2}{c}\right) \tag{31}$$

Theorem (Milne-Thomson's)

Suppose we have a flow with complex potential $w = f(z)$, where all the singularities of $f(z)$ lie in $|z| > a$. Then

$$w = f(z) + \overline{f(a^2/\overline{z})} \tag{32}$$

where an overbar denotes complex conjugate, is the complex potential of a flow with (i) the same singularities as $f(z)$ in $|z| > a$ and (ii) $|z| = a$ as a streamline.

Blasius' theorem. The forces involved.

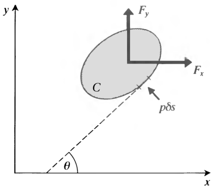

Figure: Figure taken from "Elementary Fluid Dynamics", D. J. Acheson. Sketch for proof of Blasius' theorem.

$$\mathbf{F} = (-\sin\theta, \cos\theta) p\delta s \tag{33}$$

$$F_x - iF_y = \\ = -\int p\sin\theta\delta s - i\int p\cos\theta\delta s = \\ = -\int pie^{-i\theta}\delta s \tag{34}$$

Using the Bernoulli's theorem:

$$F_x - iF_y = -i\int\left(-\rho\frac{q^2}{2} - \rho\phi_g + \rho K\right)e^{-i\theta}\delta s \tag{35}$$

where $q = (u^2+v^2)^{1/2}$, $u = q\cos\theta$, $v = q\sin\theta$, $\frac{dw}{dz} = u - iv = qe^{-i\theta}$. Only the first one is important and gives:

Blasius' theorem $$F_x - iF_y = i\int\frac{\rho}{2}\left(\frac{dw}{dz}\right)^2 dz \tag{36}$$

Application of Blasius' theorem

We can apply the result of Blasius' theorem to the flow past a circular cylinder in order to understand what kind of forces are involved. Thus, applying (35) to (8) we obtain[1]:

$$F_x - iF_y = \frac{1}{2} i\rho \oint_C \left[U(1 - \frac{a^2}{z^2}) - \frac{i\Gamma}{2\pi z} \right]^2 dz = \frac{1}{2} i\rho 2\pi i \left(-\frac{iU\Gamma}{\pi} \right) = i\rho U\Gamma \qquad (37)$$

where the last term is obtained applying the residue calculus.

[1]We're using the fact that only the term z^{-1} gives a contribution to the integral.

Conformal mapping

$$w = \phi + i\psi \tag{38}$$

Let's choose $Z = f(z)$ as some analytic function of z, with an inverse $z = F(Z)$ which is an analytic function of Z. Then: $W(Z) = w\{F(Z)\}$, and $W(Z) = \Phi(X, Y) + \Psi(X, Y)$.

$$\textbf{Joukowski transformation:} \qquad Z = z + \frac{c^2}{z} \tag{39}$$

whose inverse is:

$$z = \frac{1}{2}Z + \left(\frac{1}{4}Z^2 - c^2\right) \tag{40}$$

Thanks to the Joukowski transformation the circle $z = ae^{i\theta}$ $(0 \leq c \leq a)$ is mapped into an ellipse:

$$X + iY = \left(a + \frac{c^2}{a}\right)\cos\theta + i\left(a - \frac{c^2}{a}\right)\sin\theta \rightarrow \frac{X^2}{(a + c^2/a)^2} + \frac{Y^2}{(a - c^2/a)^2} = 1 \tag{41}$$