Path-integral methods as a universal tool in quantum field theory and statistical mechanics*

Dmitri Antonov[1, †]

[1]*Instituto Superior Técnico, Departamento Fisica / CFIF,*

Av. Rovisco Pais, 1049-001 Lisboa, Portugal

Contents

In this course, we try to make all the calculations as detailed as possible.

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†Formerly at Institut für Theor. Physik, Univ. Heidelberg, Philosophenweg 19, 69120 Heidelberg, Germany.

1. Path integrals in quantum mechanics. Euclidean formulation of quantum mechanics. Path integral for a harmonic oscillator. An analogy between the Euclidean formulation of quantum mechanics in D dimensions and statistical mechanics in D spatial and 1 temporal dimensions in equilibrium.

The concept of path integrals in quantum mechanics stems from the famous *gedankenexperiment*, that is the thought experiment, where electrons emitted by some source pass through two slots and, after that, are detected on a screen [1]. The slots are located symmetrically with respect to the line, which goes through the emitter perpendicularly to the screen. One measures the probability $P(x)$ for an electron to be detected on the screen at a distance x from this line. Since electrons are particles, one could guess that an electron definitely passes through one of the two slots, and therefore $P(x) = P_1(x) + P_2(x)$. Here, $P_1(x)$ is the probability measured with the closed slot number 2, and $P_2(x)$ is the probability measured with the closed slot number 1. Experimentally, however, when both slots are kept open, one observes a picture corresponding to the intensity distribution, which appears from the interference of two waves. For this reason, one can assume that the process is described by some (complex-valued) probability amplitude $\varphi(x)$, and it is this amplitude which is an additive quantity. That means the probability is expressed in terms of the amplitude as $P(x) = |\varphi(x)|^2$, where $\varphi(x) = \varphi_1(x) + \varphi_2(x)$.

Using light, namely the Compton scattering of photons off the electrons, one can try to detect which of the two slots a given electron passes through. As a result, one can know for sure that the electron passes through the slot number 1 or through the slot number 2, and therefore the probability itself becomes additive, $P(x) = P_1(x) + P_2(x)$. That is, however, only possible provided one can register a photon scattered off the electron. In the course of scattering, such a photon with the wavelength λ transfers to the electron a momentum of the order of $\hbar/\lambda$. Thus, an ambiguity of this order of magnitude appears in the measured momentum of the detected electron, that is the origin of the Heisenberg uncertainty principle. Only with the increase of the physical influence (used to detect which of the two slots the electron has passed through) up to the point where the interference picture is lost completely, does one arrive at the additive probability.

In the thought experiment suggested by Richard Feynman, one increases the number of slots, as well as of intermediate screens with the slots. Essentially, one can imagine the whole space consisting of such infinitesimal slots at every point. This way, one arrives at an idea of the integral over trajectories, or the path integral. The probability amplitude for a particle located at the space point $\mathbf{x}_a$ at the moment of time t_a to be, at a later moment t_b, detected at the space point $\mathbf{x}_b$ is given by the formal sum

$$\langle \mathbf{x}_b, t_b | \mathbf{x}_a, t_a \rangle = \sum_{\{\mathbf{x}(t)\}} \mathrm{e}^{iS[\mathbf{x}(t)]/\hbar}.$$

This sum runs over all paths $\{\mathbf{x}(t)\}$ connecting these two space points in such a way that $\mathbf{x}(t_a) = \mathbf{x}_a$ and $\mathbf{x}(t_b) = \mathbf{x}_b$. Here, $S[\mathbf{x}(t)] = \int_{t_a}^{t_b} dt\, \mathcal{L}(\mathbf{x}, \dot{\mathbf{x}})$ is the action of the particle, corresponding to the Lagrangian $\mathcal{L}$. The method of path integration [1] aims at a calculation of such sums for various physical systems.

We start with a calculation of the one-dimensional quantum-mechanical path integral for a free particle. First, we use the so-called Method of mathematical induction to prove that

$$\int_{-\infty}^{+\infty} dq_1 \cdots dq_n \exp\left\{ i\lambda \left[(q_1 - q')^2 + (q_2 - q_1)^2 + \cdots + (q'' - q_n)^2 \right] \right\} = \\ = \sqrt{\left(\frac{i\pi}{\lambda}\right)^n \frac{1}{n+1}} \cdot \mathrm{e}^{\frac{i\lambda (q''-q')^2}{n+1}} \tag{1}$$

For $n = 1$, using the formula $\int_{-\infty}^{+\infty} dx \mathrm{e}^{ax^2+bx} = \sqrt{\frac{\pi}{-a}} \mathrm{e}^{-\frac{b^2}{4a}}$, we have

$$\int_{-\infty}^{+\infty} dq_1 \mathrm{e}^{i\lambda[(q_1-q')^2+(q''-q_1)^2]} = \int_{-\infty}^{+\infty} dq_1 \mathrm{e}^{i\lambda[2q_1^2 - 2q_1(q'+q'') + q'^2 + q''^2]} = \\ = \mathrm{e}^{i\lambda(q'^2+q''^2)} \sqrt{\frac{\pi}{-2i\lambda}} \cdot \mathrm{e}^{-\frac{i\lambda}{2}(q'+q'')^2} = \sqrt{\frac{i\pi}{\lambda} \cdot \frac{1}{2}} \cdot \mathrm{e}^{\frac{i\lambda}{2}(q''-q')^2}.$$

Further, assuming the validity of Eq. (1), let us prove that

$$\int_{-\infty}^{+\infty} dq_1 \cdots dq_{n+1} \exp\left\{ i\lambda \left[(q_1 - q')^2 + \cdots + (q'' - q_{n+1})^2 \right] \right\} = \\ = \sqrt{\left(\frac{i\pi}{\lambda}\right)^{n+1} \frac{1}{n+2}} \cdot \mathrm{e}^{\frac{i\lambda (q''-q')^2}{n+2}}. \tag{2}$$

We have

$$\int_{-\infty}^{+\infty} dq_1 \cdots dq_{n+1} \exp\left\{ i\lambda \left[(q_1 - q')^2 + \cdots + (q_{n+1} - q_n)^2 + (q'' - q_{n+1})^2 \right] \right\} = \\ = \sqrt{\left(\frac{i\pi}{\lambda}\right)^n \frac{1}{n+1}} \int_{-\infty}^{+\infty} dq_{n+1} \exp\left[\frac{i\lambda}{n+1}(q_{n+1} - q')^2 + i\lambda(q'' - q_{n+1})^2 \right] =$$

$$= \sqrt{\left(\frac{i\pi}{\lambda}\right)^n \frac{1}{n+1}} \cdot \mathrm{e}^{i\lambda\left(\frac{q'^2}{n+1}+q''^2\right)} \cdot \sqrt{\frac{\pi(n+1)}{-i\lambda(n+2)}} \cdot \mathrm{e}^{\frac{4\lambda^2\left(\frac{q'}{n+1}+q''\right)^2(n+1)}{4i\lambda(n+2)}} =$$

$$= \sqrt{\left(\frac{i\pi}{\lambda}\right)^{n+1} \frac{1}{n+2}} \cdot \exp\left\{i\lambda\left[\frac{q'^2}{n+1} + q''^2 - \frac{1}{n+2}\left(\frac{q'^2}{n+1} + 2q'q'' + (n+1)q''^2\right)\right]\right\}.$$

Noting that

$$\frac{q'^2}{n+1} + q''^2 - \frac{1}{n+2}\left(\frac{q'^2}{n+1} + 2q'q'' + (n+1)q''^2\right) = \frac{(q'-q'')^2}{n+2},$$

we see that Eq. (2) is proven.

We now find an n-dependent constant $A(n)$ in the integration measure

$$\mathcal{D}q(t) = \lim_{n\to\infty} A(n)\, dq_1 \cdots dq_n.$$

Consider the Lagrangian of a free particle, $\mathcal{L} = \frac{m\dot{q}^2}{2}$. One has

$$\langle q'', t''|q', t'\rangle = \int \mathcal{D}q\, \mathrm{e}^{i\int_{t'}^{t''}\mathcal{L}dt} =$$

$$= \lim_{n\to\infty} A(n) \int_{-\infty}^{+\infty} dq_1 \cdots dq_n \exp\left\{\frac{im}{2\varepsilon}\left[(q_1-q')^2 + \cdots + (q''-q_n)^2\right]\right\} =$$

$$= \lim_{n\to\infty} A(n) \left(\frac{2\pi i\varepsilon}{m}\right)^{n/2} \frac{1}{\sqrt{n+1}} \cdot \mathrm{e}^{\frac{im}{2\varepsilon(n+1)}(q''-q')^2}.$$

Noting that $n+1 = \frac{t''-t'}{\varepsilon}$, we continue:

$$\langle q'', t''|q', t'\rangle == \frac{\mathrm{e}^{\frac{im(q''-q')^2}{2(t''-t')}}}{\sqrt{t''-t'}} \cdot \lim_{n\to\infty}\left[A(n)\cdot\varepsilon^{\frac{n+1}{2}}\cdot\left(\frac{2\pi i}{m}\right)^{n/2}\right].$$

Using finally the identity

$$\varepsilon^{\frac{n+1}{2}}\cdot\left(\frac{2\pi i}{m}\right)^{n/2} = \left(\frac{m}{2\pi i}\right)^{1/2}\cdot\left(\frac{2\pi i\varepsilon}{m}\right)^{\frac{n+1}{2}},$$

we obtain

$$\langle q'', t''|q', t'\rangle == \sqrt{\frac{m}{2\pi i(t''-t')}} \cdot \mathrm{e}^{\frac{im(q''-q')^2}{2(t''-t')}} \cdot \lim_{n\to\infty}\left[A(n)\cdot\left(\frac{2\pi i\varepsilon}{m}\right)^{\frac{n+1}{2}}\right].$$

A normalization condition, which eliminates the n-dependence completely, can be imposed by demanding $\lim_{n\to\infty}[\cdots]$ to be equal to a constant. This constant is conventionally fixed to 1, that yields the desired result

$$A(n) = \left(\frac{m}{2\pi i\varepsilon}\right)^{\frac{n+1}{2}}. \tag{3}$$

The amplitude of transition during the time $(t''-t')$ can be represented as an infinite product of amplitudes describing transitions over infinitesimal time intervals:

$$\langle q'',t''|q',t'\rangle = \lim_{n\to\infty}\int_{-\infty}^{+\infty} dq_1\cdots dq_n\ \langle q'',t''|q_n,t_n\rangle\langle q_n,t_n|q_{n-1},t_{n-1}\rangle\cdots\langle q_1,t_1|q',t'\rangle,$$

where $|q,t\rangle = \mathrm{e}^{iHt}\,|q\rangle$. An amplitude over an infinitesimal time interval can be evaluated as follows:

$$\begin{aligned}&\langle q_{k+1},t_{k+1}|q_k,t_k\rangle = \langle q_{k+1}|\mathrm{e}^{-i\varepsilon H}|q_k\rangle \simeq \langle q_{k+1}|(1-i\varepsilon H)|q_k\rangle = \\ &= \delta(q_{k+1}-q_k) - i\varepsilon\langle q_{k+1}|H|q_k\rangle = \int_{-\infty}^{+\infty}\frac{dp_k}{2\pi}e^{ip_k(q_{k+1}-q_k)} - i\varepsilon\langle q_{k+1}|H|q_k\rangle.\end{aligned} \tag{4}$$

For a Hamiltonian of the general form, $H=\frac{\hat{p}^2}{2m}+V(q)$, we have

$$\left\langle q_{k+1}\left|\frac{\hat{p}^2}{2m}\right|q_k\right\rangle = \int_{-\infty}^{+\infty}dp'\int_{-\infty}^{+\infty}dp\langle q_{k+1}|p'\rangle\left\langle p'\left|\frac{\hat{p}^2}{2m}\right|p\right\rangle\langle p|q_k\rangle =$$

$$= \int_{-\infty}^{+\infty}\frac{dp'dp}{2\pi}\mathrm{e}^{i(p'q_{k+1}-pq_k)}\cdot\frac{p^2}{2m}\cdot\delta(p-p') = \int_{-\infty}^{+\infty}\frac{dp}{2\pi}\mathrm{e}^{ip(q_{k+1}-q_k)}\cdot\frac{p^2}{2m},$$

$$\langle q_{k+1}|V(q)|q_k\rangle \simeq V\left(\frac{q_{k+1}+q_k}{2}\right)\cdot\langle q_{k+1}|q_k\rangle = \int_{-\infty}^{+\infty}\frac{dp}{2\pi}\mathrm{e}^{ip(q_{k+1}-q_k)}V(\bar{q}_k).$$

Here, in the second step, we have used the equality $\langle q_{k+1}|p'\rangle = \frac{\mathrm{e}^{ip'q_{k+1}}}{\sqrt{2\pi}}$, and in the last step we have denoted $\bar{q}_k \equiv \frac{q_{k+1}+q_k}{2}$, and used the equality $\langle q_{k+1}|q_k\rangle = \delta(q_{k+1}-q_k)$. Altogether, we can write

$$\langle q_{k+1}|H|q_k\rangle = \int_{-\infty}^{+\infty}\frac{dp}{2\pi}\mathrm{e}^{ip(q_{k+1}-q_k)}H(p,\bar{q}_k),$$

where $H(p,\bar{q}_k)=\frac{p^2}{2m}+V(\bar{q}_k)$, and Eq. (4) reads

$$\langle q_{k+1},t_{k+1}|q_k,t_k\rangle = \int_{-\infty}^{+\infty}\frac{dp_k}{2\pi}\mathrm{e}^{ip_k(q_{k+1}-q_k)}[1-i\varepsilon H(p_k,\bar{q}_k)].$$

Promoting the ε-term back to the exponential, we have

$$\langle q_{k+1},t_{k+1}|q_k,t_k\rangle = \int_{-\infty}^{+\infty}\frac{dp_k}{2\pi}\mathrm{e}^{ip_k(q_{k+1}-q_k)-i\varepsilon H(p_k,\bar{q}_k)}.$$

Therefore, noting that $q_0 = q'$, $q_{n+1} = q''$, one has

$$\langle q'', t''|q', t'\rangle = \lim_{n\to\infty} \int_{-\infty}^{+\infty} dq_1 \cdots dq_n \frac{dp_0}{2\pi} \cdots \frac{dp_n}{2\pi} \exp\left\{ i \sum_{k=0}^{n} \left[p_k(q_{k+1} - q_k) - \epsilon H(p_k, \bar{q}_k)\right]\right\}.$$

This expression can symbolically be written as

$$\langle q'', t''|q', t'\rangle = \int \frac{\mathcal{D}q\mathcal{D}p}{2\pi} \exp\left\{ i \int_{t'}^{t''} dt \left[p\dot{q} - H(p, q)\right]\right\}.$$

This formula is a general expression for the case when H = (any function of $\hat{p}$) + (any function of q). In the continuum limit, q is a function of t, and we are left with the integral over functions, i.e. the functional integral. Note that $p(t)$ is also a function, not an operator, and $\{q(t), p(t)\}$ are trajectories in the phase space. Furthermore, the Gaussian p-integrations in the formula

$$\langle q'', t''|q', t'\rangle =$$
$$= \lim_{n\to\infty} \int_{-\infty}^{+\infty} dq_1 \cdots dq_n \frac{dp_0}{2\pi} \cdots \frac{dp_n}{2\pi} \exp\left\{ i \sum_{k=0}^{n} \left[p_k(q_{k+1} - q_k) - \frac{p_k^2}{2m} \cdot \varepsilon - V(\bar{q}_k) \cdot \varepsilon\right]\right\}$$

can be performed explicitly:

$$\int_{-\infty}^{+\infty} \frac{dp_k}{2\pi} \exp\left\{ i \left[p_k(q_{k+1} - q_k) - \frac{p_k^2}{2m} \cdot \varepsilon\right]\right\} = \sqrt{\frac{m}{2\pi i \varepsilon}} \mathrm{e}^{\frac{im(q_{k+1}-q_k)^2}{2\varepsilon}}.$$

In particular, in the free-particle case, $V \equiv 0$, one has

$$\langle q'', t''|q', t'\rangle = \lim_{n\to\infty} \left(\frac{m}{2\pi i \varepsilon}\right)^{\frac{n+1}{2}} \int_{-\infty}^{+\infty} dq_1 \cdots dq_n \exp\left\{ \frac{im}{2\epsilon} \left[(q' - q_1)^2 + \cdots + (q'' - q_n)^2\right]\right\},$$

recovering in this way Eq. (3) for $A(n)$.

We reproduce now the Schrödinger equation in the limit $t'' \to t'$, where the particle does not manage to move somewhat significantly away from q'. We introduce the following notations: $K(q'', t''|q', t') \equiv \langle q'', t''|q', t'\rangle$, $t' = t$, $q' = y$, $q'' = x$, $t'' = t + \varepsilon$, $y - x = \xi$. Then

$$\psi(x, t + \varepsilon) = \int_{-\infty}^{+\infty} dy K(x, t + \varepsilon|y, t)\psi(y, t), \qquad \text{where}$$

$$K(x, t + \varepsilon|y, t) = \sqrt{\frac{m}{2\pi i \varepsilon}} \exp\left[\frac{im}{2\varepsilon}(x - y)^2 - i\varepsilon V\left(\frac{x + y}{2}\right)\right].$$

Expanding both sides of this integral equation for ψ up to the terms linear in ε, we have

$$\psi(x, t) + \varepsilon \frac{\partial}{\partial t}\psi(x, t) = \sqrt{\frac{m}{2\pi i \varepsilon}} \int_{-\infty}^{+\infty} d\xi \exp\left[\frac{im\xi^2}{2\varepsilon} - i\varepsilon V\left(x + \frac{\xi}{2}\right)\right] \psi(x + \xi, t) \simeq$$

$$\simeq \sqrt{\frac{m}{2\pi i\varepsilon}} \int_{-\infty}^{+\infty} d\xi \mathrm{e}^{\frac{im\xi^2}{2\varepsilon}} [1 - i\varepsilon V(x)] \left[\psi(x,t) + \xi \frac{\partial \psi}{\partial x} + \frac{\xi^2}{2}\frac{\partial^2 \psi}{\partial x^2}\right] = \psi(x,t) + \sqrt{\frac{m}{2\pi i\varepsilon}} \times$$

$$\times \left\{ \frac{\partial \psi}{\partial x} \cdot \int_{-\infty}^{+\infty} d\xi \cdot \xi \cdot \mathrm{e}^{\frac{im\xi^2}{2\varepsilon}} + \frac{1}{2}\frac{\partial^2 \psi}{\partial x^2} \cdot \int_{-\infty}^{+\infty} d\xi \cdot \xi^2 \cdot \mathrm{e}^{\frac{im\xi^2}{2\varepsilon}} - i\varepsilon V(x)\psi \cdot \int_{-\infty}^{+\infty} d\xi \mathrm{e}^{\frac{im\xi^2}{2\varepsilon}} \right\} =$$

$$= \psi + \frac{i\varepsilon}{2m}\frac{\partial^2 \psi}{\partial x^2} - i\varepsilon V(x)\psi.$$

Thus, one arrives at the Schrödinger equation

$$i\frac{\partial \psi}{\partial t} = -\frac{1}{2m} \cdot \frac{\partial^2 \psi}{\partial x^2} + V(x)\psi.$$

Indeed, when the dependence on the Planck constant $\hbar$ is restored, one has

$$K(x, t+\varepsilon|y,t) = \sqrt{\frac{m}{2\pi i\hbar\varepsilon}} \exp\left[\frac{im(x-y)^2}{2\hbar\varepsilon} - \frac{i\varepsilon}{\hbar}V\left(\frac{x+y}{2}\right)\right] \quad \Rightarrow$$

$$\Rightarrow \quad \psi + \varepsilon\frac{\partial \psi}{\partial t} = \psi + \frac{i\hbar\varepsilon}{2m}\frac{\partial^2 \psi}{\partial x^2} - \frac{i\varepsilon}{\hbar}V\psi \quad \Rightarrow \quad i\hbar\frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + V\psi,$$

i.e. the Schrödinger equation in the conventional form.

We calculate now the path integral for the harmonic oscillator of mass $m = 1$:

$$\mathcal{L} = \frac{\dot{z}^2}{2} - V(z), \qquad S = \int_0^T dt \mathcal{L}(z, \dot{z}), \qquad \langle x|\mathrm{e}^{-iHT}|y\rangle = \int (\mathcal{D}z)_{xy} \mathrm{e}^{iS[z(t)]},$$

where $(\mathcal{D}z)_{xy}$ denotes the measure of integration over paths $z(t)$ such that $z(0) = y$, $z(T) = x$. Proceeding from the states with definite coordinate to the states with definite energy, $H|n\rangle = E_n|n\rangle$, one gets the expression

$$\langle x|\mathrm{e}^{-iHT}|y\rangle = \sum_n \mathrm{e}^{-iE_nT}\langle x|n\rangle\langle n|y\rangle,$$

which involves the sum of oscillating exponents. If we are interested in the ground state, it is convenient to transform the oscillating exponents to the decreasing ones. That is achieved by the Wick rotation $t \to -i\tau$, which yields the Euclidean quantum mechanics. Then, in the limit $T \to \infty$, only the term $\mathrm{e}^{-E_0T}\psi_0(x)\psi_0^*(y)$ in the sum $\sum\limits_n$ survives. One further has

$$iS = i\int_0^T dt \left[\frac{1}{2}\left(i\frac{dz}{d\tau}\right)^2 - V\right] = \int_0^T d\tau \left[-\frac{1}{2}\left(\frac{dz}{d\tau}\right)^2 - V\right],$$

where it has been used that $dt = -id\tau$. We denote the Euclidean action $S_E \equiv \int_0^T d\tau \left[\frac{1}{2}\left(\frac{dz}{d\tau}\right)^2 + V(z)\right]$ as just S. Then, the integral of interest, $\int \mathcal{D}z\mathrm{e}^{-S[z(\tau)]}$, is accumulated in the regions near the minima of S. We denote by $\bar{z}(\tau)$ the path corresponding to the minimal action S_0 (also called the extremal path or the stationary point of the path integral): $S_0 = S[\bar{z}(\tau)]$. Then $\int \mathcal{D}z \mathrm{e}^{-S} \sim \mathrm{e}^{-S_0}$. (For S possessing several stationary points, the right-hand side of this relation is replaced by $\sum\limits_i \mathrm{e}^{-S[\bar{z}_i(\tau)]}$.) As a next step, we fix the pre-exponential factor. For simplicity, we consider the case where only one stationary point of S exists. Decomposing z into $\bar{z}$ and a small fluctuation $\xi \equiv \delta z$ as $z = \bar{z} + \xi$, one has

$$\dot{z}^2 = \dot{\bar{z}}^2 + 2\dot{\bar{z}}\dot{\xi} + \dot{\xi}^2 \quad \Rightarrow \quad \int_0^T d\tau \frac{d\dot{z}^2}{d\xi} = -\int_0^T d\tau (2\ddot{\bar{z}} + \ddot{\xi}) \quad \Rightarrow$$

$$\Rightarrow \quad \delta S = \int_0^T d\tau \cdot \xi \cdot \left[-\ddot{\bar{z}} - \frac{1}{2}\ddot{\xi} + V'(\bar{z}) + \frac{1}{2}\xi \cdot V''(\bar{z}) \right].$$

After the Wick rotation, the potential has changed the sign, and $\ddot{\bar{z}} = V'(\bar{z})$ is the classical equation of motion in the potential $-V(z)$. The action expanded around S_0 takes the form

$$S = S_0 + \frac{1}{2}\int_0^T d\tau \cdot \xi \cdot \left[-\ddot{\xi} + V''(\bar{z}) \cdot \xi \right].$$

Suppose that we know a complete set of eigenfunctions and eigenvalues of the equation

$$-\ddot{z}_n(\tau) + V''(\bar{z}) z_n(\tau) = \varepsilon_n z_n(\tau).$$

Then, by using the Hilbert-Schmidt orthogonalization procedure, these functions can be made orthonormal, $\int_0^T d\tau z_n(\tau) z_m(\tau) = \delta_{mn}$. An arbitrary function $\xi(\tau)$ can be represented as $\xi(\tau) = \sum\limits_n c_n z_n(\tau)$, and we have

$$\int_0^T d\tau \cdot \xi \cdot \left[-\ddot{\xi} + V''(\bar{z}) \cdot \xi \right] =$$

$$= \int_0^T d\tau \left\{ \sum_n c_n z_n \cdot \left[-\sum_m c_m \ddot{z}_m + V''(\bar{z}) \cdot \sum_m c_m z_m \right] \right\} = \sum_n c_n^2 \varepsilon_n \quad \Rightarrow$$

$$\Rightarrow \quad S = S_0 + \frac{1}{2}\sum_n \varepsilon_n c_n^2,$$

where we have replaced $-\sum\limits_m c_m \ddot{z}_m + V''(\bar{z}) \cdot \sum\limits_m c_m z_m$ by $\sum\limits_m c_m \varepsilon_m z_m$, and used the orthonormality of z_n's. Next, one may always replace $\mathcal{D}z$ by $\prod\limits_n \frac{dc_n}{\sqrt{2\pi}}$, since the proportionality constant between these two measures has some meaning only when the overall normalization of the path integral is fixed. (That will be done below.) Then

$$\int_{-\infty}^{+\infty} \frac{dc_n}{\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}\varepsilon_n c_n^2} = \frac{1}{\sqrt{\varepsilon_n}} \quad \Rightarrow \quad \int \mathcal{D}z \mathrm{e}^{-S} \propto \mathrm{e}^{-S_0} \prod_n \varepsilon_n^{-1/2}. \tag{5}$$

Note that, symbolically, by an analogy to the case of finite-dimensional matrices, one can write

$$\prod_n \varepsilon_n^{-1/2} = \left[\det\left(-\frac{d^2}{d\tau^2} + V''(\bar{z}(\tau))\right)\right]^{-1/2}.$$

We can now specify the potential of an oscillator:

$$V = \left.\frac{m\omega^2 z^2}{2}\right|_{m=1} = \frac{\omega^2 z^2}{2} \quad \Rightarrow \quad V'' = \omega^2.$$

For the eigenvalues and eigenfunctions we have

$$\left.\begin{array}{c} \left(-\frac{d^2}{d\tau^2} + \omega^2\right) z_n = \varepsilon_n z_n \\ z_n(0) = z_n(T) = 0 \end{array}\right\} \quad \Rightarrow \quad \varepsilon_n = \left(\frac{\pi n}{T}\right)^2 + \omega^2, \qquad z_n = A \sin\left(\frac{\pi n \tau}{T}\right).$$

Replacing the symbol "∝" in Eq. (5) by an unknown constant N, one has

$$N \prod_{n=1}^{\infty} \left[\left(\frac{\pi n}{T}\right)^2 + \omega^2\right]^{-1/2} = N \left[\prod_{n=1}^{\infty} \left(\frac{\pi n}{T}\right)^2\right]^{-1/2} \cdot \left[\prod_{n=1}^{\infty} \left(1 + \frac{\omega^2 T^2}{\pi^2 n^2}\right)\right]^{-1/2}.$$

We introduce yet another normalization factor $\mathcal{N} = N \left[\prod\limits_{n=1}^{\infty} \left(\frac{\pi n}{T}\right)^2\right]^{-1/2}$, which does not depend on ω and therefore corresponds to the free motion of a particle.

Next, the so-called Fundamental theorem of algebra states that a polynomial of degree n has exactly n (in general, complex-valued) roots, and can therefore be represented as $P_n(x) = a_0 \left(1 - \frac{x}{x_1}\right) \cdots \left(1 - \frac{x}{x_n}\right)$, where $x_1, \ldots, x_n$ are the roots of P_n, and a_0 is its free term. Consider a generalization of this theorem to the case $n \to \infty$, and take as an infinite-degree polynomial the Taylor series of the function $\frac{\sinh x}{x}$:

$$\frac{\sinh x}{x} = 1 + \frac{x^2}{6} + \cdots \qquad \Rightarrow a_0 = 1.$$

Since $\sinh x = -i \sin(ix)$, the roots of this "polynomial" are the same as those of $\sin(ix)$, i.e. $\pm i\pi, \pm 2\pi i, \ldots$. Therefore,

$$\frac{\sinh x}{x} = \left(1 - \frac{x}{i\pi}\right)\left(1 + \frac{x}{i\pi}\right) \cdot \left(1 - \frac{x}{2\pi i}\right)\left(1 + \frac{x}{2\pi i}\right) \cdots =$$

$$= \left(1 + \frac{x^2}{\pi^2}\right)\left(1 + \frac{x^2}{(2\pi)^2}\right) \cdots = \prod_{n=1}^{\infty} \left(1 + \frac{x^2}{(\pi n)^2}\right) \qquad \Rightarrow$$

$$\Rightarrow \quad \prod_{n=1}^{\infty} \left(1 + \frac{\omega^2 T^2}{\pi^2 n^2}\right) = \frac{\sinh(\omega T)}{\omega T} \quad \Rightarrow \quad \int (\mathcal{D}z)_{xy} \mathrm{e}^{-S} = \mathcal{N} \mathrm{e}^{-S_0} \left[\frac{\sinh(\omega T)}{\omega T}\right]^{-1/2},$$

where S_0 can be found from the classical equation of motion, and reads

$$S_0 = \frac{\omega}{2\sinh(\omega T)}\left[(x^2+y^2)\cosh(\omega T) - 2xy\right].$$

In the limit $\omega \to 0$, the path integral for a free particle, $\frac{1}{\sqrt{2\pi T}}\mathrm{e}^{-\frac{(x-y)^2}{2T}}$, should be recovered, yielding $\mathcal{N} = \frac{1}{\sqrt{2\pi T}}$. This completes the calculation. Finally, one finds

$$\langle x=0|\mathrm{e}^{-HT}|y=0\rangle = \frac{1}{\sqrt{2\pi T}}\left[\frac{\omega T}{\sinh(\omega T)}\right]^{1/2} =$$

$$= \left[\frac{\omega}{2\pi\sinh(\omega T)}\right]^{1/2} \longrightarrow \sqrt{\frac{\omega}{\pi}}\mathrm{e}^{-\frac{\omega T}{2}}\left(1+\frac{1}{2}\mathrm{e}^{-2\omega T}+\cdots\right) \quad \text{at} \quad T\to\infty.$$

The leading term of this expression reproduces the known quantum-mechanical results for the eigenenergy and the wave-function of the ground state of the oscillator: $E_0 = \frac{\omega}{2}$, $[\psi_0(0)]^2 = \sqrt{\frac{\omega}{\pi}}$. The next-to-leading term corresponds to the $(n=2)$-state of the oscillator, while odd n's do not contribute, since for them $\psi_n(0) = 0$.

We will now discuss an analogy between the Euclidean formulation of quantum mechanics in D dimensions and statistical mechanics in D spatial and 1 temporal dimensions in equilibrium. To find such an analogy, we notice that the thermodynamic properties of an equilibrium system in $(D+1)$ dimensions are determined by the thermal partition function

$$\mathcal{Z}(\beta, V) = \sum_n \mathrm{e}^{-\beta E_n} \equiv \mathrm{tr}\,\mathrm{e}^{-\beta H}, \qquad \text{where} \qquad \beta = \frac{1}{\text{temperature}},$$

and the trace is taken over the complete set of states $\{\psi_n\}$, such that $H\psi_n = E_n\psi_n$. For the rest of this Section, we will not use temperature in the formulae, since it is denoted by the same letter T as the proper time in the path integral. To avoid possible confusions related to that, we will rather use the inverse temperature β.

A statistical-mechanics counterpart of the propagator is the thermal density matrix

$$\langle\mathbf{x}|\mathrm{e}^{-\beta H}|\mathbf{y}\rangle = \sum_n \mathrm{e}^{-\beta E_n}\psi_n^*(\mathbf{x})\psi_n(\mathbf{y}),$$

where from now on in this Section we denote D-dimensional vectors as $\mathbf{x}$ and the eigenstates of the position operator as $|\mathbf{x}\rangle$. Thus, an analogy between Euclidean quantum mechanics in D dimensions and statistical mechanics in D spatial and one temporal dimensions, whose time-dependence disappears in equilibrium, can be established as follows. The path-integral representation for the thermal density matrix is given by

$$\int\limits_{\substack{\mathbf{z}(0)=\mathbf{y}\\ \mathbf{z}(\beta)=\mathbf{x}}} \mathcal{D}\mathbf{z}(t)\,\mathrm{e}^{-S_E}$$

with $S_E = \int_0^\beta dt\left[\frac{1}{2}\left(\frac{d\mathbf{z}}{dt}\right)^2 + \bar{V}(\mathbf{z})\right]$, where $\bar{V}$ denotes the potential (to distinguish it from the volume V). From now on, unless the opposite is explicitly stated, t denotes the Euclidean time. The partition function is the spatial integral of the diagonal element of the thermal density matrix:

$$\mathcal{Z}(\beta, V) = \int\limits_V d^D x\,\langle\mathbf{x}|\mathrm{e}^{-\beta H}|\mathbf{x}\rangle = V\cdot \int\limits_{\mathbf{z}(0)=\mathbf{z}(\beta)} \mathcal{D}\mathbf{z}(t)\,\mathrm{e}^{-S_E}.$$

Recalling that, for a free particle of mass 1, $\int_{\mathbf{z}(0)=\mathbf{z}(T)} \mathcal{D}\mathbf{z}(t)\,\mathrm{e}^{-S_E} = \frac{1}{(2\pi T)^{D/2}}$, we obtain for a particle of mass m: $\mathcal{Z}(\beta, V) = V\left(\frac{m}{2\pi\beta}\right)^{D/2}$. One can see that this expression does coincide with the one following from the Boltzmann distribution in classical statistics, where $E(\mathbf{p}) = \frac{\mathbf{p}^2}{2m}$ is the energy of a free nonrelativistic particle, which reads

$$\mathcal{Z}(\beta, V) = V\cdot\int \frac{d^D p}{(2\pi)^D}\mathrm{e}^{-\beta E(\mathbf{p})} = \frac{V}{(2\pi)^D}\left(\frac{2\pi m}{\beta}\right)^{D/2} = V\left(\frac{m}{2\pi\beta}\right)^{D/2}.$$

Furthermore, as follows from the path integral, the thermal density matrix in the free case reads

$$\langle\mathbf{x}|\mathrm{e}^{-\beta H}|\mathbf{y}\rangle = \left(\frac{m}{2\pi\beta}\right)^{D/2}\mathrm{e}^{-\frac{m(\mathbf{x}-\mathbf{y})^2}{2\beta}}.$$

Accordingly, an alternative derivation, which uses the Boltzmann statistics, is based on the formula

$$\langle\mathbf{x}|\mathrm{e}^{-\beta H}|\mathbf{y}\rangle = \sum_n \mathrm{e}^{-\beta E_n}\psi_n^*(\mathbf{x})\psi_n(\mathbf{y}).$$

Using for eigenfunctions of the free Hamiltonian plane waves normalized in the spatial volume V, i.e. $\psi_n(\mathbf{y}) = \psi_{\mathbf{p}}(\mathbf{y}) = \frac{1}{\sqrt{V}}\mathrm{e}^{-i\mathbf{p}\mathbf{y}}$, we do reproduce the above result:

$$\langle\mathbf{x}|\mathrm{e}^{-\beta H}|\mathbf{y}\rangle = \int \frac{d^D p}{(2\pi)^D}\,\mathrm{e}^{i\mathbf{p}(\mathbf{x}-\mathbf{y})-\frac{\beta\mathbf{p}^2}{2m}} = \left(\frac{m}{2\pi\beta}\right)^{D/2}\mathrm{e}^{-\frac{m(\mathbf{x}-\mathbf{y})^2}{2\beta}}.$$

In particular, the thermal density matrix for a harmonic oscillator reads

$$\langle\mathbf{x}|\mathrm{e}^{-\beta H}|\mathbf{y}\rangle = \left[\frac{m\omega}{2\pi\sinh(\omega\beta)}\right]^{D/2}\exp\left\{-\frac{m\omega}{2\sinh(\omega\beta)}\left[(\mathbf{x}^2+\mathbf{y}^2)\cosh(\omega\beta) - 2\mathbf{x}\mathbf{y}\right]\right\}.$$

In conclusion of this Section, we make the following final remark. Consider a one-dimensional chain of point-like masses connected by springs. The potential energy of such a chain reads

$$E_{\rm pot}=\sum_{i=1}^{N}\left[\frac{m}{2}\cdot\frac{(z_i-z_{i-1})^2}{t_i-t_{i-1}}+\bar V(z_{i-1})\cdot(t_i-t_{i-1})\right],$$

where we have assumed an additional interaction, with energy-density $\bar V$, between the neighbors. At temperature $1/\beta$, the partition function of this system has the form

$$\mathcal{Z}=\int\prod_{i=1}^{N}dz_i\,{\rm e}^{-\beta E_{\rm pot}}\to\int\mathcal{D}z(t)\,{\rm e}^{-\beta E_{\rm pot}[z(t)]}\quad \text{at}\quad N\to\infty.$$

Therefore, one observes the following correspondence. Consider two states of a particle and a quantum transition between these two states, which occurs during the Euclidean time T. Then, the amplitude of this transition is equal to the statistical sum of a one-dimensional classical string of length T [in general, in an external potential $\bar V(x)$], at the temperature $\frac{1}{\beta}=\hbar$.

2. A free-boson propagator at finite temperature. A path-integral derivation of the partition function of an ideal Bose gas in quantum statistics.

To calculate the path integral

$$\int\limits_{\substack{z_\mu(0)=y_\mu\\ z_\mu(T)=x_\mu}}\mathcal{D}z_\mu(t)\,{\rm e}^{-\frac12\int_0^T dt\dot z_\mu^2(t)},\tag{6}$$

it is useful to proceed to the integration over closed paths as follows: $z_\mu\to\xi_\mu=z_\mu+\frac{y_\mu-x_\mu}{T}t-y_\mu$. Then $\xi_\mu(0)=\xi_\mu(T)=0$, while the exponent takes the form

$$\int_0^T dt\dot z_\mu^2=\int_0^T dt\left(\dot\xi_\mu+\frac{x_\mu-y_\mu}{T}\right)^2=\int_0^T dt\,\dot\xi_\mu^2+\frac{(x-y)^2}{T}+\frac{2}{T}(x_\mu-y_\mu)\cdot[\xi_\mu(T)-\xi_\mu(0)]$$

$$\Rightarrow\qquad \text{Eq. (6)}={\rm e}^{-\frac{(x-y)^2}{2T}}\cdot f(T).$$

We have denoted

$$f(T)\equiv\int\limits_{\xi_\mu(0)=\xi_\mu(T)=0}\mathcal{D}\xi_\mu\,{\rm e}^{-\frac12\int_0^T dt\dot\xi_\mu^2(t)},$$

that is possible since this path integral is a function of T only. It can most easily be calculated by the comparison with the proper-time representation of the Euclidean propagator for a free massless boson. Indeed, the propagator obeys the equation

$$-\partial^2 G(x-y) = \delta^{(D)}(x-y),$$

which, through the Fourier transform, leads to the following proper-time representation:

$$G(x-y) = \int \frac{d^Dp}{(2\pi)^D} \frac{\mathrm{e}^{ip(y-x)}}{p^2} = \int \frac{d^Dp}{(2\pi)^D} \mathrm{e}^{ip(y-x)} \cdot \frac{1}{2}\int_0^\infty dT \mathrm{e}^{-\frac{p^2T}{2}} =$$
$$= \frac{1}{2}\int_0^\infty dT \mathrm{e}^{-\frac{(x-y)^2}{2T}} \cdot \frac{1}{(2\pi T)^{D/2}}.$$

Comparing it with the representation of the propagator in terms of the path integral,

$$G(x-y) = \frac{1}{2}\int_0^\infty dT \int\limits_{\substack{z_\mu(0)=y_\mu \\ z_\mu(T)=x_\mu}} \mathcal{D}z_\mu(t)\, \mathrm{e}^{-\frac{1}{2}\int_0^T dt \dot{z}_\mu^2(t)},$$

we conclude that

$$f(T) = \frac{1}{(2\pi T)^{D/2}}.$$

A new element of this derivation was the introduction of the *proper* time T.

Next, one can prove that, for a Hermitian operator $\hat{D}$, the equality

$$\ln \det \hat{D} = \mathrm{tr}\, \ln \hat{D} \tag{7}$$

holds. That can be seen by reducing $\hat{D}$ to a diagonal form by a unitary transformation. Denoting by D_i's (positive) eigenvalues of $\hat{D}$, one has $\ln \prod\limits_i D_i = \sum\limits_i \ln D_i$, that proves the desired equality. Then, the representation

$$\langle x|\frac{1}{(-\partial^2+m^2)}|y\rangle = \frac{1}{2}\int_0^\infty dT \mathrm{e}^{-\frac{1}{2}m^2T} \int\limits_{\substack{z_\mu(0)=y_\mu \\ z_\mu(T)=x_\mu}} \mathcal{D}z_\mu(t)\, \mathrm{e}^{-\frac{1}{2}\int_0^T dt \dot{z}_\mu^2(t)}$$

can be used to calculate $\ln\det(-\partial^2+m^2)$. Indeed, formally integrating the equation

$$\langle x|\frac{1}{\hat{D}}|y\rangle = \frac{1}{2}\int_0^\infty dT \langle x|\mathrm{e}^{-\frac{T\hat{D}}{2}}|y\rangle$$

over $\hat{D}$, we obtain (up to an inessential constant of integration):

$$\langle x|\ln \hat{D}\,|y\rangle = -\int_0^\infty \frac{dT}{T}\langle x|\mathrm{e}^{-\frac{T\hat{D}}{2}}|y\rangle. \tag{8}$$

Using Eqs. (7) and (8), one obtains

$$\ln\det(-\partial^2+m^2)=\mathrm{tr}\,\ln(-\partial^2+m^2)=-\int_0^\infty\frac{dT}{T}\langle x|\mathrm{e}^{-\frac{T}{2}(-\partial^2+m^2)}|x\rangle=$$

$$=-\int_0^\infty\frac{dT}{T}\mathrm{e}^{-\frac{1}{2}m^2T}\int\limits_{z_\mu(0)=z_\mu(T)}\mathcal{D}z_\mu(t)\mathrm{e}^{-\frac{1}{2}\int_0^T dt\dot{z}_\mu^2(t)}.$$

We consider now the following representation of the Green function of the Laplacian in D Euclidean dimensions:

$$\langle x|(-\partial^{-2})|y\rangle=\int_0^\infty dsP(s,R),\tag{9}$$

where $R_\mu=x_\mu-y_\mu$, $R=|x-y|$, and $\frac{T}{2}\equiv s$ is the Schwinger proper time (while the symbol T below in this Section will be reserved for temperature). In this representation, the principal quantity is the probability for a random walker to evolve, during the proper time s of the random walk, to the distance R from the starting point. This probability reads

$$P(s,R)=\frac{\mathrm{e}^{-\frac{R^2}{4s}}}{(4\pi s)^{D/2}}$$

One can readily see that $P(s,R)$ respects the conservation law

$$\int d^DR\,P(s,R)=1\tag{10}$$

and the initial condition

$$\lim_{s\to 0}P(s,R)=\delta^{(D)}(R).\tag{11}$$

In particular, performing the s-integration in Eq. (9), one obtains the D-dimensional Coulomb (or Newton) law:

$$\langle x|(-\partial^{-2})|y\rangle=\frac{\Gamma\left(\frac{D}{2}-1\right)}{4\pi^{D/2}R^{D-2}}.$$

For $D=3$ and $D=4$, this law takes the conventional forms, $\langle x|(-\partial^{-2})|y\rangle\Big|_{D=3}=\frac{1}{4\pi R}$ and

$$\langle x|(-\partial^{-2})|y\rangle\Big|_{D=4}=\frac{1}{4\pi^2R^2}.\tag{12}$$

With these preliminaries, we proceed to the theory of a free massive scalar field at temperature T, in the spatial volume V. The logarithm of the partition function of this theory reads

$$\ln\mathcal{Z}(T,V)=\ln\int\limits_{\varphi(\mathbf{x},\beta)=\varphi(\mathbf{x},0)}\mathcal{D}\varphi(\mathbf{x},t)\exp\left\{-\int_0^\beta dt\int_V d^Dx\left[\frac{1}{2}(\partial_\mu\varphi)^2+\frac{m^2}{2}\varphi^2\right]\right\}=$$

$$= \ln\left\{\left[\det(-\partial_\mu^2+m^2)\right]^{-1/2}\right\} = -\frac{1}{2}\ln\det(-\partial_\mu^2+m^2) = -\frac{1}{2}\mathrm{tr}\,\ln(-\partial_\mu^2+m^2) =$$

$$= -\frac{1}{2}\beta V\int\frac{d^Dp}{(2\pi)^D}\mathrm{tr}_t\ln(-\partial_t^2+\omega^2), \tag{13}$$

where $\omega^2 = \mathbf{p}^2+m^2$. At finite temperature $T\equiv\frac{1}{\beta}$, the coordinate $x_4\equiv t$ becomes periodic with the period β. This means the theory is compactified onto a circle of circumference β.

We now make a digression, and consider a quantum-mechanical problem on a particle moving along a circle of the radius R. (In our case, $R=\frac{\beta}{2\pi}$.) In this digression, t denotes the physical, rather than Euclidean, time. The corresponding Lagrangian reads

$$\mathcal{L} = \frac{mR^2\dot{\phi}^2}{2}, \qquad \text{where} \qquad \dot{\phi}\equiv\frac{d\phi}{dt}.$$

The Schrödinger equation, with $\hbar = 1$, and the periodic boundary conditions yield the spectrum and the corresponding eigenfunctions:

$$\left.\begin{array}{rcl} -\frac{1}{2mR^2}\frac{d^2\psi_l}{d\phi^2} & = & E_l\psi_l, \\ \psi_l(0) & = & \psi_l(2\pi) \\ \psi_l'(0) & = & \psi_l'(2\pi) \end{array}\right\} \Rightarrow\; E_l = \frac{l^2}{2mR^2},\quad \psi_l\sim \mathrm{e}^{il\phi},\quad l\in\mathbf{Z}.$$

The normalization condition for a particle on a line is $\int_{-\infty}^{+\infty}dx\psi^*(x)\psi(x)=1$. In the case of a circle, it becomes $R\int_0^{2\pi}d\phi\psi_l^*(\phi)\psi_l(\phi)=1$, and yields the normalized eigenfunctions

$$\psi_l(\phi) = \frac{\mathrm{e}^{il\phi}}{\sqrt{2\pi R}}.$$

By using these functions and the general formula for the propagation kernel,

$$K(q'',t''|q',t') = \sum_n \psi_n^*(q'')\psi_n(q')\mathrm{e}^{-iE_n(t''-t')},$$

one can write the kernel in the form

$$K(\phi'',t''|\phi',t') = \frac{1}{2\pi R}\sum_{l=-\infty}^{+\infty}\mathrm{e}^{-il\phi-it\frac{l^2}{2mR^2}}, \tag{14}$$

where $\phi\equiv\phi''-\phi'$, $t\equiv t''-t'$. Consider the $(t\to 0)$-limit of this expression. It can be taken with the use of the Poisson sum formula, $\sum\limits_{l=-\infty}^{+\infty}\mathrm{e}^{-il\phi} = 2\pi\sum\limits_{n=-\infty}^{+\infty}\delta(\phi-2\pi n)$. One obtains the formula

$$K(\phi'',t''|\phi',t')\to\frac{1}{R}\sum_{n=-\infty}^{+\infty}\delta(\phi-2\pi n),$$

which means that the particle can get from the point ϕ' to the point ϕ'', passing the circle an arbitrary number of times. The number n is called the winding number. Furthermore, for an arbitrary t, we can transform Eq. (14) by using the following generalization of the Poisson sum formula (which can be called a discrete version of the Gaussian integral):

$$\sum_{l=-\infty}^{+\infty} \mathrm{e}^{-Al^2+iBl} = \sqrt{\frac{\pi}{A}} \cdot \sum_{n=-\infty}^{+\infty} \mathrm{e}^{-\frac{1}{4A}(B-2\pi n)^2}, \quad \text{where } \; A, B \in \mathbf{C}.$$

In our case, $A = \frac{it}{2mR^2}$, $B = -\phi$, and we have

$$K(\phi'', t''|\phi', t') = \sqrt{\frac{m}{2\pi i t}} \sum_{n=-\infty}^{+\infty} \mathrm{e}^{\frac{imR^2}{2t}(\phi+2\pi n)^2}. \tag{15}$$

Our digression on a particle moving along a circle is finished at this point.

In order to return to the path integral for a particle at finite temperature T, we perform the Wick rotation $it \to s$, and fix the mass $m = \frac{1}{2}$. In this way, we arrive at the Schrödinger equation $-\frac{d^2\psi}{dR_4^2} = E\psi$, where $R_4 = R\phi$. Recalling also that $2\pi R = \beta$, we obtain from Eq. (15):

$$K(R_4, s) = \frac{1}{\sqrt{4\pi s}} \sum_{n=-\infty}^{+\infty} \mathrm{e}^{-\frac{1}{4s}(R_4+\beta n)^2}.$$

Denoting $x_\mu - y_\mu = R_\mu = (\mathbf{R}, R_4)$, we have for the propagator at finite temperature T in $(D = 4)$ dimensions:

$$\langle x|(-\partial^{-2})|y\rangle = \sum_{n=-\infty}^{+\infty} \int_0^\infty ds P_n(s, R_\mu).$$

Here, the probabilities

$$P_n(s, R_\mu) = \frac{1}{(4\pi s)^2} \exp\left[-\frac{\mathbf{R}^2 + (R_4 - \beta n)^2}{4s}\right]$$

obey the conservation law $\int d^4R\, P_n(s, R_\mu) = 1$ and the initial condition $\lim_{s\to 0} P_n(s, R_\mu) = \delta^{(3)}(\mathbf{R})\delta(R_4 - \beta n)$, which can be compared with their zero-temperature counterparts, Eqs. (10) and (11). Further, denoting $\xi = \frac{1}{s}$, we obtain

$$\langle x|(-\partial^{-2})|y\rangle =$$

$$= \frac{1}{(4\pi)^2} \sum_{n=-\infty}^{+\infty} \int_0^\infty d\xi \exp\left[-\frac{\mathbf{R}^2 + (R_4 - \beta n)^2}{4} \cdot \xi\right] = \frac{1}{4\pi^2} \sum_{n=-\infty}^{+\infty} \frac{1}{\mathbf{R}^2 + (R_4 - \beta n)^2}.$$

Comparing this expression with Eq. (12), we notice that the presence of finite temperature effectively leads to the substitution $R_\mu^2 \to \mathbf{R}^2 + (R_4 - \beta n)^2$, and the subsequent summation over n.

Consider now the sum $\sum\limits_{k=-\infty}^{+\infty} \mathrm{e}^{-s\omega_k^2 - i\omega_k R_4}$, where $\omega_k \equiv 2\pi T k$ are the so-called Matsubara frequencies. Applying again the formula

$$\sum_{k=-\infty}^{+\infty} \mathrm{e}^{-Ak^2+iBk} = \sqrt{\frac{\pi}{A}} \sum_{n=-\infty}^{+\infty} \mathrm{e}^{-\frac{1}{4A}(B-2\pi n)^2},$$

where $A = (2\pi T)^2 s$, $B = -2\pi T R_4$, we obtain

$$\sum_{k=-\infty}^{+\infty} \mathrm{e}^{-s\omega_k^2 - i\omega_k R_4} = \frac{1}{2T\sqrt{\pi s}} \sum_{n=-\infty}^{+\infty} \exp\left[-\frac{1}{4s}(R_4 - \beta n)^2\right] = \beta \sum_{n=-\infty}^{+\infty} P_n(s, R_4),$$

where

$$P_n(s, R_4) = \frac{1}{2\sqrt{\pi s}} \exp\left[-\frac{(R_4 - \beta n)^2}{4s}\right].$$

Therefore,

$$\sum_{n=-\infty}^{+\infty} P_n(s, R_4) = T \sum_{k=-\infty}^{+\infty} \mathrm{e}^{-s\omega_k^2 - i\omega_k R_4}. \tag{16}$$

We can now finish the calculation of the partition function of a free massive scalar field at finite temperature. Differentiating the trace in Eq. (13) with respect to ω^2, we have

$$\frac{\partial}{\partial \omega^2} \mathrm{tr}_t \ln(-\partial_t^2 + \omega^2) = \mathrm{tr}_t \frac{1}{(-\partial_t^2 + \omega^2)} = T \sum_{k=-\infty}^{+\infty} \int_0^\infty ds \mathrm{e}^{-s(\omega_k^2 + \omega^2)}. \tag{17}$$

Due to the trace, in the last equality we have used Eq. (16) with $R_4 = 0$. Therefore, upon the s-integration,

$$\frac{\partial}{\partial \omega^2} \mathrm{tr}_t \ln(-\partial_t^2 + \omega^2) = T \sum_{k=-\infty}^{+\infty} \frac{1}{(2\pi T k)^2 + \omega^2}.$$

To calculate this sum, we rewrite it as an integral in the complex z-plane over the contour C which encircles the imaginary axis counterclockwise. In the vicinity of the pole $z_k = i\omega_k$, one can approximate

$$\frac{T}{z - z_k} \simeq \frac{1}{2} \coth\left(\frac{z}{2T}\right),$$

and use the Cauchy theorem to write

$$\frac{\partial}{\partial \omega^2} \mathrm{tr}_t \ln(-\partial_t^2 + \omega^2) = -\frac{T}{2\pi i} \oint_C \frac{dz}{z - z_k} \cdot \frac{1}{z^2 - \omega^2} = -\frac{1}{4\pi i} \oint_C dz \coth\left(\frac{z}{2T}\right) \frac{1}{z^2 - \omega^2}.$$

One can now continuously deform the contour in such a way that the deformed contour encircles (clockwise) only the isolated poles $z = \pm\omega$. Their contribution has the form

$$\frac{\partial}{\partial \omega^2} \mathrm{tr}_t \ln(-\partial_t^2 + \omega^2) =$$

$$= -\frac{1}{4\pi i}\left[(-2\pi i)\cdot \mathrm{Res}_{z=\omega}\frac{\coth\left(\frac{z}{2T}\right)}{z^2-\omega^2} - 2\pi i\cdot \mathrm{Res}_{z=-\omega}\frac{\coth\left(\frac{z}{2T}\right)}{z^2-\omega^2}\right] = \frac{1}{2\omega}\coth\left(\frac{\omega}{2T}\right).$$

Integrating this equation and accounting for the prefactor of β from Eq. (13), one has

$$\beta\cdot \mathrm{tr}_t\ \ln(-\partial_t^2+\omega^2) = \beta\int^{\omega^2} d\omega'^2\frac{1}{2\omega'}\coth\left(\frac{\omega'}{2T}\right) = \beta\int^{\omega} d\omega'\coth\left(\frac{\omega'}{2T}\right) =$$

$$= 2\ln\ \sinh\left(\frac{\omega}{2T}\right) + (\omega - \text{independent constant}) =$$

$$= \frac{\omega}{T} + 2\ln\left(1-\mathrm{e}^{-\omega/T}\right) + (\omega - \text{independent constant}).$$

Thus, one arrives at the following expression:

$$\ln\mathcal{Z}(T,V) = -V\int\frac{d^Dp}{(2\pi)^D}\left[\frac{\omega}{2T} + \ln\left(1-\mathrm{e}^{-\omega/T}\right)\right].$$

It reproduces the standard result for an ideal Bose gas in quantum statistics, and additionally contains the term $\frac{\omega}{2T}$, which is associated with the zero-point energy of the vacuum (recognizable by the ground-state energy of an oscillator, equal to $\frac{\omega}{2}$).

3. Instantons in quantum mechanics. An analogy with 1D Ising model. Basics of Yang-Mills instantons.

Preliminaries from quantum mechanics.

Let us start with recollecting some elements of quasiclassics. One seeks a solution to the Schrödinger equation

$$\frac{\hbar^2}{2m}\partial^2\psi + (E-U)\psi = 0$$

in the form $\psi = \mathrm{e}^{i\sigma/\hbar}$, where $\sigma(x)$ is an unknown function. For $\partial^2\psi$ one has

$$\partial^2\psi = \frac{i}{\hbar}\partial_k\left[(\partial_k\sigma)\mathrm{e}^{i\sigma/\hbar}\right] = \frac{i}{\hbar}\left[\partial^2\sigma + \frac{i}{\hbar}(\partial_k\sigma)^2\right]\mathrm{e}^{i\sigma/\hbar},$$

that upon the insertion into the Schrödinger equation yields

$$\frac{1}{2m}(\partial_k\sigma)^2 - \frac{i\hbar}{2m}\partial^2\sigma = E-U.$$

From now on, we consider a 1D motion, in which case this equation takes the form

$$\frac{1}{2m}\sigma'^2 - \frac{i\hbar}{2m}\sigma'' = E-U.$$

One seeks now a solution to this equation in the form of a series in powers of the Planck constant: $\sigma = \sigma_0 + \frac{\hbar}{i}\sigma_1 + \cdots$. To the order $\hbar^0$, one gets

$$\frac{1}{2m}\sigma_0'^{\,2} = E - U \quad \Rightarrow \quad \sigma_0 = \pm \int dx \sqrt{2m[E - U(x)]} = \pm \int dx\, p,$$

where $p = \sqrt{2m[E - U(x)]}$ is the classical momentum of a particle. To the order $\hbar^1$, one has

$$\frac{1}{2m} \cdot \frac{2\hbar}{i}\sigma_0'\sigma_1' - \frac{i\hbar}{2m}\sigma_0'' = 0 \;\Rightarrow\; \sigma_0'\sigma_1' + \frac{\sigma_0''}{2} = 0 \;\Rightarrow \sigma_1' = -\frac{\sigma_0''}{2\sigma_0'} = -\frac{p'}{2p} \;\Rightarrow \sigma_1 = -\ln\sqrt{p} \;\Rightarrow$$

$$\psi = \exp\left(\frac{i}{\hbar}\sigma_0 + \sigma_1\right) = \frac{c_1}{\sqrt{p}} \mathrm{e}^{\frac{i}{\hbar}\int dxp} + \frac{c_2}{\sqrt{p}} \mathrm{e}^{-\frac{i}{\hbar}\int dxp},$$

where $c_{1,2}$ are the constants of integration.

Let us choose two coordinates b and a, such that $b < a$. Suppose that $U(b) = E$ and $U(a) = E$, and furthermore that $U(x) > E$ for $x < b$ and $x > a$, while $U(x) < E$ for $b < x < a$. This means b and a are the turning points of the classical motion of a particle. When one crosses these points, $\sqrt{p} \propto |U - E|^{1/4}$ goes over to $(|U - E|\mathrm{e}^{\pm i\pi})^{1/4}$. Here, the sign of the phase acquired depends on whether a turning point is encircled clockwise or counterclockwise in the complex x-plane [2]. As a result, for $x = b+0$ one has $\psi_1 = \frac{c}{\sqrt{p}}\cos\beta$, where $\beta \equiv \frac{1}{\hbar}\int_b^x dx'p - \frac{\pi}{4}$, while for $x = a-0$ one has $\psi_2 = \frac{c'}{\sqrt{p}}\cos\alpha$, where $\alpha \equiv \frac{1}{\hbar}\int_x^a dx'p - \frac{\pi}{4}$. Suppose that $\alpha + \beta = \pi n$, where n is an integer. Since $\cos\beta = (-1)^n\cos\alpha$, one has $\psi_1 = \psi_2$ by choosing $c' = (-1)^n c$. The condition $\alpha + \beta = \pi n$ reads $\frac{1}{\hbar}\int_b^a dxp - \frac{\pi}{2} = \pi n$. Classically, the particle would have been performing a periodic motion with the period (= time of motion from b to a and back) $T = 2\int_b^a \frac{dx}{v} = 2m\int_b^a \frac{dx}{p}$. Therefore, denoting the integration over the period by $\oint$, one obtains $\oint dxp = 2\int_b^a dxp = 2\hbar(\frac{\pi}{2} + \pi n)$, or

$$\frac{1}{2\pi\hbar}\oint dxp = n + \frac{1}{2},$$

that is the Bohr-Sommerfeld quantization condition.

Since, at $x < b$ and $x > a$, ψ falls off exponentially, for its normalization it suffices to integrate $|\psi|^2$ over $x \in [b, a]$. Furthermore, since $\frac{1}{\hbar}\int_b^x dx'p - \frac{\pi}{4}$ is varying rapidly, it also suffices to approximate

$$\cos^2\left(\frac{1}{\hbar}\int_b^x dx'p - \frac{\pi}{4}\right) \simeq \left\langle \cos^2\left(\frac{1}{\hbar}\int_b^x dx'p - \frac{\pi}{4}\right)\right\rangle = \frac{1}{2}.$$

Then the normalization condition reads

$$\int_b^a dx|\psi|^2 \simeq \frac{c^2}{2}\int_b^a \frac{dx}{p} = \frac{c^2}{2} \cdot \frac{T}{2m} = 1.$$

Introducing the classical frequency $\omega = 2\pi/T$, one obtains

$$c = \sqrt{\frac{2m\omega}{\pi}}. \qquad (18)$$

Level splitting in two symmetric potential wells.

Consider $U(x)$ formed by two symmetric potential wells (I and II) separated by a barrier. Had the barrier been unpenetrable for the particle, the energy levels E_0 equal for both wells would be existing. These levels would correspond to the motion of the particle in one of the two wells. Let $\psi_0(x)$ denote the respective semi-classical wave function in well I, normalized by the condition $\int_0^\infty dx\psi_0^2 = 1$. A possibility of the underbarrier penetration leads to the splitting of E_0 in two levels, E_1 and E_2. These new levels correspond to the states in which the particle moves in the two wells simultaneously. In the zeroth approximation, the wave functions corresponding to the levels E_1 and E_2 are respectively the symmetric and the antisymmetric combinations of $\psi_0(x)$ and $\psi_0(-x)$:

$$\psi_1(x) = \frac{1}{\sqrt{2}}[\psi_0(x) + \psi_0(-x)], \quad \psi_2(x) = \frac{1}{\sqrt{2}}[\psi_0(x) - \psi_0(-x)].$$

Consider the Schrödinger equations

$$\psi_0'' + \frac{2m}{\hbar^2}(E_0 - U)\psi_0 = 0 \ \text{ and } \ \psi_1'' + \frac{2m}{\hbar^2}(E_1 - U)\psi_1 = 0.$$

Subtracting from the first equation multiplied by ψ_1 the second equation multiplied by ψ_0, one has

$$\psi_1\psi_0'' - \psi_0\psi_1'' + \frac{2m}{\hbar^2}(E_0 - E_1)\psi_0\psi_1 = 0. \qquad (19)$$

Integration in the range from zero to infinity yields

$$\int_0^\infty dx(\psi_1\psi_0'' - \psi_0\psi_1'') = \psi_1\psi_0'\Big|_0^\infty - \int_0^\infty dx\psi_1'\psi_0' - \psi_0\psi_1'\Big|_0^\infty + \int_0^\infty dx\psi_0'\psi_1' =$$

$$= -\psi_1(0)\psi_0'(0) + \psi_0(0)\psi_1'(0) = -\sqrt{2}\psi_0(0)\psi_0'(0), \qquad (20)$$

where at the last step it has been taken into account that $\psi_1(0) = \sqrt{2}\psi_0(0)$, $\psi_1'(0) = 0$.

In the region I, $\psi_0(-x)$ is exponentially small compared to $\psi_0(x)$, while in the region II it is the other way around. For this reason, the product $\psi_0(x)\psi_0(-x)$ is exponentially small everywhere. Therefore,

$$\int_0^\infty dx\psi_0\psi_1 \simeq \frac{1}{\sqrt{2}}\int_0^\infty dx\psi_0^2 = \frac{1}{\sqrt{2}}. \qquad (21)$$

Inserting Eqs. (20) and (21) into the (integrated) Eq. (19), one gets

$$\frac{\sqrt{2}m}{\hbar^2}(E_1 - E_0) = -\sqrt{2}\psi_0(0)\psi_0'(0) \quad \Rightarrow \quad E_1 - E_0 = -\frac{\hbar^2}{m}\psi_0(0)\psi_0'(0). \tag{22}$$

Analogously to Eq. (19), one obtains an equation involving ψ_2 instead of ψ_1:

$$\psi_2\psi_0'' - \psi_0\psi_2'' + \frac{2m}{\hbar^2}(E_0 - E_2)\psi_0\psi_2 = 0.$$

Integrating, one has

$$\int_0^\infty dx(\psi_2\psi_0'' - \psi_0\psi_2'') = -\psi_2(0)\psi_0'(0) + \psi_0(0)\psi_2'(0) = \sqrt{2}\psi_0(0)\psi_0'(0),$$

where at the last stage it has been used that $\psi_2(0) = 0$, $\psi_2'(0) = \sqrt{2}\psi_0'(0)$. In the same way as in Eq. (21), $\int_0^\infty dx\psi_0\psi_2 \simeq \frac{1}{\sqrt{2}}$, that yields

$$\frac{\sqrt{2}m}{\hbar^2}(E_2 - E_0) = \sqrt{2}\psi_0(0)\psi_0'(0) \quad \Rightarrow \quad E_2 - E_0 = \frac{\hbar^2}{m}\psi_0(0)\psi_0'(0). \tag{23}$$

Subtracting Eq. (22) from Eq. (23), one obtains

$$\Delta E \equiv E_2 - E_1 = \frac{2\hbar^2}{m}\psi_0(0)\psi_0'(0). \tag{24}$$

Next, under the barrier, i.e. at $|x| < a$,

$$\psi_0(x) = \frac{c}{2\sqrt{|p|}}\exp\left(-\frac{1}{\hbar}\int_x^a dx'|p|\right) = \sqrt{\frac{\omega}{2\pi v}}\exp\left(-\frac{1}{\hbar}\int_x^a dx'|p|\right),$$

where Eq. (18) has been used. Therefore,

$$\psi_0(0) = \sqrt{\frac{\omega}{2\pi v_0}}\exp\left(-\frac{1}{\hbar}\int_0^a dx'|p|\right),$$

where $v_0 = \sqrt{\frac{2}{m}(U_0 - E_0)}$. Accordingly,

$$\psi_0'(x) = \frac{1}{\hbar}|p(x)|\psi_0(x) \;\Rightarrow\; \psi_0'(0) = \frac{mv_0}{\hbar}\psi_0(0),$$

and one obtains for the level splitting, Eq. (24):

$$\Delta E = \frac{2\hbar^2}{m}\cdot\frac{mv_0}{\hbar}\psi_0^2(0) = 2\hbar v_0\cdot\frac{\omega}{2\pi v_0}\exp\left(-\frac{2}{\hbar}\int_0^a dx|p|\right) = \frac{\hbar\omega}{\pi}\exp\left(-\frac{1}{\hbar}\int_{-a}^a dx|p|\right). \tag{25}$$

We specify now the double-well potential as

$$V[x] = \frac{\lambda}{4}\left(x^2 - \frac{\mu^2}{\lambda}\right)^2, \tag{26}$$

and use the units where $\hbar = 1$. The Euclidean action of a particle reads

$$S[x] = \int d\tau \left[\frac{1}{2}\dot{x}^2(\tau) + V(x(\tau)) \right]. \tag{27}$$

The dimensionality of the proper time is $[\tau] = m^{-2}$. As $S[x]$ is dimensionless, the dimensionality of x^2 is the same as that of τ, and therefore $[x] = m^{-1}$. Furthermore, since $[\lambda x^4] = m^2$, the dimensionality of λ is $[\lambda] = m^6$, and since $[\mu^2] = [\lambda x^2]$, the dimensionality of μ is $[\mu] = m^2$. The minima of the potential are defined by the equation

$$V' = 0 \;\Rightarrow\; \lambda x^3 - \mu^2 x = 0 \;\Rightarrow\; x_0^{\pm} = \pm\frac{\mu}{\sqrt{\lambda}}.$$

We consider the limit

$$\lambda \ll \mu^3, \tag{28}$$

where the vacua at the points $x_0^{\pm}$ are degenerate (i.e. the particle has the same energy E_0 in both wells) to all orders of perturbation theory. By perturbation theory we mean here an expansion near one of the minima, $x(\tau) = \pm\frac{\mu}{\sqrt{\lambda}} + \chi(\tau)$, where $|\chi| \ll \frac{\mu}{\sqrt{\lambda}}$. Thus, the correlation function of two position operators goes to a constant at large proper times:

$$\langle x(0)x(\tau)\rangle \longrightarrow \frac{\mu^2}{\lambda} \quad \text{at} \quad \tau \to \infty.$$

The fact that this limit is nonvanishing means that the particle is localized in one of the two vacua. The next terms of the perturbative expansion in λ do not change this result, since the potential $V[x]$ near the minima reads

$$V[x] = \frac{\lambda}{4}\left(\chi^2 \pm \frac{2\mu}{\sqrt{\lambda}}\chi\right)^2 = \frac{\lambda}{4}\chi^4 \pm \mu\sqrt{\lambda}\chi^3 + \mu^2\chi^2, \tag{29}$$

and $\mu^2 > 0$, so that the perturbation theory is a normal one, for each of the vacua at $x_0^{\pm}$. However, nonperturbatively, $\langle x(0)x(\tau)\rangle = \sum\limits_n |x_{n0}|^2 \mathrm{e}^{-(E_n - E_0)\tau}$, and thus

$$\langle x(0)x(\tau)\rangle \sim \mathrm{e}^{-\Delta E\cdot\tau} \quad \text{at} \quad \tau \to \infty,$$

where $\Delta E = E_2 - E_1$ is the energy splitting between the two lowest (symmetric and antisymmetric) states. Therefore, the reflection symmetry, $x \leftrightarrow -x$, which is broken in perturbation theory, is restored nonperturbatively at $\tau \to \infty$.

We derive now ΔE explicitly. At $\lambda \to 0$, Eq. (29) yields the potential of the harmonic oscillator with the frequency $\omega = \mu\sqrt{2}$. Thus, its ground-state energy is $E_0 = \frac{\omega}{2} = \frac{\mu}{\sqrt{2}}$.

Equation (25) with $m = 1$ [cf. Eq. (27)] reads

$$\Delta E = \frac{\mu\sqrt{2}}{\pi} \exp\left(-\int_{-a}^{a} dx \sqrt{2(V - E_0)}\right). \tag{30}$$

One can introduce instead of $x(\tau)$ the dimensionless position operator $z(\tau) = \frac{\sqrt{\lambda}}{\mu}x(\tau)$, in terms of which $V = \frac{\mu^4}{4\lambda}(z^2-1)^2$. Introducing furthermore the variable $h = \sqrt{\frac{\lambda}{\mu^3\sqrt{2}}}$, one has

$$V - E_0 = \frac{\mu}{\sqrt{2}}\left[\frac{\mu^3\sqrt{2}}{4\lambda}(z^2-1)^2 - 1\right] = \frac{\mu}{4\sqrt{2}\cdot h^2}\left[(z^2-1)^2 - 4h^2\right].$$

The factor emerging from $dx\sqrt{2(V-E_0)}$ in Eq. (30) is

$$\sqrt{2}\cdot\frac{\mu}{\sqrt{\lambda}}dz\cdot\frac{\sqrt{\mu}}{h}\cdot\frac{1}{2^{5/4}} = \frac{dz}{2h^2},$$

where we have used the relation $\mu^{3/2} = \frac{\sqrt{\lambda}}{2^{1/4}\cdot h}$. The points z_{left} and z_{right}, where the particle goes under the barrier, are determined by the equation $V = E_0$, that is $(z^2-1)^2 = 4h^2$. In the limit (28) under study, $h \ll 1$, and this equation yields $z_{\text{left}} \simeq -1 + h$ and $z_{\text{right}} \simeq 1 - h$. Thus, the integral in Eq. (30) reads

$$\int_{-a}^{a} dx\sqrt{2(V-E_0)} = \frac{1}{2h^2}\int_{-1+h}^{1-h} dz\sqrt{(1-z^2)^2 - 4h^2} \simeq \frac{1}{2h^2}\int_{-1+h}^{1-h} dz\left(1 - z^2 - \frac{2h^2}{1-z^2}\right),$$

where at the last step we have used the smallness of h to expand the square root. Elementary integrations over z in this formula yield

$$\int_{-a}^{a} dx\sqrt{2(V-E_0)} = \frac{2}{3h^2} + \ln h + \mathcal{O}(1).$$

Accordingly, Eq. (30) to the same accuracy yields the following final result for the splitting of the energy levels:

$$\Delta E \simeq \frac{\mu\sqrt{2}}{\pi}\exp\left(-\frac{2}{3h^2} - \ln h\right) = \frac{\mu\sqrt{2}}{\pi h}\mathrm{e}^{-\frac{2}{3h^2}} = \frac{\mu}{\pi}\sqrt{\frac{2\sqrt{2}\mu^3}{\lambda}}\exp\left(-\frac{2\sqrt{2}\mu^3}{3\lambda}\right). \tag{31}$$

This result is non-analytic in λ. Thus, it cannot be obtained in perturbation theory, i.e. it is non-perturbative.

Instantons in the double-well potential.

Minima of the action (27) can be obtained from the classical equation of motion

$$-\ddot{x} + V' = -\ddot{x} - \mu^2 x + \lambda x^3 = 0, \tag{32}$$

which is obviously obeyed by $x_0^{\pm} = \pm\mu/\sqrt{\lambda}$. Besides these trivial minima, also the nontrivial ones, called instantons, exist:

$$x_{\rm inst}(\tau) = \frac{\mu}{\sqrt{\lambda}} \tanh \frac{\mu(\tau - \tau_0)}{\sqrt{2}}, \tag{33}$$

where τ_0 is the position (or the center) of the instanton. An instanton interpolates between the two minima of $V[x]$ when τ varies from $-\infty$ to $+\infty$, i.e. it connects the minima by an underbarrier motion during an infinite period of time. A solution, which interpolates between $\mu/\sqrt{\lambda}$ at $\tau = -\infty$ and $-\mu/\sqrt{\lambda}$ at $\tau = +\infty$ is called anti-instanton:

$$x_{\rm ainst}(\tau) = -\frac{\mu}{\sqrt{\lambda}} \tanh \frac{\mu(\tau - \tau_0)}{\sqrt{2}}.$$

Using the fact that $(\tanh t)' = 1/\cosh^2 t$, one can readily check that $x_{\rm inst}(\tau)$ is indeed a solution to the equation of motion, Eq. (32).

Setting $\tau_0 = 0$, we find the action of an (anti-)instanton. We have

$$\dot{x}_{\rm inst} = \frac{\mu^2}{\sqrt{2\lambda}} \cdot \frac{1}{\cosh^2(\mu\tau/\sqrt{2})}, V[x_{\rm inst}] = \frac{\lambda}{4} \cdot \frac{\mu^4}{\lambda^2} \cdot \left[\tanh^2(\mu\tau/\sqrt{2}) - 1\right]^2 = \frac{\mu^4}{4\lambda} \cdot \frac{1}{\cosh^4(\mu\tau/\sqrt{2})}.$$

Substituting these expressions into Eq. (27), one has

$$S_{\rm inst} = \int_{-\infty}^{+\infty} d\tau \left[\frac{1}{2} \cdot \frac{\mu^4}{2\lambda} \cdot \frac{1}{\cosh^4(\mu\tau/\sqrt{2})} + \frac{\mu^4}{4\lambda} \cdot \frac{1}{\cosh^4(\mu\tau/\sqrt{2})}\right] = \frac{\mu^4}{2\lambda} \int_{-\infty}^{+\infty} \frac{d\tau}{\cosh^4(\mu\tau/\sqrt{2})}.$$

Introducing a dimensionless integration variable $\xi = \mu\tau/\sqrt{2}$, and noticing that

$$\int_{-\infty}^{+\infty} \frac{d\xi}{\cosh^4 \xi} = \frac{4}{3},$$

one obtains [cf. the exponent in Eq. (31)]

$$S_{\rm inst} = \frac{2\sqrt{2}}{3} \cdot \frac{\mu^3}{\lambda}, \tag{34}$$

that is again non-analytic in λ.

The fact that $e^{-S_{\rm inst}}$ is exponentially small for $\lambda \ll \mu^3$ leads to a naïve expectation that the $(x \leftrightarrow -x)$-symmetry cannot be restored by tunneling. However, it turns out that the full contribution of all the trajectories with various values of τ_0 is sufficient for the symmetry restoration. An arbitrary trajectory can be represented as a sum

$$x(\tau) = x_{\rm inst}(\tau) + \sum_n c_n x_n(\tau),$$

where x_n's describe small fluctuations around $x_{\rm inst}$. The equation for x_n's has been derived in Section 1:

$$\left(-\frac{d^2}{d\tau^2}+V''[x_{\rm inst}]\right)x_n=\varepsilon_n x_n. \tag{35}$$

From the explicit form of the potential, Eq. (26), one has $V''=3\lambda x^2-\mu^2$, and Eq. (35) takes the form

$$\frac{d^2x_n}{d\tau^2}+\left(\mu^2-3\lambda x_{\rm inst}^2\right)x_n=-\varepsilon_n x_n.$$

Due to the translation invariance, reflected in the fact that $x_{\rm inst}(\tau-\tau_0)$ with an arbitrary τ_0 is a solution to the classical equation of motion, there exists the so-called zero mode x_0 corresponding to $\varepsilon_0=0$. Accordingly, $\int\mathcal{D}x(\tau)$ is equivalent to $\int d\tau_0\cdot\prod\limits_n\int dc_n$, where $\int d\tau_0$ was not formerly present. To find the Jacobian corresponding to the change of variables $x(\tau)\to\tau_0,\{c_n\}$, one considers the square of the norm in the functional space:

$$\int d\tau(\delta x(\tau))^2\simeq(\delta\tau_0)^2\int d\tau(\dot x_{\rm inst})^2+\sum_n(\delta c_n)^2,$$

that yields

$$\mathcal{D}x(\tau)=\sqrt{\int d\tau(\dot x_{\rm inst})^2}\int d\tau_0\cdot\prod_n\int dc_n.$$

The action S can be expanded around the action S_0 of a classical solution as $S=S_0+\frac12\sum\limits_n\varepsilon_nc_n^2$. This expansion can be used in order to calculate the correlation function

$$\langle x(\tau_1)x(\tau_2)\rangle=\frac{\int\mathcal{D}x(\tau)x(\tau_1)x(\tau_2){\rm e}^{-S}}{\int\mathcal{D}x(\tau){\rm e}^{-S}}.$$

The action S_0 of the classical solutions $x=\pm\frac{\mu}{\sqrt\lambda}$ vanishes, whereas for the classical solution $x=x_{\rm inst}$ it is given by Eq. (34). Therefore,

$$\langle x(\tau_1)x(\tau_2)\rangle=$$

$$=\frac{\frac{\mu^2}{\lambda}\cdot\int\prod\limits_n dc_n{\rm e}^{-\frac12\sum\limits_n\varepsilon_n^{(0)}c_n^2}+{\rm e}^{-\frac{2\sqrt2}{3}\frac{\mu^3}{\lambda}}\sqrt{\int d\tau(\dot x_{\rm inst})^2}\prod\limits_n dc_n{\rm e}^{-\frac12\sum\limits_n\varepsilon_nc_n^2}\int d\tau_0x_{\rm inst}(\tau_1)x_{\rm inst}(\tau_2)}{\int\prod\limits_n dc_n{\rm e}^{-\frac12\sum\limits_n\varepsilon_n^{(0)}c_n^2}+{\rm e}^{-\frac{2\sqrt2}{3}\frac{\mu^3}{\lambda}}\sqrt{\int d\tau(\dot x_{\rm inst})^2}\prod\limits_n dc_n{\rm e}^{-\frac12\sum\limits_n\varepsilon_nc_n^2}\int d\tau_0},$$

where $\varepsilon_n^{(0)}$ and ε_n are the spectra of fluctuations respectively around the trivial and the instanton solutions. We now use the exponential smallness of ${\rm e}^{-\frac{2\sqrt2}{3}\frac{\mu^3}{\lambda}}$ in the denominator, and expand it as $\frac{1}{A+\xi}\simeq\frac1A(1-\frac{\xi}{A})$, where $|\xi|\ll|A|$. Furthermore, we write explicitly

$$\int d\tau_0x_{\rm inst}(\tau_1)x_{\rm inst}(\tau_2)=\frac{\mu^2}{\lambda}\int d\tau_0\tanh\frac{\mu(\tau_1-\tau_0)}{\sqrt2}\tanh\frac{\mu(\tau_2-\tau_0)}{\sqrt2}.$$

This yields

$$\langle x(\tau_1)x(\tau_2)\rangle \simeq \frac{\mu^2}{\lambda}\times$$

$$\times\left[1+\mathrm{e}^{-\frac{2\sqrt{2}}{3}\frac{\mu^3}{\lambda}}\sqrt{\int d\tau(\dot{x}_{\mathrm{inst}})^2\frac{\int\prod\limits_n dc_n \mathrm{e}^{-\frac{1}{2}\sum\limits_n\varepsilon_n c_n^2}}{\int\prod\limits_n dc_n \mathrm{e}^{-\frac{1}{2}\sum\limits_n\varepsilon_n^{(0)} c_n^2}}}\int d\tau_0 \tanh\frac{\mu(\tau_1-\tau_0)}{\sqrt{2}}\tanh\frac{\mu(\tau_2-\tau_0)}{\sqrt{2}}\right]\times$$

$$\times\left[1-\mathrm{e}^{-\frac{2\sqrt{2}}{3}\frac{\mu^3}{\lambda}}\sqrt{\int d\tau(\dot{x}_{\mathrm{inst}})^2\frac{\int\prod\limits_n dc_n \mathrm{e}^{-\frac{1}{2}\sum\limits_n\varepsilon_n c_n^2}}{\int\prod\limits_n dc_n \mathrm{e}^{-\frac{1}{2}\sum\limits_n\varepsilon_n^{(0)} c_n^2}}}\int d\tau_0\right]\simeq$$

$$\simeq\frac{\mu^2}{\lambda}\left(1-C\cdot\mathrm{e}^{-\frac{2\sqrt{2}}{3}\frac{\mu^3}{\lambda}}|\tau_1-\tau_2|\right), \qquad (36)$$

where

$$C=$$

$$=-\sqrt{\int d\tau(\dot{x}_{\mathrm{inst}})^2\frac{\int\prod\limits_n dc_n \mathrm{e}^{-\frac{1}{2}\sum\limits_n\varepsilon_n c_n^2}}{\int\prod\limits_n dc_n \mathrm{e}^{-\frac{1}{2}\sum\limits_n\varepsilon_n^{(0)} c_n^2}}}\int_{-\infty}^{+\infty}\frac{d\tau_0}{|\tau_1-\tau_2|}\left[\tanh\frac{\mu(\tau_1-\tau_0)}{\sqrt{2}}\tanh\frac{\mu(\tau_2-\tau_0)}{\sqrt{2}}-1\right].$$

This expression for C can be simplified further, by introducing the following variables: $x=\tau_0/|\tau_1-\tau_2|$, $t_1=\mu\tau_1/\sqrt{2}$, and $t_2=\mu\tau_2/\sqrt{2}$. The τ_0-integral entering C then reads

$$\int_{-\infty}^{+\infty}dx\left[\tanh(t_1-x|t_1-t_2|)\tanh(t_2-x|t_1-t_2|)-1\right].$$

Furthermore, upon one more change of variables, $t_1-t_2=t$, $t_1+t_2=T$, this integral takes the form

$$\int_{-\infty}^{+\infty}dx\left[\tanh\left(\frac{T+t}{2}-x|t|\right)\tanh\left(\frac{T-t}{2}-x|t|\right)-1\right].$$

Consider now time intervals $|\tau_1-\tau_2|$ large compared to the instanton size, $\sqrt{2}/\mu$ [cf. Eq. (33)]. In terms of the variable t, this is the limit $|t|\gg 1$. If $|x|\gtrsim 1/2$, then

$$\tanh\left(\frac{T+t}{2}-x|t|\right)\tanh\left(\frac{T-t}{2}-x|t|\right)\simeq\tanh^2(x|t|)\simeq 1,$$

and the integrand vanishes. If $|x|\lesssim 1/2$, then $x|t|$ in the arguments of the two tanh's can be disregarded, and $\tanh\frac{T+t}{2}\tanh\frac{T-t}{2}\to -1$. Therefore, at $|t|\gg 1$,

$$\int_{-\infty}^{+\infty}dx\left[\tanh\left(\frac{T+t}{2}-x|t|\right)\tanh\left(\frac{T-t}{2}-x|t|\right)-1\right]\simeq\int_{-1/2}^{1/2}dx(-2)=-2,$$

and the constant C reads

$$C \simeq 2\sqrt{\int d\tau(\dot{x}_{\rm inst})^2 \frac{\int \prod\limits_n dc_n {\rm e}^{-\frac{1}{2}\sum\limits_n \varepsilon_n c_n^2}}{\int \prod\limits_n dc_n {\rm e}^{-\frac{1}{2}\sum\limits_n \varepsilon_n^{(0)} c_n^2}}}.$$

Thus $C > 0$. Furthermore, by an explicit calculation [3], one can get for C the following remarkable result:

$$C = \Delta E \cdot {\rm e}^{\frac{2\sqrt{2}}{3}\frac{\mu^3}{\lambda}}, \quad \text{i.e.} \quad C \cdot {\rm e}^{-\frac{2\sqrt{2}}{3}\frac{\mu^3}{\lambda}} = \Delta E, \tag{37}$$

where ΔE is given by Eq. (31).

Besides the one-instanton contribution to the correlation function $\langle x(\tau_1)x(\tau_2)\rangle$, one should also consider contributions of the multi-instanton configurations, corresponding to multiple underbarrier penetrations of the particle from one vacuum to the other and back. According to Eq. (36), an average separation in Euclidean time between the objects constituting this configuration is $C^{-1}{\rm e}^{\frac{2\sqrt{2}}{3}\frac{\mu^3}{\lambda}} = \frac{1}{\Delta E}$. That, according to Eq. (31), is exponentially larger than the size of an instanton, $\sqrt{2}/\mu$. For this reason, interactions between (anti-)instantons can be disregarded, and one arrives at the instanton-gas configuration

$$x^{(N)}(\tau) = \frac{\mu}{\sqrt{\lambda}} \prod_{i=1}^{N} {\rm sign}\left(\tau - \tau_0^{(i)}\right),$$

whose action is equal to the sum of actions of individual instantons: $S^{(N)} = N \cdot \frac{2\sqrt{2}\mu^3}{3\lambda}$. One can choose for concreteness the following sequence of the centers of instantons: $\tau_0^{(1)} > \tau_0^{(2)} > \cdots > \tau_0^{(N)}$. Summing over many-instanton configurations in the dilute-gas approximation, one obtains an exponentiation of the one-instanton contribution to the correlation function $\langle x(\tau_1)x(\tau_2)\rangle$. Equations (36) and (37) yield

$$\langle x(0)x(\tau)\rangle = \frac{\mu^2}{\lambda}\sum_{N=0}^{\infty}(-\Delta E)^N \int_0^\tau d\tau_1 \int_0^{\tau_1} d\tau_2 \cdots \int_0^{\tau_{N-1}} d\tau_N = \frac{\mu^2}{\lambda}{\rm e}^{-\tau\Delta E},$$

where at the last step we have used an apparent fact that $\int_0^\tau d\tau_1 \int_0^{\tau_1} d\tau_2 \cdots \int_0^{\tau_{N-1}} d\tau_N = \frac{\tau^N}{N!}$. The exponentiation of single-instanton contributions is analogous to the exponentiation of a single-particle contribution to the partition function in the case of an ideal gas in statistical mechanics.

The $(x \leftrightarrow -x)$-symmetry is now restored at $\tau \to \infty$. This restoration is produced by instantons. The system develops a large but finite correlation length, $1/\Delta E$, and becomes similar to 1D Ising model at finite temperature. Narrow instantons (represented by sign-functions) form configurations whose counting is equivalent to the combinatorics of spin orientations in the chain. Thus, the quantum-mechanical model for one particle in a double-well potential resembles the spin chain, that is a macroscopic object.

A reminder on 1D Ising model.

The partition function of 1D Ising model reads

$$\mathcal{Z} = \sum_{\{\sigma_i\}} \mathrm{e}^{-\beta E[\sigma_i]} = \sum_{\sigma_1=\pm 1} \cdots \sum_{\sigma_N=\pm 1} \prod_i \mathrm{e}^{K\sigma_i\sigma_{i+1}}, \tag{38}$$

where $\beta = 1/T$ is the inverse temperature, $E[\sigma_i] = -J\sum_i \sigma_i\sigma_{i+1}$, $K = \beta J$, and one usually uses the units where $J = 1$. The model is Z_2-symmetric, i.e. it is symmetric under the replacement of all σ_i's by $-\sigma_i$'s. One can introduce the so-called transfer-matrix $\hat{T}$ with the elements $T_{\sigma\sigma'} = \mathrm{e}^{\beta\sigma\sigma'}$, so that $T_{11} = T_{22} = \mathrm{e}^{\beta}$ and $T_{12} = T_{21} = \mathrm{e}^{-\beta}$. The characteristic equation $\det(\hat{T} - \lambda\hat{I}) = 0$ yields the eigenvalues of $\hat{T}$: $\lambda_1 = 2\cosh\beta$, $\lambda_2 = 2\sinh\beta$. Furthermore, one usually imposes the periodic boundary conditions, where the $(N+1)$-st node of the chain coincides with the 1-st node. Then the product in Eq. (38) can be written as

$$\prod_i \mathrm{e}^{K\sigma_i\sigma_{i+1}} = T_{\sigma_1\sigma_2}T_{\sigma_2\sigma_3}\cdots T_{\sigma_N\sigma_1},$$

and the partition function takes the form

$$\mathcal{Z} = \mathrm{tr}\,\hat{T}^N = (2\cosh\beta)^N + (2\sinh\beta)^N.$$

In the thermodynamic limit, one gets

$$\mathcal{Z} = (2\cosh\beta)^N\left[1 + (\tanh\beta)^N\right] \to (2\cosh\beta)^N, \quad \text{at} \quad N \to \infty. \tag{39}$$

In the ground state, all the spins are aligned. An arbitrary state, where some of them are flipped, can be characterized by the number n of links joining differently oriented spins. Since the full number of links is N, the corresponding statistical weight of such a state is $\mathrm{e}^{-2\beta n} \cdot \mathrm{e}^{\beta N}$. The factor of 2 in $\mathrm{e}^{-2\beta n}$ is because, for a given link, there exist two states, in which two spins at the end-points of this link have different orientations. Since the number of states with a given n is equal to the number of combinations C_N^n, the partition function can be represented as

$$\mathcal{Z} = \mathrm{e}^{\beta N} \sum_{n=0}^{N} C_N^n \mathrm{e}^{-2\beta n}.$$

One can recognize in this representation the Taylor expansion of Eq. (39), since $(2\cosh\beta)^N = (\mathrm{e}^{\beta} + \mathrm{e}^{-\beta})^N = \mathrm{e}^{\beta N}(1 + \mathrm{e}^{-2\beta})^N$. Such a representation makes it straightforward to calculate the average number of flipped spins,

$$\bar{n} = \frac{\sum\limits_{n=0}^{N} n C_N^n \mathrm{e}^{-2\beta n}}{\sum\limits_{n=0}^{N} C_N^n \mathrm{e}^{-2\beta n}} = \frac{N\mathrm{e}^{-2\beta} \sum\limits_{n=1}^{N} \frac{(N-1)!\mathrm{e}^{-2\beta(n-1)}}{(n-1)!(N-1-(n-1))!}}{\sum\limits_{n=0}^{N} \frac{N!\mathrm{e}^{-2\beta n}}{n!(N-n)!}},$$

where the definition $C_N^n = \frac{N!}{n!(N-n)!}$ has been used. Changing the n-summation in the numerator by the k-summation, where $k = n - 1$, one has

$$\bar{n} = N\mathrm{e}^{-2\beta} \cdot \frac{\sum\limits_{k=0}^{N-1} \frac{(N-1)!\mathrm{e}^{-2\beta k}}{k!(N-1-k)!}}{\sum\limits_{n=0}^{N} \frac{N!\mathrm{e}^{-2\beta n}}{n!(N-n)!}} \to N\mathrm{e}^{-2\beta} \quad \text{at} \quad N \gg 1.$$

Accordingly, the correlation length

$$r_c = \frac{N}{\bar{n}} = \mathrm{e}^{2\beta}$$

gets exponentially large at small temperatures, where the spontaneous magnetization and the long-range order become significant.

To calculate the spontaneous magnetization explicitly, one considers interactions of spins with the magnetic field H. The energy of the chain in the magnetic field is $E[\sigma_i] = -\sum\limits_{i=1}^{N} \sigma_i \sigma_{i+1} - H \sum\limits_{i=1}^{N} \sigma_i$. The partition function is again given by the formula

$$\mathcal{Z} = \sum_{\sigma_1 = \pm 1} \cdots \sum_{\sigma_N = \pm 1} T_{\sigma_1 \sigma_2} \cdots T_{\sigma_N \sigma_1},$$

where the elements of the transfer-matrix now are

$$T_{\sigma_i \sigma_{i+1}} = \exp\left[\beta\left(\sigma_i \sigma_{i+1} + H \frac{\sigma_i + \sigma_{i+1}}{2}\right)\right].$$

Explicitly, they read $T_{11} = \mathrm{e}^{\beta(1+H)}$, $T_{22} = \mathrm{e}^{\beta(1-H)}$, $T_{12} = T_{21} = \mathrm{e}^{-\beta}$. The characteristic equation $\det(\hat{T} - \lambda\hat{I}) = 0$ has the form

$$\lambda^2 - 2\lambda\mathrm{e}^{\beta}\cosh(\beta H) + 2\sinh(2\beta) = 0.$$

Its roots $\lambda_{1,2} = \mathrm{e}^{\beta}\cosh(\beta H) \pm \sqrt{\mathrm{e}^{2\beta}\cosh^2(\beta H) - 2\sinh(2\beta)}$ can be written as

$$\lambda_{1,2} = \mathrm{e}^{\beta}\cosh(\beta H) \pm \sqrt{\mathrm{e}^{2\beta}\sinh^2(\beta H) + \mathrm{e}^{-2\beta}}.$$

Similarly to the $(H = 0)$-case, the partition function reads

$$\mathcal{Z} = \mathrm{tr}\, \hat{T}^N = \lambda_1^N + \lambda_2^N = \lambda_1^N \left[1 + (\lambda_2/\lambda_1)^N\right] \to$$

$$\to \lambda_1^N = \left[\mathrm{e}^{\beta}\cosh(\beta H) + \sqrt{\mathrm{e}^{2\beta}\sinh^2(\beta H) + \mathrm{e}^{-2\beta}}\right]^N$$

in the thermodynamic limit $N \to \infty$. Thus, the partition function in this limit has no singularities, and no phase transitions occur. The spontaneous magnetization is defined by the formula

$$I = -\frac{1}{N}\frac{\partial F}{\partial H}, \quad \text{where} \quad F = -T\ln\mathcal{Z} = -TN\ln\left[\mathrm{e}^{\beta}\cosh(\beta H) + \sqrt{\mathrm{e}^{2\beta}\sinh^2(\beta H) + \mathrm{e}^{-2\beta}}\right]$$

is the free energy. A straightforward differentiation yields

$$I = \frac{\sinh(\beta H)}{\sqrt{\sinh^2(\beta H) + \mathrm{e}^{-4\beta}}}.$$

Thus, at $T = 0$, all the spins are aligned, and $I = 1$. Rather, with the increase of T, I gets smaller than 1, that resembles the restoration of the $(x \leftrightarrow -x)$-symmetry by instantons in the double-well quantum-mechanical problem.

Basics of Yang-Mills instantons.

Consider a Euclidean Yang-Mills theory with the gauge group SU(2). The covariant derivative and the field-strength tensor in the fundamental representation read

$$\nabla_\mu \mathcal{O} = \partial_\mu \mathcal{O} + [A_\mu, \mathcal{O}], \quad F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha + A_\alpha A_\beta - A_\beta A_\alpha,$$

where $A_\alpha \equiv A_\alpha^a T^a$, $T^a = \frac{\sigma^a}{2}$, and σ^a's are the Pauli matrices. To get used to these notations, let us start with proving the Bianchi identity $\nabla_\mu \tilde{F}_{\mu\nu} = 0$, where $\tilde{F}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}F_{\alpha\beta}$ is the dual field-strength tensor. Due to the antisymmetry of the ε-tensor, one has

$$\frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}F_{\alpha\beta} = \varepsilon_{\mu\nu\alpha\beta}(\partial_\alpha A_\beta + A_\alpha A_\beta). \quad (40)$$

Therefore,

$$\nabla_\mu \tilde{F}_{\mu\nu} = \varepsilon_{\mu\nu\alpha\beta} \left\{ \partial_\mu(\partial_\alpha A_\beta + A_\alpha A_\beta) + A_\mu(\partial_\alpha A_\beta + A_\alpha A_\beta) - (\partial_\alpha A_\beta + A_\alpha A_\beta)A_\mu \right\} =$$

$$= \varepsilon_{\mu\nu\alpha\beta} \left\{ \partial_\mu\partial_\alpha A_\beta + \partial_\mu A_\alpha \cdot A_\beta + A_\alpha\partial_\mu A_\beta + A_\mu\partial_\alpha A_\beta + A_\mu A_\alpha A_\beta - \partial_\alpha A_\beta \cdot A_\mu - A_\alpha A_\beta A_\mu \right\} \equiv$$

$$\equiv \varepsilon_{\mu\nu\alpha\beta} \left\{ \partial_\mu\partial_\alpha A_\beta + (\partial_\mu A_\alpha \cdot A_\beta - \partial_\alpha A_\beta \cdot A_\mu) + \right.$$

$$\left. + (A_\alpha\partial_\mu A_\beta + A_\mu\partial_\alpha A_\beta) + (A_\mu A_\alpha A_\beta - A_\alpha A_\beta A_\mu) \right\},$$

where at the last step we have only regrouped the terms. Now, in the term $-\varepsilon_{\mu\nu\alpha\beta}\partial_\alpha A_\beta \cdot A_\mu$, one can redefine the indices as $\alpha \to \mu$, $\beta \to \alpha$, $\mu \to \beta$, use the antisymmetry of the ε-tensor to write

$$-\varepsilon_{\mu\nu\alpha\beta}\partial_\alpha A_\beta \cdot A_\mu = -\varepsilon_{\mu\nu\alpha\beta}\partial_\mu A_\alpha \cdot A_\beta,$$

and cancel it with the other term in the same brackets. The same cancellation occurs in the second brackets, where one can rename the indices $\mu \leftrightarrow \alpha$ in the term $\varepsilon_{\mu\nu\alpha\beta}A_\mu\partial_\alpha A_\beta$ and write it as

$$\varepsilon_{\mu\nu\alpha\beta}A_\mu\partial_\alpha A_\beta = -\varepsilon_{\mu\nu\alpha\beta}A_\alpha\partial_\mu A_\beta.$$

Finally, in the term $-\varepsilon_{\mu\nu\alpha\beta}A_\alpha A_\beta A_\mu$, we can rename the indices as $\mu \to \beta$, $\alpha \to \mu$, $\beta \to \alpha$, write it as

$$-\varepsilon_{\mu\nu\alpha\beta}A_\alpha A_\beta A_\mu = -\varepsilon_{\mu\nu\alpha\beta}A_\mu A_\alpha A_\beta,$$

and cancel it with the other term in the third brackets. Therefore, altogether, $\nabla_\mu \tilde{F}_{\mu\nu} = 0$.

Consider now the operator $\mathrm{tr}(F_{\mu\nu}\tilde{F}_{\mu\nu})$. Let us demonstrate that it can be written as a derivative $\partial_\mu \mathcal{K}_\mu$, and find the current $\mathcal{K}_\mu$. Using the definition of $F_{\mu\nu}$, we have

$$\mathrm{tr}(F_{\mu\nu}\tilde{F}_{\mu\nu}) = \mathrm{tr}\left\{ (\partial_\mu A_\nu - \partial_\nu A_\mu)\tilde{F}_{\mu\nu} + (A_\mu A_\nu - A_\nu A_\mu)\tilde{F}_{\mu\nu} \right\}.$$

With the use of the cyclic symmetry of the trace-operation, this expression can further be written as

$$\mathrm{tr}(F_{\mu\nu}\tilde{F}_{\mu\nu}) = \mathrm{tr}\left\{ (\partial_\mu A_\nu - \partial_\nu A_\mu)\tilde{F}_{\mu\nu} + A_\mu A_\nu \tilde{F}_{\mu\nu} - A_\mu \tilde{F}_{\mu\nu} A_\nu \right\} =$$

$$= \mathrm{tr}\left\{ (\partial_\mu A_\nu - \partial_\nu A_\mu)\tilde{F}_{\mu\nu} + A_\mu[A_\nu, \tilde{F}_{\mu\nu}] \right\}.$$

Due to the Bianchi identity, $\nabla_\nu \tilde{F}_{\mu\nu} = 0$, and the definition of the covariant derivative, the commutator $[A_\nu, \tilde{F}_{\mu\nu}]$ is equal to $-\partial_\nu \tilde{F}_{\mu\nu}$. Using this fact, one can proceed further and write

$$\mathrm{tr}(F_{\mu\nu}\tilde{F}_{\mu\nu}) = \mathrm{tr}\left\{ \partial_\mu A_\nu \cdot \tilde{F}_{\mu\nu} - \partial_\nu(A_\mu \tilde{F}_{\mu\nu}) \right\} = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}\,\mathrm{tr}\left\{ \partial_\mu A_\nu \cdot F_{\alpha\beta} - \partial_\nu(A_\mu F_{\alpha\beta}) \right\}.$$

By means of Eq. (40), this expression can be written as

$$\mathrm{tr}(F_{\mu\nu}\tilde{F}_{\mu\nu}) = \varepsilon_{\mu\nu\alpha\beta}\,\mathrm{tr}\left\{\partial_\mu A_\nu\cdot(\partial_\alpha A_\beta + A_\alpha A_\beta) - \partial_\nu(A_\mu\partial_\alpha A_\beta + A_\mu A_\alpha A_\beta)\right\} =$$

$$= \varepsilon_{\mu\nu\alpha\beta}\,\mathrm{tr}\left\{\partial_\mu A_\nu\cdot\partial_\alpha A_\beta + \partial_\mu A_\nu\cdot A_\alpha A_\beta - \partial_\nu A_\mu\cdot\partial_\alpha A_\beta - \partial_\nu(A_\mu A_\alpha A_\beta)\right\}. \qquad (41)$$

Consider now the following expression:

$$\varepsilon_{\mu\nu\alpha\beta}\,\mathrm{tr}\,\partial_\mu(A_\nu A_\alpha A_\beta) = \varepsilon_{\mu\nu\alpha\beta}\,\mathrm{tr}(\partial_\mu A_\nu\cdot A_\alpha A_\beta + A_\nu\partial_\mu A_\alpha\cdot A_\beta + A_\nu A_\alpha\partial_\mu A_\beta) =$$

$$= \varepsilon_{\mu\nu\alpha\beta}\,\mathrm{tr}(\partial_\mu A_\nu\cdot A_\alpha A_\beta + \partial_\mu A_\alpha\cdot A_\beta A_\nu + \partial_\mu A_\beta\cdot A_\nu A_\alpha),$$

where at the last step the cyclic symmetry of the trace-operation has been used. Renaming the indices in the second term in the brackets as $\alpha\to\nu$, $\beta\to\alpha$, $\nu\to\beta$, and in the third term as $\beta\to\nu$, $\nu\to\alpha$, $\alpha\to\beta$, one obtains

$$\varepsilon_{\mu\nu\alpha\beta}\,\mathrm{tr}\,\partial_\mu(A_\nu A_\alpha A_\beta) = 3\varepsilon_{\mu\nu\alpha\beta}\,\mathrm{tr}(\partial_\mu A_\nu\cdot A_\alpha A_\beta).$$

This formula, once read from the right to the left, can be used for a different representation of the second term in the brackets on the right-hand side of Eq. (41). The third term in that brackets, written (up to the trace-operation) as

$$\varepsilon_{\mu\nu\alpha\beta}(-\partial_\nu A_\mu\cdot\partial_\alpha A_\beta) = \varepsilon_{\mu\nu\alpha\beta}\partial_\mu A_\nu\cdot\partial_\alpha A_\beta,$$

can be combined together with the first term. Altogether, one has for Eq. (41):

$$\mathrm{tr}(F_{\mu\nu}\tilde{F}_{\mu\nu}) = \varepsilon_{\mu\nu\alpha\beta}\,\mathrm{tr}\left\{2\partial_\mu A_\nu\cdot\partial_\alpha A_\beta + \frac{1}{3}\partial_\mu(A_\nu A_\alpha A_\beta) - \partial_\nu(A_\mu A_\alpha A_\beta)\right\}.$$

In the last term on the right-hand side, one can again rename the indices $\mu\leftrightarrow\nu$ to obtain

$$\mathrm{tr}(F_{\mu\nu}\tilde{F}_{\mu\nu}) = \varepsilon_{\mu\nu\alpha\beta}\,\mathrm{tr}\left\{2\partial_\mu A_\nu\cdot\partial_\alpha A_\beta + \frac{4}{3}\partial_\mu(A_\nu A_\alpha A_\beta)\right\}.$$

One can finally construct the full derivative by adding a vanishing term $\propto\varepsilon_{\mu\nu\alpha\beta}\partial_\mu\partial_\alpha$:

$$\mathrm{tr}(F_{\mu\nu}\tilde{F}_{\mu\nu}) = \varepsilon_{\mu\nu\alpha\beta}\,\mathrm{tr}\left\{2\partial_\mu A_\nu\cdot\partial_\alpha A_\beta + 2A_\nu\partial_\mu\partial_\alpha A_\beta + \frac{4}{3}\partial_\mu(A_\nu A_\alpha A_\beta)\right\} = \partial_\mu\mathcal{K}_\mu, \qquad (42)$$

where the desired current reads

$$\mathcal{K}_\mu = 2\varepsilon_{\mu\nu\alpha\beta}\,\mathrm{tr}\left(A_\nu\partial_\alpha A_\beta + \frac{2}{3}A_\nu A_\alpha A_\beta\right).$$

Next, by means of the equality $\mathrm{tr}\, T^a T^b = \frac{1}{2}\delta^{ab}$, the Yang-Mills action can be written as

$$S = \frac{1}{2}\,\mathrm{tr}\int d^4x F_{\mu\nu}^2.$$

Furthermore, since

$$F_{\mu\nu}^2 = F_{\mu\nu}\tilde{F}_{\mu\nu} + \frac{1}{2}(F_{\mu\nu} - \tilde{F}_{\mu\nu})^2$$

and $(F_{\mu\nu} - \tilde{F}_{\mu\nu})^2 \geq 0$, one can use Eq. (42) to write

$$S \geq \frac{1}{2}\int d^4x \partial_\mu \mathcal{K}_\mu.$$

Using the Gauss' law, this expression can be rewritten as an integral over a 3D hypersurface, so that

$$S \geq \frac{1}{2}\int ds_\mu \mathcal{K}_\mu = Q \cdot \frac{8\pi^2}{g^2},$$

where the last expression originates from the solid angle $2\pi^2$ of S^3. Here Q is an integer, called topological charge, which characterizes the mapping $S^3 \to SU(2) \simeq S^3$. The action S acquires its minimum at self-dual solutions, for which $\tilde{F}^a_{\mu\nu} = F^a_{\mu\nu}$ and

$$A_\mu \to \frac{i}{g}\Omega\partial_\mu\Omega^\dagger, \quad \text{at} \quad |x| \to \infty, \tag{43}$$

where Ω is a unitary matrix depending on the angles, $\det\Omega = 1$. Fields yielding a finite action S correspond to Ω's not reducible to the unity matrix. In particular, the matrix

$$\Omega_1 = \frac{x_4 + i\vec{x}\,\vec{\sigma}}{|x|}$$

corresponds to $Q = 1$, and $(\Omega_1)^n$ corresponds to $Q = n = 0, \pm 1, \ldots$. The value of the action $S = 8\pi^2/g^2$ of the self-dual solution with $Q = 1$, called the Yang-Mills instanton, will be obtained explicitly at the end of this Subsection.

It appears convenient to use the representation $x_4 + i\vec{x}\,\vec{\sigma} = i\sigma^+_\mu x_\mu$ with $\sigma^\pm_\mu \equiv (\vec{\sigma}, \mp i)$, and

$$\sigma^+_\mu\sigma^-_\nu = \delta_{\mu\nu} + i\eta_{a\mu\nu}\sigma^a, \quad \sigma^-_\mu\sigma^+_\nu = \delta_{\mu\nu} + i\bar{\eta}_{a\mu\nu}\sigma^a. \tag{44}$$

Here

$$\eta_{a\mu\nu} = \begin{cases} \varepsilon_{a\mu\nu}, & \mu,\nu = 1,2,3, \\ -\delta_{a\nu}, & \mu = 4, \\ \delta_{a\mu}, & \nu = 4, \\ 0, & \mu = \nu = 4, \end{cases}$$

and the symbols $\bar{\eta}_{a\mu\nu}$ differ from $\eta_{a\mu\nu}$ by the signs in front of $\delta_{a\nu}$ and $\delta_{a\mu}$. The symbols $\eta_{a\mu\nu}$ and $\bar{\eta}_{a\mu\nu}$ are called 't Hooft symbols [4]. Accordingly, the matrices Ω_1 and $\Omega_1^\dagger$ can be represented as $\Omega_1 = i\frac{\sigma_\alpha^+ x_\alpha}{|x|}$ and $\Omega_1^\dagger = -i\frac{\sigma_\nu^- x_\nu}{|x|}$, so that

$$\partial_\mu \Omega_1^\dagger = -i\sigma_\nu^- \left(\frac{\delta_{\mu\nu}}{|x|} - \frac{x_\mu x_\nu}{|x|^3}\right).$$

Using also Eq. (44), one has for Eq. (43)

$$A_\mu \to \frac{i}{g} \cdot \frac{\sigma_\alpha^+ x_\alpha}{|x|}\sigma_\nu^- \cdot \frac{1}{|x|}\left(\delta_{\mu\nu} - \frac{x_\mu x_\nu}{x^2}\right) = \frac{i}{g}\left(\delta_{\alpha\nu} + i\eta_{a\alpha\nu}\sigma^a\right)\frac{x_\alpha}{x^2}\left(\delta_{\mu\nu} - \frac{x_\mu x_\nu}{x^2}\right) =$$

$$= \frac{i}{gx^2}\left(x_\nu + i\eta_{a\alpha\nu}x_\alpha\sigma^a\right)\left(\delta_{\mu\nu} - \frac{x_\mu x_\nu}{x^2}\right) = -\frac{1}{gx^2}\eta_{a\alpha\nu}x_\alpha\sigma^a\left(\delta_{\mu\nu} - \frac{x_\mu x_\nu}{x^2}\right) \quad \text{at } |x| \to \infty.$$

Using further the antisymmetry of the symbol $\eta_{a\alpha\nu}$ with respect to the interchange of indices $\alpha \leftrightarrow \nu$, one arrives at

$$A_\mu \equiv A_\mu^a \frac{\sigma^a}{2} \to -\frac{1}{gx^2}\eta_{a\alpha\mu}x_\alpha\sigma^a,$$

and thus

$$A_\mu^a \to \frac{2}{g}\eta_{a\mu\nu}\frac{x_\nu}{x^2} \quad \text{at} \quad |x| \to \infty.$$

Assuming the same angular dependence of A_μ^a for any $|x|$, it is natural to seek $A_\mu^a(x)$ in the form

$$A_\mu^a(x) = \frac{2}{g}\eta_{a\mu\nu}x_\nu\frac{f(x^2)}{x^2}, \quad \text{where } f(x^2) \to 1 \text{ at } x^2 \to \infty, \quad \text{and } f(x^2) \to \text{const}\cdot x^2 \text{ at } x^2 \to 0.$$

The latter condition corresponds to the requirement of having no singularity at the origin. It turns out that one can indeed construct a self-dual solution corresponding to this *ansatz*. This can be done by explicitly calculating $F_{\mu\nu}^a$ and $\tilde{F}_{\mu\nu}^a$, and equating the results to each other. Let us start with calculating $F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g\varepsilon^{abc}A_\mu^b A_\nu^c$. For the derivative term one has

$$\partial_\mu A_\nu^a =$$

$$= \frac{2}{g}\eta_{a\nu\alpha}\partial_\mu\left(x_\alpha\frac{f}{x^2}\right) = \frac{2}{g}\left[\eta_{a\nu\mu}\frac{f}{x^2} + \eta_{a\nu\alpha}x_\alpha\partial_\mu\frac{f}{x^2}\right] = \frac{2}{gx^2}\left[-\eta_{a\mu\nu}f + 2\eta_{a\nu\alpha}x_\mu x_\alpha\left(f' - \frac{f}{x^2}\right)\right],$$

and therefore

$$\partial_\mu A_\nu^a - \partial_\nu A_\mu^a = \frac{2}{gx^2}\left[-2\eta_{a\mu\nu}f + 2\left(f' - \frac{f}{x^2}\right)x_\alpha\left(x_\mu\eta_{a\nu\alpha} - x_\nu\eta_{a\mu\alpha}\right)\right].$$

The term

$$g\varepsilon^{abc}A_\mu^b A_\nu^c = \frac{4}{g}\varepsilon^{abc}\eta_{b\mu\alpha}\eta_{c\nu\beta}x_\alpha x_\beta\frac{f^2}{|x|^4}$$

can be simplified by using the equality

$$\varepsilon^{abc}\eta_{b\mu\alpha}\eta_{c\nu\beta} = \delta_{\mu\nu}\eta_{a\alpha\beta} - \delta_{\mu\beta}\eta_{a\alpha\nu} - \delta_{\alpha\nu}\eta_{a\mu\beta} + \delta_{\alpha\beta}\eta_{a\mu\nu},$$

which yields

$$g\varepsilon^{abc}A^b_\mu A^c_\nu = \frac{4}{g}\left(x^2\eta_{a\mu\nu} - x_\mu x_\alpha \eta_{a\alpha\nu} - x_\nu x_\beta \eta_{a\mu\beta}\right)\frac{f^2}{|x|^4}.$$

Thus, the field-strength tensor reads

$$F^a_{\mu\nu} = \frac{4}{gx^2}\times$$

$$\times\left[-\eta_{a\mu\nu}f + (x_\alpha x_\mu \eta_{a\nu\alpha} - x_\alpha x_\nu \eta_{a\mu\alpha})\left(f' - \frac{f}{x^2}\right) + \frac{f^2}{x^2}\left(x^2\eta_{a\mu\nu} - x_\mu x_\alpha \eta_{a\alpha\nu} - x_\nu x_\alpha \eta_{a\mu\alpha}\right)\right] =$$

$$= \frac{4}{gx^2}\left[\eta_{a\mu\nu}f(f-1) + (x_\mu x_\alpha \eta_{a\nu\alpha} - x_\nu x_\alpha \eta_{a\mu\alpha})\left(f' + \frac{f^2 - f}{x^2}\right)\right] =$$

$$= -\frac{4}{g}\left\{\eta_{a\mu\nu}\frac{f(1-f)}{x^2} + \frac{x_\mu x_\alpha \eta_{a\nu\alpha} - x_\nu x_\alpha \eta_{a\mu\alpha}}{|x|^4}[f(1-f) - x^2 f']\right\}. \quad (45)$$

For the dual field-strength tensor this formula yields

$$\tilde{F}^a_{\mu\nu} = -\frac{2}{g}\left\{\varepsilon_{\mu\nu\lambda\rho}\eta_{a\lambda\rho}\frac{f(1-f)}{x^2} + \frac{x_\lambda x_\alpha \varepsilon_{\mu\nu\lambda\rho}\eta_{a\rho\alpha} - x_\rho x_\alpha \varepsilon_{\mu\nu\lambda\rho}\eta_{a\lambda\alpha}}{|x|^4}[f(1-f) - x^2 f']\right\} =$$

$$= -\frac{2}{g}\left\{\varepsilon_{\mu\nu\lambda\rho}\eta_{a\lambda\rho}\frac{f(1-f)}{x^2} + 2\frac{x_\lambda x_\alpha \varepsilon_{\mu\nu\lambda\rho}\eta_{a\rho\alpha}}{|x|^4}[f(1-f) - x^2 f']\right\},$$

where, in order to obtain the last formula, we renamed the indices $\lambda \leftrightarrow \rho$ in the term proportional to $x_\rho x_\alpha \varepsilon_{\mu\nu\lambda\rho}\eta_{a\lambda\alpha}$. The expression obtained can be handled by means of the formula

$$\varepsilon_{\mu\nu\lambda\rho}\eta_{a\rho\alpha} = \delta_{\alpha\nu}\eta_{a\mu\lambda} - \delta_{\alpha\mu}\eta_{a\nu\lambda} - \delta_{\alpha\lambda}\eta_{a\mu\nu},$$

which in particular yields $\varepsilon_{\mu\nu\lambda\rho}\eta_{a\lambda\rho} = 2\eta_{a\mu\nu}$. As a result one obtains

$$\tilde{F}^a_{\mu\nu} = -\frac{4}{g}\left\{\eta_{a\mu\nu}\frac{f(1-f)}{x^2} - \frac{x^2\eta_{a\mu\nu} + x_\mu x_\lambda \eta_{a\nu\lambda} - x_\nu x_\lambda \eta_{a\mu\lambda}}{|x|^4}[f(1-f) - x^2 f']\right\} =$$

$$= -\frac{4}{g}\left\{\eta_{a\mu\nu}f' - \frac{x_\mu x_\lambda \eta_{a\nu\lambda} - x_\nu x_\lambda \eta_{a\mu\lambda}}{|x|^4}[f(1-f) - x^2 f']\right\}.$$

Comparing this formula with Eq. (45), one concludes that the self-duality condition $F^a_{\mu\nu} = \tilde{F}^a_{\mu\nu}$ is reduced to the equation

$$x^2 f' - f(1-f) = 0$$

for the unknown function $f(x^2)$. One can straightforwardly check that the solution to this equation reads

$$f(x^2) = \frac{x^2}{x^2 + \rho^2}, \tag{46}$$

where ρ is a constant of integration called the instanton size or the instanton radius. Due to the translation invariance, one can obtain an instanton solution with the center at an arbitrary point x_0:

$$A_\mu^a(x) = \frac{2}{g}\eta_{a\mu\nu}\frac{(x - x_0)_\nu}{(x - x_0)^2 + \rho^2},$$

whose field-strength tensor is readily seen from Eqs. (45) and (46) to have the form

$$F_{\mu\nu}^a(x) = -\frac{4}{g}\eta_{a\mu\nu}\frac{\rho^2}{[(x - x_0)^2 + \rho^2]^2}.$$

It can now be verified that the instanton action is indeed equal to $8\pi^2/g^2$. One has

$$S = \frac{1}{4}\int d^4x (F_{\mu\nu}^a)^2 = \frac{1}{4}\cdot\frac{16}{g^2}\,\eta_{a\mu\nu}^2\int d^4x \frac{\rho^4}{(x^2 + \rho^2)^4}.$$

Noticing that $\eta_{a\mu\nu}^2 = 12$ and introducing a dimensionless integration variable $\xi = |x|/\rho$, one obtains

$$S = \frac{48}{g^2}\cdot 2\pi^2\int_0^\infty d\xi \frac{\xi^3}{(\xi^2 + 1)^4} = \frac{8\pi^2}{g^2},$$

where at the final step the value 1/12 of the ξ-integral has been used.

The Yang-Mills running coupling in this formula is defined at the distance equal to the instanton size, that is $g = g(\rho)$. Due to color confinement in Yang-Mills theory, $g(\rho)$ grows with the increase of ρ, and the instanton action S vanishes. Therefore, the weight of large instantons in the partition function, $\sim \mathrm{e}^{-S}$, is no longer exponentially small in the infra-red limit. Hence, naïvely, instantons of large sizes should proliferate in the vacuum, and the problem arises as how to stabilize arbitrarily large instantons. It turns out that this problem can be solved [5] by accounting for the interaction of instantons with soft background fields, which might provide confinement. Such fields stop an infinite growth of $g(\rho)$ to the infra-red region [6].

4. One-loop effective action of a particle in a gauge field. A path-integral derivation of the Euler-Heisenberg Lagrangian. Schwinger formula and the decay of a metastable vacuum. World-line instantons.

One-loop effective action of a particle in a gauge field.

Consider the QCD partition function in Euclidean space,

$$\mathcal{Z} = \int \mathcal{D}A_\mu^a \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathrm{e}^{-\int d^4x \mathcal{L}},$$

where the Lagrangian, the non-Abelian field-strength tensor, and the covariant derivative are defined respectively as

$$\mathcal{L} = \frac{1}{4}(F_{\mu\nu}^a)^2 + \bar{\psi}(\gamma_\mu D_\mu + m)\psi, \quad F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + igf^{abc}A_\mu^b A_\nu^c, \quad D_\mu = \partial_\mu + igT^a A_\mu^a.$$

Integrating over the quark fields and using Eq. (7), one obtains

$$\mathcal{Z} = \int \mathcal{D}A_\mu^a \mathrm{e}^{-\frac{1}{4}\int d^4x (F_{\mu\nu}^a)^2} \det(\gamma_\mu D_\mu + m) = \left\langle \exp\left[V \cdot \mathrm{tr}\ \ln(\gamma_\mu D_\mu + m)\right]\right\rangle_{A_\mu^a},$$

where

$$\langle \cdots \rangle_{A_\mu^a} \equiv \int \mathcal{D}A_\mu^a (\cdots) \mathrm{e}^{-\frac{1}{4}\int d^4x (F_{\mu\nu}^a)^2},$$

and V is the four-dimensional volume occupied by the system. One can further approximate the averaged exponent by the first cumulant, and introduce the notion of the averaged one-loop effective action

$$\left\langle \Gamma[A_\mu^a]\right\rangle_{A_\mu^a} = \left\langle \mathrm{tr}\ \ln(\gamma_\mu D_\mu + m)\right\rangle_{A_\mu^a},$$

so that, in this one-loop approximation, the partition function takes the form

$$\mathcal{Z} \simeq \exp\left[-V \cdot \left\langle \Gamma[A_\mu^a]\right\rangle_{A_\mu^a}\right].$$

In the diagrammatic language, the one-loop approximation means that $\Gamma[A_\mu^a]$ describes a loop of a quark with infinitely many external lines of the A_μ^a-field, but does not describe exchanges by the A_μ^a-field inside the loop and/or interactions of two and more such loops.

Up to an inessential additive constant, the averaged effective action can be rewritten as

$$\left\langle \Gamma[A_\mu^a]\right\rangle_{A_\mu^a} =$$

$$= \left\langle \mathrm{tr}\ \ln(-i\gamma_\mu\partial_\mu + g\gamma_\mu T^a A_\mu^a - im)\right\rangle_{A_\mu^a} = \frac{1}{2}\left\langle \mathrm{tr}\ \ln\left[(-i\gamma_\mu\partial_\mu + g\gamma_\mu T^a A_\mu^a)^2 + m^2\right]\right\rangle_{A_\mu^a}.$$

One can further use the anticommutation relation for the Euclidean γ-matrices, $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$ (where, for brevity, we avoid writing the unit 4×4-matrix explicitly), to represent the product of two γ-matrices as

$$\gamma_\mu \gamma_\nu = \frac{1}{2}(\gamma_\mu \gamma_\nu + \gamma_\mu \gamma_\nu + \gamma_\mu \gamma_\nu - \gamma_\mu \gamma_\nu) = \delta_{\mu\nu} + \sigma_{\mu\nu},$$

where $\sigma_{\mu\nu} \equiv \frac{1}{2}[\gamma_\mu, \gamma_\nu]$. Using the Kronecker $\delta_{\mu\nu}$ in this decomposition, one can single out the square of the covariant derivative as

$$(-i\partial_\mu + gT^a A^a_\mu)(-i\partial_\nu + gT^b A^b_\nu)(\delta_{\mu\nu} + \sigma_{\mu\nu}) =$$

$$= -D^2_\mu + \sigma_{\mu\nu}\left[-igT^a(\partial_\mu A^a_\nu + A^a_\nu \partial_\mu + A^a_\mu \partial_\nu) + g^2 T^a T^b A^a_\mu A^b_\nu\right]. \tag{47}$$

Due to the antisymmetry of $\sigma_{\mu\nu}$, the term $\sigma_{\mu\nu}(A^a_\nu \partial_\mu + A^a_\mu \partial_\nu)$ vanishes. For the same reason, we can write $\sigma_{\mu\nu}\partial_\mu A^a_\nu = \frac{1}{2}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)$, so that

$$\text{Eq. (47)} = -D^2_\mu - ig\sigma_{\mu\nu}\left[\frac{T^a}{2}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) + \frac{ig}{2}(T^a T^b + T^b T^a + T^a T^b - T^b T^a) A^a_\mu A^b_\nu\right].$$

The symmetric part of the product $T^a T^b$ yields the tensor

$$\mathcal{O}_{\mu\nu} \equiv T^a A^a_\mu \cdot T^b A^b_\nu + T^b A^b_\nu \cdot T^a A^a_\mu.$$

Accordingly, $\mathcal{O}_{\nu\mu} = T^b A^b_\mu \cdot T^a A^a_\nu + T^a A^a_\nu \cdot T^b A^b_\mu$, where we have changed the order of the two terms. Renaming the indices $a \leftrightarrow b$, we obtain $\mathcal{O}_{\nu\mu} = \mathcal{O}_{\mu\nu}$, i.e. $\mathcal{O}_{\mu\nu}$ is a symmetric tensor. Therefore,

$$\sigma_{\mu\nu}\mathcal{O}_{\mu\nu} = 0.$$

The antisymmetric part of the product $T^a T^b$ yields $T^a T^b - T^b T^a = if^{abc}T^c$, and we can continue by writing

$$\text{Eq. (47)} = -D^2_\mu - \frac{ig}{2}\sigma_{\mu\nu}T^c(\partial_\mu A^c_\nu - \partial_\nu A^c_\mu + igf^{abc}A^a_\mu A^b_\nu).$$

Renaming the indices in the last term as $c \to a$, $a \to b$, $b \to c$, and noticing that $f^{bca} = f^{abc}$, we obtain

$$\text{Eq. (47)} = -D^2_\mu - \frac{ig}{2}\sigma_{\mu\nu}T^a F^a_{\mu\nu}.$$

Using now Eq. (8), we can represent the averaged effective action as

$$\left\langle \Gamma[A^a_\mu] \right\rangle_{A^a_\mu} = -\frac{1}{2}\,\mathrm{tr}\int_0^\infty \frac{dT}{T}\mathrm{e}^{-m^2 T}\left\langle \langle x| \exp\left\{-T\left[-D^2_\mu - \frac{ig}{2}\sigma_{\mu\nu}T^a F^a_{\mu\nu}\right]\right\}|x\rangle \right\rangle_{A^a_\mu}.$$

Similarly to the 1D case [cf. the equation next to Eq. (3)], the matrix element in this formula can be represented as an infinite product of the amplitudes of transitions, which occur during infinitesimal intervals of proper time:

$$\lim_{n\to\infty}\int d^4x_1\cdots d^4x_n\,\langle x,T|x_n,\tau_n\rangle\,\langle x_n,\tau_n|x_{n-1},\tau_{n-1}\rangle\cdots\langle x_1,\tau_1|x,0\rangle\,.$$

For each such matrix element, we can give meaning to the square of the covariant derivative in the exponent in the same way as it was done after Eq. (4) for the ordinary derivative. We have (not distinguishing upper and lower Lorentz indices)

$$\langle x_{k+1},\tau_{k+1}|x_k,\tau_k\rangle=\langle x_{k+1}|\,\mathrm{e}^{\varepsilon D_\mu^2}\,|x_k\rangle=\int\frac{d^4p_k}{(2\pi)^4}\mathrm{e}^{ip_k^\mu\Delta x_k^\mu-\varepsilon(p_k^\mu+gT^aA_\mu^a)^2},$$

where $\Delta x_k\equiv x_{k+1}-x_k$ and $A_\mu^a\equiv A_\mu^a(\frac{x_{k+1}+x_k}{2})$. Introducing instead of p_k^μ a new momentum, $q_k^\mu=p_k^\mu+gT^aA_\mu^a$, we can readily perform the Gaussian integration over q_k^μ:

$$\int\frac{d^4q_k}{(2\pi)^4}\mathrm{e}^{i(q_k^\mu-gT^aA_\mu^a)(\Delta x_k)^\mu-\varepsilon q_k^2}=\mathrm{e}^{-igT^aA_\mu^a(\Delta x_k)^\mu}\cdot\frac{1}{(4\pi\varepsilon)^2}\mathrm{e}^{-\frac{(\Delta x_k)^2}{4\varepsilon}}\,.$$

Thus, we obtain

$$\langle x|\,\mathrm{e}^{TD_\mu^2}\,|x\rangle=\lim_{n\to\infty}\int\frac{d^4x_1}{(4\pi\varepsilon)^2}\cdots\frac{d^4x_n}{(4\pi\varepsilon)^2}\mathrm{e}^{-\sum\limits_k\left[\frac{(\Delta x_k)^2}{4\varepsilon}+igT^aA_\mu^a(\Delta x_k)^\mu\right]}=$$

$$=\int_P\mathcal{D}x_\mu\mathrm{e}^{-\frac{1}{4}\int_0^T d\tau\dot{x}_\mu^2}\cdot\mathrm{tr}\,P\,\exp\left(-ig\int_0^T d\tau T^aA_\mu^a\dot{x}_\mu\right).$$

In the last formula,

$$\int_P\equiv\int\limits_{x_\mu(T)=x_\mu(0)}\;,\quad\text{where trajectories } x_\mu(\tau)\text{ are such that }\int_0^T d\tau x_\mu(\tau)=0. \qquad (48)$$

Here P stands for "periodic", while "P exp" denotes the path-ordered exponent (with P standing for "path"), similarly to the time-ordered "T exp", which represents an S-matrix. Altogether, we have for the averaged effective action:

$$\left\langle\Gamma[A_\mu^a]\right\rangle_{A_\mu^a}=-\frac{1}{2}\int_0^\infty\frac{dT}{T}\mathrm{e}^{-m^2T}\times$$

$$\times\int_P\mathcal{D}x_\mu\mathrm{e}^{-\frac{1}{4}\int_0^T d\tau\dot{x}_\mu^2}\left\langle\mathrm{tr}\,P\,\exp\left[-ig\int_0^T d\tau T^a\left(A_\mu^a\dot{x}_\mu-\frac{\sigma_{\mu\nu}}{2}F_{\mu\nu}^a\right)\right]\right\rangle_{A_\mu^a}. \qquad (49)$$

The term $\propto\sigma_{\mu\nu}F_{\mu\nu}^a$ in the P-exponent, called spin factor, can be described by means of antiperiodic functions $\psi^\mu(\tau)$. They represent γ-matrices in the sense of the substitution

$\frac{\gamma^\mu}{\sqrt{2}} \to \psi^\mu$. With the use of these functions, the averaged one-loop effective action can be written as

$$\left\langle \Gamma[A_\mu^a] \right\rangle_{A_\mu^a} = -(2s+1) \int_0^\infty \frac{dT}{T} \mathrm{e}^{-m^2 T} \int_P \mathcal{D}x_\mu \int_A \mathcal{D}\psi_\mu \mathrm{e}^{-\int_0^T d\tau \left(\frac{1}{4}\dot{x}_\mu^2 + \frac{1}{2}\psi_\mu \dot{\psi}_\mu\right)} \times$$
$$\times \left\langle \mathrm{tr}\, P \exp\left[-ig \int_0^T d\tau T^a \left(A_\mu^a \dot{x}_\mu - \psi_\mu \psi_\nu F_{\mu\nu}^a\right)\right]\right\rangle_{A_\mu^a}.$$

Here

$$\int_A \equiv \int\limits_{\psi_\mu(T) = -\psi_\mu(0)}$$

(A stands for "antiperiodic"), and s is the spin of the fermion (e.g., for a quark, $s = 1/2$).

Using the non-Abelian Stokes' theorem, the spin factor can be expressed through the other A_μ^a-dependent term,

$$\langle W[x_\mu(\tau)] \rangle_{A_\mu^a} \equiv \langle W(C) \rangle_{A_\mu^a} = \left\langle \mathrm{tr}\, P \exp\left(-ig \int_0^T d\tau T^a A_\mu^a \dot{x}_\mu\right)\right\rangle_{A_\mu^a}, \tag{50}$$

as

$$\exp\left[-2 \int_0^T d\tau \psi_\mu \psi_\nu \frac{\delta}{\delta s_{\mu\nu}(x(\tau))}\right] \langle W(C) \rangle_{A_\mu^a},$$

where the variation with respect to the surface element lying on the contour recovers the field-strength tensor. Therefore, the whole dependence of the (unaveraged) effective action $\Gamma[A_\mu^a]$ on the gauge field A_μ^a is reduced to that of the phase factor $W(C)$, which is defined at a closed contour C parametrized by the vector-function $x_\mu(\tau)$. This gauge-invariant phase factor is called Wilson loop. Often, its average, Eq. (50), is for brevity also called just a Wilson loop (not Wilson-loop average). The same reduction of the A_μ^a-dependence to that of a Wilson loop holds for any gauge-invariant amplitude describing vacuum→vacuum transition.

In a quantum theory, unlike the classical one, Wilson-loop averages are observable. In the simplest, Abelian, case, this is illustrated by the Aharonov-Bohm experiment. It is a modification of the famous experiment with two screens discussed at the beginning of the course, where one of the screens has two slots, and a detector measures at the other screen the interference picture produced by the electron beams, which pass through these slots. In the Aharonov-Bohm experiment, one places between the two screens a solenoid, perpendicular to the line, which connects the slots. The magnetic field (which has only one component) is non-vanishing, $B \neq 0$, only inside the solenoid, while everywhere outside it

$B = 0$. Electrons, in general (i.e. unless their beams cross the solenoid, that can happen with the vanishing probability if we make the solenoid infinitely thin), pass through the region where $B = 0$, but the vector-potential is non-vanishing, $A_\mu \neq 0$. That turns out to be sufficient for the interference picture to change with the change of B.

To understand the reason for that, consider the amplitude of probability for a (massless and spinless, for simplicity) electron to propagate from the source at the point x to some point y at the screen where the detector is located. This amplitude reads

$$\Psi(x, y|A_\mu) = \sum_{C_1} \mathrm{e}^{ie\int_{C_1} dz_\mu A_\mu} + \sum_{C_2} \mathrm{e}^{ie\int_{C_2} dz_\mu A_\mu}.$$

Here $(-e)$ is the electron charge, C_1 and C_2 are the trajectories of the beams passing through the two slots and having the solenoid between them, and

$$\sum_C \equiv \int_0^\infty dT \int\limits_{\substack{z(0)=x \\ z(T)=y}} \mathcal{D}z_\mu \mathrm{e}^{-\frac{1}{4}\int_0^T d\tau \dot{z}_\mu^2(\tau)},$$

where C is either C_1 or C_2 The intensity of the interference pattern, contained in $|\Psi(x, y|A_\mu)|^2$, is described by the cross term

$$\mathrm{e}^{ie\int_{C_1} dz_\mu A_\mu} \cdot \mathrm{e}^{-ie\int_{C_2} dz_\mu A_\mu} = \exp\left[ie \oint\limits_{C_1\cup(C_2)^{-1}} dz_\mu A_\mu\right],$$

where $(C_2)^{-1}$ denotes contour C_2 passed in the opposite direction, i.e. from y to x. This expression is just the Wilson loop, and it does not depend on the shape of the closed contour $C_1 \cup (C_2)^{-1}$ because, due to the (Abelian) Stokes' theorem, it equals to

$$\mathrm{e}^{\frac{ie}{2}\int ds_{\mu\nu} F_{\mu\nu}} = \mathrm{e}^{ieBS},$$

where S is the area of the cross-section of the solenoid (and BS is the magnetic flux through the solenoid). This is the formal reason why the interference picture changes with the change of B. We also see that, by means of the path-integral representation for the probability amplitude, Wilson-loop averages are indeed observable in a quantum theory, i.e. their values can be measured experimentally.

A path-integral derivation of the Euler-Heisenberg Lagrangian. Schwinger formula.

Following the method described in Ref. [7], we calculate now the one-loop effective action of an electron in a constant Abelian electromagnetic field. Here "constant" means constancy (i.e. independence of a space-time point) of the field-strength tensor $F_{\mu\nu}$. Accordingly, the vector-potential of such a field has the form

$$A_\mu(x) = \frac{1}{2} x_\nu F_{\nu\mu}. \tag{51}$$

The field $A_\mu(x)$ is an external classical field, therefore there is no average over it, and no notion of the *averaged* effective action. Omitting "tr" and "P", which apply only to the non-Abelian case, we can write the effective action as

$$\Gamma[A_\mu] = -2 \int_0^\infty \frac{dT}{T} \mathrm{e}^{-m^2 T} \times$$

$$\times \int_P \mathcal{D}x_\mu \int_A \mathcal{D}\psi_\mu \exp\left[-\int_0^T d\tau \left(\frac{1}{4}\dot{x}_\mu^2 + \frac{1}{2}\psi_\mu\dot{\psi}_\mu + ieA_\mu\dot{x}_\mu - ie\psi_\mu\psi_\nu F_{\mu\nu} \right) \right].$$

For the sake of generality, we perform now all the calculations in a D-dimensional Euclidean space.

We start with the calculation of the corresponding bosonic determinant. Note that, for a free boson, the path integral in D dimensions has been calculated in the 2nd lecture, and reads

$$\int_P \mathcal{D}x_\mu \exp\left[-\frac{1}{4}\int_0^T d\tau \dot{x}_\mu^2 \right] = \left[\det{}_P(-\partial_\tau^2) \right]^{-1/2} = (4\pi T)^{-D/2}. \tag{52}$$

For the vector-potential (51), the term $ieA_\mu\dot{x}_\mu$ yields the bosonic path integral of interest

$$\int_P \mathcal{D}x_\mu \exp\left[-\frac{1}{4}\int_0^T d\tau \left(\dot{x}_\mu^2 - 2ieF_{\mu\nu}\dot{x}_\mu x_\nu \right) \right].$$

Due to the periodic boundary condition, $x_\mu(T) = x_\mu(0)$, one has $\int_0^T d\tau \dot{x}_\mu x_\nu = -\int_0^T d\tau x_\mu \dot{x}_\nu$, and the path integral can be written as

$$\int_P \mathcal{D}x_\mu \exp\left[-\frac{1}{4}\int_0^T d\tau x_\mu \left(-\partial_\tau^2 \delta_{\mu\nu} + 2ieF_{\mu\nu}\partial_\tau \right) x_\nu \right] = \left[\det{}_P(-\partial_\tau^2\,\hat{1} + 2ie\hat{F}\partial_\tau) \right]^{-1/2} =$$

$$= (4\pi T)^{-D/2} \cdot \left[\det{}_P(\hat{1} - 2ie\hat{F}\partial_\tau^{-1}) \right]^{-1/2}.$$

Here $\hat{\mathcal{O}}$ denotes a 4×4-matrix, and in the last step we have used Eq. (52). Further, representing det as $\mathrm{e}^{\mathrm{tr}\,\ln}$ and using the known Taylor series

$$\ln(1+x) = \sum_{n=1}^\infty \frac{(-1)^{n+1}}{n} \cdot x^n,$$

one has

$$\left[\det{}_P(\hat{1} - 2ie\hat{F}\partial_\tau^{-1}) \right]^{-1/2} = \exp\left[-\frac{1}{2}\sum_{n=1}^\infty \frac{(-1)^{n+1}}{n} (-2ie)^n (\mathrm{tr}\,\hat{F}^n) \cdot (\mathrm{Tr}_P\,\partial_\tau^{-n}) \right]. \tag{53}$$

The functional trace in the periodic case, Tr_P, can be calculated similarly to the trace in the antiperiodic case, Tr_A. To do so, consider the following quantity depending on some parameter C:

$$\sigma^{(n)}_{A,P}(C) \equiv \mathrm{Tr}_{A,P}[(\partial_\tau - C)^{-n}]. \tag{54}$$

By means of the Taylor expansion, it can be expressed through $\sigma^{(1)}_{A,P}(C)$ as

$$\sigma^{(n)}_{A,P}(C) = \frac{1}{(n-1)!}\left(\frac{d}{dC}\right)^{n-1}\sigma^{(1)}_{A,P}(C). \tag{55}$$

Now, since the spectrum of the operator $(\partial_\tau - C)$ is known in both the periodic and the antiperiodic cases, further calculation can be done by a direct summation of the corresponding eigenvalues.

We start with the antiperiodic case. There we have

$$\sigma^{(1)}_A(C) = \sum_{n=-\infty}^{+\infty}\frac{1}{i\cdot\frac{2\pi}{T}(n+\frac{1}{2}) - C} = \sum_{n=-\infty}^{+\infty}\frac{-i\cdot\frac{2\pi}{T}(n+\frac{1}{2}) - C}{(\frac{2\pi}{T})^2(n+\frac{1}{2})^2 + C^2}.$$

In the sum corresponding to the first addendum in the numerator, $(n = 0)$-term cancels with the $(n = -1)$-term, $(n = 1)$-term cancels with the $(n = -2)$-term, and so on, i.e. in general every n-th term with $n \geq 0$ cancels with the $(-n-1)$-th term. Thus, only the second addendum in the numerator contributes, and we have

$$\sigma^{(1)}_A(C) = -C\sum_{n=-\infty}^{+\infty}\frac{1}{(\frac{2\pi}{T})^2(n+\frac{1}{2})^2 + C^2}.$$

Noting again that $(n+\frac{1}{2})^2\Big|_{n\geq 0} = [(-n-1)+\frac{1}{2}]^2$, we can finally write this sum as

$$\sigma^{(1)}_A(C) = -2C\sum_{n=0}^{\infty}\frac{1}{(\frac{2\pi}{T})^2(n+\frac{1}{2})^2 + C^2}.$$

One can further use the following known representation of the hyperbolic tangent:

$$\tanh\left(\frac{\pi x}{2}\right) = \frac{x}{\pi}\cdot\sum_{n=0}^{\infty}\frac{1}{(n+\frac{1}{2})^2 + (\frac{x}{2})^2}.$$

One obtains

$$\sigma^{(1)}_A(C) = -2C\left(\frac{T}{2\pi}\right)^2\cdot\sum_{n=0}^{\infty}\frac{1}{(n+\frac{1}{2})^2 + \left(\frac{CT}{2\pi}\right)^2} = -\frac{T}{2}\tanh\left(\frac{CT}{2}\right).$$

Using Eq. (55), we can now calculate $\sigma_A^{(n)}(C)$ as

$$\sigma_A^{(n)}(C) = -\frac{T}{2}\frac{1}{(n-1)!}\left(\frac{d}{dC}\right)^{n-1}\tanh\left(\frac{CT}{2}\right).$$

Introducing instead of C a new parameter $x = CT/2$, one can write this expression as

$$\sigma_A^{(n)}(C) = -\frac{1}{(n-1)!}\left(\frac{T}{2}\right)^n\left(\frac{d}{dx}\right)^{n-1}\tanh x\Big|_{x=\frac{CT}{2}}. \tag{56}$$

To perform the differentiation, one can Taylor expand the tanh-function as

$$\tanh x = \sum_{k=1}^{\infty}\frac{2^{2k}(2^{2k}-1)B_{2k}}{(2k)!}x^{2k-1},$$

where B_{2k} are Bernoulli numbers, and apply the formula

$$\left(\frac{d}{dx}\right)^n x^m = m(m-1)\cdots(m-n+1)x^{m-n}.$$

This yields

$$\left(\frac{d}{dx}\right)^{n-1}x^{2k-1} = (2k-1)\cdots(2k-n+1)x^{2k-n} = \frac{(2k-1)!}{(2k-n)!}\cdot\left(\frac{CT}{2}\right)^{2k-n} \quad \text{for } k \geq \frac{n}{2},$$

and 0 for $k < \frac{n}{2}$. Equation (56) then leads to

$$\sigma_A^{(n)}(C) = -\frac{1}{(n-1)!}\left(\frac{T}{2}\right)^n\sum_{k=n/2}^{\infty}\frac{2^{2k}(2^{2k}-1)B_{2k}}{(2k)!}\cdot\frac{(2k-1)!}{(2k-n)!}\cdot\left(\frac{CT}{2}\right)^{2k-n} =$$

$$= -\frac{1}{(n-1)!}\sum_{k=n/2}^{\infty}\frac{(2^{2k}-1)B_{2k}}{2k\cdot(2k-n)!}\cdot T^{2k}\cdot C^{2k-n}.$$

We recall now the initial Eq. (54), and take the limit $C \to 0$. (That is why the Taylor expansion of tanh was legitimate.) In this limit, only the ($k = n/2$)-term in the sum survives, and one obtains the desired result

$$\mathrm{Tr}_A(\partial_\tau^{-n}) = -\frac{1}{n!}(2^n-1)B_nT^n \quad \text{for even} \quad n. \tag{57}$$

For the periodic boundary conditions, we should use in Eq. (55) $\sigma_P^{(1)}(C)$ without the ($n = 0$)-mode, i.e.

$$\sigma_P^{(1)}(C) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty}\frac{1}{i\cdot\frac{2\pi}{T}n - C}.$$

The reason for that is because this mode, corresponding to the translation of the contour as a whole, has already been taken into account by demanding that not only $x_\mu(T) = x_\mu(0)$, but also that $\int_0^T d\tau x_\mu(\tau) = 0$ [cf. Eq. (48)]. In other words, $x_\mu(\tau)$ describes the shape of the contour, whereas the position of the contour corresponds to the $(n = 0)$-mode. The position-vector, once integrated over, yields a factor of volume in the initial relation $\mathcal{Z} \simeq \exp\Big[-V \cdot \big\langle \Gamma[A_\mu^a]\big\rangle_{A_\mu^a}\Big]$.

Proceeding further, we have

$$\sigma_P^{(1)}(C) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty} \frac{-i \cdot \frac{2\pi}{T}n - C}{\left(\frac{2\pi}{T}n\right)^2 + C^2} = -C\left(\frac{T}{2\pi}\right)^2 \sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty} \frac{1}{n^2 + \left(\frac{CT}{2\pi}\right)^2}.$$

Such a sum was calculated at the end of the 2nd lecture:

$$\sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty} \frac{1}{n^2 + \left(\frac{CT}{2\pi}\right)^2} = \frac{2\pi}{CT}\left[\pi \coth\left(\frac{CT}{2}\right) - \frac{2\pi}{CT}\right].$$

One gets

$$\sigma_P^{(1)}(C) = -\frac{T}{2}\left(\coth x - \frac{1}{x}\right)\Big|_{x=CT/2},$$

and, by means of Eq. (55),

$$\sigma_P^{(n)}(C) = -\frac{1}{(n-1)!}\left(\frac{T}{2}\right)^n \left(\frac{d}{dx}\right)^{n-1} \left(\coth x - \frac{1}{x}\right)\Big|_{x=CT/2}.$$

The differentiation can again be performed by using the known Taylor series,

$$\coth x - \frac{1}{x} = \sum_{k=1}^{\infty} \frac{2^{2k}}{(2k)!} B_{2k} x^{2k-1},$$

so that

$$\sigma_P^{(n)}(C) = -\frac{1}{(n-1)!}\left(\frac{T}{2}\right)^n \sum_{k=n/2}^{\infty} \frac{2^{2k}}{(2k)!} B_{2k} \cdot \frac{(2k-1)!}{(2k-n)!}\left(\frac{CT}{2}\right)^{2k-n} =$$

$$= -\frac{1}{(n-1)!}\sum_{k=n/2}^{\infty} \frac{B_{2k}}{2k \cdot (2k-n)!} T^{2k} C^{2k-n}.$$

In the limit $C \to 0$ of interest, only the $(k = n/2)$-term in the sum survives, and we obtain the desired result:

$$\mathrm{Tr}_P(\partial_\tau^{-n}) = -\frac{1}{n!} B_n T^n \quad \text{for} \quad \text{even} \quad n. \tag{58}$$

Using Eqs. (57) and (58), we can now proceed with the calculation of the bosonic and fermionic determinants. For the bosonic determinant, Eq. (53), we have

$$\left[\det{}_P(\hat{1} - 2ie\hat{F}\partial_\tau^{-1})\right]^{-1/2} = \exp\left[-\frac{1}{2}\sum_{\substack{n=2 \\ n \text{ even}}}^{\infty} \frac{(-1)^{n+1}}{n}(-2ie)^n(\mathrm{tr}\hat{F}^n) \cdot \left(-\frac{T^n}{n!}B_n\right)\right] =$$

$$= \exp\left[-\frac{1}{2}\sum_{\substack{n=2\\ n\ \text{even}}}^{\infty}\frac{B_n}{n!\,n}(2ieT)^n(\mathrm{tr}\,\hat{F}^n)\right]. \tag{59}$$

The fermionic determinant appears from the corresponding path integral as follows:

$$\int_A \mathcal{D}\psi_\mu \exp\left[-\int_0^T d\tau\left(\frac{1}{2}\psi_\mu\dot{\psi}_\mu - ie\psi_\mu\psi_\nu F_{\mu\nu}\right)\right] =$$

$$= \int_A \mathcal{D}\psi_\mu \exp\left[-\frac{1}{2}\int_0^T d\tau\psi_\mu\left(\partial_\tau\delta_{\mu\nu} - 2ieF_{\mu\nu}\right)\psi_\nu\right] = \left[\det{}_A(\partial_\tau\hat{1} - 2ie\hat{F})\right]^{1/2}.$$

Using the normalization of the free path integral,

$$\int_A \mathcal{D}\psi_\mu \mathrm{e}^{-\frac{1}{2}\int_0^T d\tau\psi_\mu\dot{\psi}_\mu} = [\det{}_A\,\partial_\tau]^{1/2} = 1,$$

we continue:

$$\left[\det{}_A(\partial_\tau\hat{1} - 2ie\hat{F})\right]^{1/2} = \left[\det{}_A(\hat{1} - 2ie\hat{F}\partial_\tau^{-1})\right]^{1/2} =$$

$$= \exp\left[\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(-2ie)^n(\mathrm{tr}\,\hat{F}^n)(\mathrm{Tr}_A\,\partial_\tau^{-n})\right].$$

Using finally Eq. (57), one obtains

$$\left[\det{}_A(\partial_\tau\hat{1} - 2ie\hat{F})\right]^{1/2} = \exp\left[\frac{1}{2}\sum_{\substack{n=2\\ n\ \text{even}}}^{\infty}\frac{B_n}{n!\,n}(2^n-1)(2ieT)^n(\mathrm{tr}\,\hat{F}^n)\right]. \tag{60}$$

To accomplish the derivation, we should calculate the sums in Eqs. (59) and (60). We proceed in Minkowski space-time, where

$$\mathrm{tr}\,(\hat{F}^2) = 2(a^2 - b^2),$$

where

$$a^2 = \frac{1}{2}\left[\mathbf{E}^2 - \mathbf{H}^2 + \sqrt{(\mathbf{E}^2 - \mathbf{H}^2)^2 + 4(\mathbf{EH})^2}\right],$$

and

$$b^2 = \frac{1}{2}\left[\mathbf{H}^2 - \mathbf{E}^2 + \sqrt{(\mathbf{E}^2 - \mathbf{H}^2)^2 + 4(\mathbf{EH})^2}\right],$$

are related to the two invariants of the electromagnetic field as $a^2 - b^2 = \mathbf{E}^2 - \mathbf{H}^2$, $a^2b^2 = (\mathbf{EH})^2$. Similarly, one has

$$\mathrm{tr}\,(\hat{F}^{2n}) = 2[(a^2)^n + (-b^2)^n].$$

Therefore, one can represent the sum in Eq. (59) as

$$\frac{1}{2}\sum_{\substack{n=2\\ n\ \text{even}}}^{\infty}\frac{B_n}{n!\,n}(2ieT)^n(\mathrm{tr}\,\hat{F}^n) = \sum_{\substack{n=2\\ n\ \text{even}}}^{\infty}\frac{B_n}{n!\,n}(2ieT)^n[(a^2)^{n/2} + (-b^2)^{n/2}]. \tag{61}$$

Using now the Taylor series

$$\ln\frac{\sin x}{x}=\sum_{k=1}^{\infty}\frac{(-1)^k 2^{2k-1}B_{2k}x^{2k}}{k(2k)!}=\sum_{\substack{n=2\\ n\ \text{even}}}^{\infty}\frac{(2ix)^n B_n}{n\cdot n!},$$

one has

$$\text{Eq. (61)}=\ln\frac{\sin(eaT)}{eaT}+\ln\frac{\sin(iebT)}{iebT}=\ln\frac{\sin(eaT)\sinh(ebT)}{(eaT)(ebT)}.$$

Plugging this expression into Eq. (59), one obtains the bosonic determinant

$$\left[\det_P(\hat{1}-2ie\hat{F}\partial_\tau^{-1})\right]^{-1/2}=\frac{(eaT)(ebT)}{\sin(eaT)\sinh(ebT)}.$$

Similarly, using the Taylor series

$$\ln\cos x=\sum_{\substack{n=2\\ n\ \text{even}}}^{\infty}\frac{(2ix)^n(2^n-1)}{n\cdot n!}B_n,$$

one has for the sum in Eq. (60):

$$\frac{1}{2}\sum_{\substack{n=2\\ n\ \text{even}}}^{\infty}\frac{B_n}{n!\,n}(2^n-1)(2ieT)^n(\operatorname{tr}\hat{F}^n)=\sum_{\substack{n=2\\ n\ \text{even}}}^{\infty}\frac{B_n}{n!\,n}(2^n-1)(2ieT)^n[(a^2)^{n/2}+(-b^2)^{n/2}]=$$

$$=\ln\cos(eaT)+\ln\cos(iebT)=\ln[\cos(eaT)\cosh(ebT)].$$

Therefore, the fermionic determinant reads

$$\left[\det_A(\partial_\tau\hat{1}-2ie\hat{F})\right]^{1/2}=\cos(eaT)\cosh(ebT).$$

Altogether, one obtains the proper-time representation for the (unsubtracted) Euler-Heisenberg one-loop effective action of a spin-$\frac{1}{2}$ particle in 4D Minkowski space-time:

$$\Gamma[A_\mu]_{\text{spin}-\frac{1}{2}}=-\frac{2}{(4\pi)^2}\int_0^\infty\frac{dT}{T^3}\mathrm{e}^{-m^2T}\frac{(eaT)(ebT)}{\tan(eaT)\tanh(ebT)}.$$

Accordingly, for a spin-0 particle, the effective action reads

$$\Gamma[A_\mu]_{\text{spin}-0}=\frac{1}{(4\pi)^2}\int_0^\infty\frac{dT}{T^3}\mathrm{e}^{-m^2T}\frac{(eaT)(ebT)}{\sin(eaT)\sinh(ebT)}.$$

Consider now the case of vanishing magnetic field, $\mathbf{H}=0$. For a spin-0 particle, one has

$$\Gamma[E]_{\text{spin}-0}=\frac{1}{(4\pi)^2}\int_0^\infty\frac{dT}{T^3}\mathrm{e}^{-m^2T}\frac{eET}{\sin(eET)}=\left(\frac{eE}{4\pi}\right)^2\int_0^\infty\frac{dx}{x^2}\cdot\frac{\mathrm{e}^{-\frac{m^2x}{eE}}}{\sin x},$$

where $x = eET$ is the dimensionless integration variable. In the vicinity of the points $x = \pi n$, one can write

$$\sin x = \sin(x - \pi n + \pi n) = (-1)^n \sin(x - \pi n) \simeq (-1)^n \cdot (x - \pi n).$$

One can now calculate the imaginary part of the effective action by shifting the poles downwards from the real axis:

$$\mathrm{Im}\, \frac{1}{x - \pi n + i0} = -\pi \cdot \delta(x - \pi n).$$

One obtains

$$\mathrm{Im}\, \Gamma[E]_{\mathrm{spin-0}} = \frac{(eE)^2}{16\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(\pi n)^2} \mathrm{e}^{-\frac{\pi m^2 n}{eE}} = \frac{(eE)^2}{16\pi^3} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \mathrm{e}^{-\frac{\pi m^2 n}{eE}}. \tag{62}$$

Similarly, in the spin-$\frac{1}{2}$ case, one has

$$\Gamma[E]_{\mathrm{spin-}\frac{1}{2}} = -\frac{2}{(4\pi)^2}(eE)^2 \int_0^{\infty} \frac{dx}{x^2} \cdot \frac{\mathrm{e}^{-\frac{m^2 x}{eE}}}{\tan x}.$$

The imaginary part of the function $\frac{1}{\tan x}$ near the pole $x = \pi n - i0$ is $-\pi \cdot \delta(x - \pi n)$. Therefore,

$$\mathrm{Im}\, \Gamma[E]_{\mathrm{spin-}\frac{1}{2}} = \frac{(eE)^2}{8\pi} \sum_{n=1}^{\infty} \frac{\mathrm{e}^{-\frac{\pi m^2 n}{eE}}}{(\pi n)^2} = \frac{(eE)^2}{8\pi^3} \sum_{n=1}^{\infty} \frac{\mathrm{e}^{-\frac{\pi m^2 n}{eE}}}{n^2}. \tag{63}$$

Equations (62) and (63) are called Schwinger formulae. They yield the rates of production of particle-antiparticle pairs, i.e. the number of pairs produced in a unit 4D volume. The Schwinger formulae are explicitly nonperturbative, i.e. nonanalytic in eE. The probability for the vacuum of a given, either scalar or spinor, field theory (contained in the 4D volume $V^{(4)}$ in the presence of a constant electric field E) not to decay, called vacuum persistence probability, is given by the expression $\mathrm{e}^{-2V^{(4)} \cdot \mathrm{Im}\, \Gamma[E]}$. This quantity is exponentially small. Accordingly, the probability for the vacuum to decay by converting its energy to the production of particle-antiparticle pairs, i.e. the probability of vacuum non-persistence to the applied electric field, is $\left(1 - \mathrm{e}^{-2V^{(4)} \cdot \mathrm{Im}\, \Gamma[E]}\right)$.

The factor of 2 in the exponents above is because the probability is equal to the square of the pair-creation amplitude. It appears naturally in the following interpretation of the Schwinger formula [8]. One can split the full momentum $\mathbf{p}$ of a produced pair as $\mathbf{p} = (\mathbf{p}_\perp, p_\parallel)$, where $\mathbf{p}_\perp = \frac{\mathbf{p} \times \mathbf{E}}{E}$ is the part perpendicular to $\mathbf{E}$, while $p_\parallel = \frac{\mathbf{pE}}{E}$ is the projection of $\mathbf{p}$ on $\mathbf{E}$. In the spin-0 case, consider the quantity

$$\ln[1 + \rho(\mathbf{p}_\perp)] = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \rho^n(\mathbf{p}_\perp), \tag{64}$$

where

$$\rho(\mathbf{p}_\perp) = \exp\left(-\pi \cdot \frac{\mathbf{p}_\perp^2 + m^2}{eE}\right).$$

One can integrate Eq. (64) over d^3p by using the following equalities:

$$\int \frac{d^3p}{(2\pi)^3}\rho^n(\mathbf{p}_\perp) = \frac{eET}{(2\pi)^3}\mathrm{e}^{-\frac{\pi m^2 n}{eE}} \int d^2p_\perp \mathrm{e}^{-\frac{\pi p_\perp^2 n}{eE}} = \frac{(eE)^2T}{(2\pi)^3} \cdot \frac{\mathrm{e}^{-\frac{\pi m^2 n}{eE}}}{n}.$$

Here T is the time of observation, and the factor eET is produced by the "empty" integration over $p_\parallel$. Accordingly, Eq. (64) yields

$$V^{(3)} \int \frac{d^3p}{(2\pi)^3} \ln[1 + \rho(\mathbf{p}_\perp)] = \frac{(eE)^2V^{(4)}}{(2\pi)^3} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}\mathrm{e}^{-\frac{\pi m^2 n}{eE}}, \tag{65}$$

where the 4D volume occupied by the system is a product of the 3D volume and the time of observation, i.e. $V^{(4)} = V^{(3)}T$. Equations (62) and (65) lead together to the following formula:

$$2V^{(4)} \cdot \mathrm{Im}\,\Gamma[E]_{\text{spin}-0} = V^{(3)} \int \frac{d^3p}{(2\pi)^3} \ln[1 + \rho(\mathbf{p}_\perp)]. \tag{66}$$

Similarly, in the spin-$\frac{1}{2}$ case, one can use the expansion

$$\ln[1 - \rho(\mathbf{p}_\perp)] = -\sum_{n=1}^{\infty} \frac{\rho^n(\mathbf{p}_\perp)}{n}$$

to obtain

$$V^{(3)} \int \frac{d^3p}{(2\pi)^3} \ln[1 - \rho(\mathbf{p}_\perp)] = -\frac{(eE)^2V^{(4)}}{(2\pi)^3} \sum_{n=1}^{\infty} \frac{\mathrm{e}^{-\frac{\pi m^2 n}{eE}}}{n^2}.$$

Comparing this expression with Eq. (63), one concludes that

$$2V^{(4)} \cdot \mathrm{Im}\,\Gamma[E]_{\text{spin}-\frac{1}{2}} = -2V^{(3)} \int \frac{d^3p}{(2\pi)^3} \ln[1 - \rho(\mathbf{p}_\perp)], \tag{67}$$

where the prefactor of (-2) on the right-hand side is due to the spin and statistics of a fermion. One can now see from the representations (66) and (67) that the $(n = 1)$-term in $2 \cdot \mathrm{Im}\,\Gamma[E]$, namely

$$\bar{\rho} \equiv (2s + 1)\frac{(eE)^2}{(2\pi)^3}\mathrm{e}^{-\frac{\pi m^2}{eE}},$$

is equal to the mean number of pairs in the unit of the four-volume $V^{(4)}$. Terms with $n \geq 2$ describe an additional Fermi-repulsion or a Bose-attraction of produced particles at their given mean four-density $2\bar{\rho}$. These higher-order terms represent quantum-mechanical exchange corrections, and emerge due to the coherent pair creation, that is the creation of $n \geq 2$ pairs in the elementary four-volume of pair formation $\sim (eE)^{-2}$. Schwinger formulae (62) and (63) represent the virial expansion of $2\cdot\text{Im}\,\Gamma[E]$ in powers of the parameter $e^{-\frac{\pi m^2}{eE}} \sim \bar{\rho}(eE)^{-2}$, which is thus the mean number of pairs in the elementary four-volume of pair formation. Such an expansion is analogous to that of the pressures of the ideal Fermi and Bose gases in powers of the mean number of particles in the three-volumes.

The pre-exponents and the sums over n in the Schwinger formulae emerge from the determinant of quantum fluctuations around the classical trajectory of a created particle-antiparticle pair in the electric field. As such, they can only be obtained by some nontrivial methods, like the one described above. Instead, the exponent $e^{-\frac{\pi m^2 n}{eE}}$ is of the semi-classical origin, and can be obtained by using elementary quantum-mechanical methods. Indeed, the creation of the pair can be viewed as a transition of a particle from the initial state with the energy $\varepsilon_{\text{in}} = -\sqrt{p^2(z)+m^2}+eEz$ to the final state with the energy $\varepsilon_{\text{f}} = \sqrt{p^2(z)+m^2}+eEz$. Here, $p(z)$ is the semi-classical momentum of the particle, and we have assumed that the electric field points to the z-direction, so that the corresponding potential reads $\varphi = -Ez$. The transition from the first state to the second one occurs under the potential barrier, where the momentum of the particle, $p(z) = \sqrt{(\varepsilon - eEz)^2 - m^2}$, is an imaginary-valued function. The semi-classical probability of the under-barrier penetration reads

$$w \propto \exp\left[-2\int_{\frac{\varepsilon-m}{eE}}^{\frac{\varepsilon+m}{eE}} dz|p(z)|\right],$$

where the turning points are defined by the equation $p(z) = 0$. Introducing, instead of z, a new integration variable $x = \varepsilon - eEz$, one has

$$w \propto \exp\left[-\frac{2}{eE}\int_{-m}^{m} dx\sqrt{m^2-x^2}\right] = e^{-\frac{2}{eE}\cdot\frac{\pi m^2}{2}} = e^{-\frac{\pi m^2}{eE}}.$$

Thus, the semi-classical exponent from the Schwinger formulae is reproduced.

World-line instantons. An interpretation of the Schwinger formula as a decay of a metastable vacuum.

Consider the one-loop effective action of a scalar particle in an Abelian background gauge field:

$$\Gamma[A_\mu] = \int_0^\infty \frac{dT}{T}\, e^{-m^2T}\int_P \mathcal{D}x_\mu \exp\left[-\int_0^T d\tau\left(\frac{\dot{x}_\mu^2}{4} + ieA_\mu\dot{x}_\mu\right)\right].$$

Introducing, instead of τ, a new integration variable $u = \tau/T$, one has

$$\int_0^T d\tau \left(\frac{\dot{x}_\mu^2}{4} + ieA_\mu \dot{x}_\mu \right) = \frac{1}{4T} \int_0^1 du \dot{x}_\mu^2 + ie \int_0^1 du A_\mu \dot{x}_\mu .$$

Introducing also, instead of T, a dimensionless variable

$$T_{\text{new}} = m^2 T, \tag{68}$$

and denoting it henceforth as just T, one further has

$$\Gamma[A_\mu] = \int_0^\infty \frac{dT}{T} \mathrm{e}^{-T} \int_P \mathcal{D}x_\mu \exp\left[- \left(\frac{m^2}{4T} \int_0^1 du \dot{x}_\mu^2 + ie \int_0^1 du A_\mu \dot{x}_\mu \right) \right] .$$

One can now perform the T-integration first, before the path integration. That yields

$$\Gamma[A_\mu] = 2 \int_P \mathcal{D}x_\mu \cdot K_0(2T_*) \cdot \exp\left(-ie \int_0^1 du A_\mu \dot{x}_\mu \right) ,$$

where K_0 is the MacDonald function, and

$$T_* \equiv \frac{m}{2} \sqrt{\int_0^1 du \dot{x}_\mu^2} \tag{69}$$

is the saddle point of the T-integral. One can furthermore assume that $T_* \gg 1$ to obtain

$$\Gamma[A_\mu] \simeq \sqrt{\frac{2\pi}{m}} \int_P \mathcal{D}x_\mu \frac{1}{\left(\int_0^1 du \dot{x}_\mu^2 \right)^{1/4}} \exp\left[- \left(m \sqrt{\int_0^1 du \dot{x}_\mu^2} + ie \int_0^1 du A_\mu \dot{x}_\mu \right) \right] .$$

It will be shown below that the condition $T_* \gg 1$ corresponds to the weak-field approximation.

In the equation of motion,

$$\frac{d}{du} \frac{\delta S}{\delta \dot{x}_\mu} = \frac{\delta S}{\delta x_\mu},$$

one should now use the derived world-line action [9]

$$S = m \sqrt{\int_0^1 du \dot{x}_\mu^2} + ie \int_0^1 du A_\mu \dot{x}_\mu .$$

We consider again constant fields, $A_\mu(x) = \frac{1}{2} x_\nu F_{\nu\mu}$, so that

$$S = m \sqrt{\int_0^1 du \dot{x}_\mu^2} + \frac{ie}{2} F_{\nu\mu} \int_0^1 du x_\nu \dot{x}_\mu . \tag{70}$$

The resulting equation of motion,

$$m \cdot \frac{d}{du} \frac{\dot{x}_\mu}{\sqrt{\int_0^1 du' \dot{x}_\nu^2}} + \frac{ie}{2} F_{\nu\mu} \dot{x}_\nu = \frac{ie}{2} F_{\nu\lambda} \frac{\delta}{\delta x_\mu} \int_0^1 du' x_\nu \dot{x}_\lambda = \frac{ie}{2} F_{\mu\lambda} \dot{x}_\lambda,$$

takes a simple form

$$m \frac{\ddot{x}_\mu}{\sqrt{\int_0^1 du' \dot{x}_\nu^2}} = ieF_{\mu\nu} \dot{x}_\nu. \tag{71}$$

A periodic solution $x_\mu(u)$ to this equation is called a world-line instanton [9, 10]. Multiplying Eq. (71) by $\dot{x}_\mu$ and summing over μ, one gets $\dot{x}_\mu \ddot{x}_\mu = 0$, that leads to the constancy of the velocity:

$$\dot{x}_\mu^2 = \text{ constant } \equiv a^2. \tag{72}$$

By means of this formula, the condition $T_* \gg 1$ takes the form

$$ma \gg 1. \tag{73}$$

We consider now a class of time-dependent background fields, for which it is possible to find the stationary instanton paths explicitly. In Euclidean space, such fields are

$$A_3 = A_3(x_4), \qquad A_\mu = 0 \text{ for } \mu \neq 3. \tag{74}$$

Since $F_{\mu 1} = F_{\mu 2} = 0$, the equation of motion (71) yields $\ddot{x}_1 = \ddot{x}_2 = 0$, and therefore $\dot{x}_1 =$ constant, $\dot{x}_2 =$ constant. For x_1 and x_2 to be periodic functions of u, one requires $\dot{x}_1 = \dot{x}_2 = 0$. Thus, Eq. (72) yields

$$a^2 = \dot{x}_3^2 + \dot{x}_4^2. \tag{75}$$

The equation of motion takes the form

$$\ddot{x}_\mu = \frac{iea}{m} F_{\mu\nu} \dot{x}_\nu,$$

and reduces to the equations

$$\ddot{x}_3 = \frac{iea}{m} F_{34} \dot{x}_4 = -\frac{iea}{m} \cdot \frac{dA_3}{dx_4} \dot{x}_4 \quad \text{and} \quad \ddot{x}_4 = -\frac{iea}{m} F_{34} \dot{x}_3. \tag{76}$$

The first of these equations can be integrated, that yields

$$\dot{x}_3 = -\frac{iea}{m} \cdot A_3(x_4), \tag{77}$$

where the additive constant of integration has been set to 0 due to the periodicity of $\dot{x}_3$. Accordingly, one has

$$|\dot{x}_4| = \sqrt{a^2 - \dot{x}_3^2} = a \cdot \sqrt{1 + \left[\frac{eA_3(x_4)}{m}\right]^2}. \tag{78}$$

By virtue of Eq. (72) and the periodicity of $x_{3,4}(u)$, the world-line action (70), calculated at the stationary (i.e. obeying the above equations of motion) solutions, reads

$$S = ma + ie \int_0^1 du x_3 F_{34} \dot{x}_4. \tag{79}$$

One can furthermore apply (twice) integration by parts, along with the second of two Eqs. (76), to write

$$\int_0^1 du(\dot{x}_4)^2 = -\int_0^1 du x_4 \ddot{x}_4 = \frac{iea}{m} \int_0^1 du x_4 F_{34} \dot{x}_3 = -\frac{iea}{m} F_{34} \int_0^1 du x_3 \dot{x}_4.$$

Therefore,

$$\frac{m}{a} \int_0^1 du(\dot{x}_4)^2 = -ieF_{34} \int_0^1 du x_3 \dot{x}_4.$$

Using this expression for the second term on the right-hand side of Eq. (79), we have

$$S = ma - \frac{m}{a} \int_0^1 du(\dot{x}_4)^2 = \frac{m}{a} \left[a^2 - \int_0^1 du(\dot{x}_4)^2\right] = \frac{m}{a} \int_0^1 du(\dot{x}_3)^2, \tag{80}$$

where, at the last stage, Eq. (75) has been used.

Consider now a constant electric field E, for which

$$A_3(x_4) = -iEx_4. \tag{81}$$

Accordingly, Eq. (78) takes the form

$$\left|\frac{dx_4}{du}\right| = a \cdot \sqrt{1 - \left(\frac{eE}{m}\right)^2 x_4^2},$$

and its solution reads

$$x_4(u) = \frac{m}{eE} \sin\left(\frac{eEa}{m} u\right). \tag{82}$$

Using this formula, we further have for Eq. (77):

$$\frac{dx_3}{du} = -\frac{iea}{m} \cdot A_3(x_4) = -\frac{eEa}{m} \cdot x_4(u) = -a \cdot \sin\left(\frac{eEa}{m} u\right).$$

Integrating this equation, we obtain

$$x_3(u) = \frac{m}{eE} \cos\left(\frac{eEa}{m} u\right). \tag{83}$$

For the solutions (82) and (83) to be periodic with period 1, the constant a should have the form

$$a = \frac{m}{eE} \cdot 2\pi n, \qquad \text{where} \qquad n = 1, 2, \ldots.$$

With this expression for a, condition (73) becomes

$$1 \ll ma = \frac{m^2}{eE} \cdot 2\pi n.$$

This inequality should hold for any integer n, even for $n = 1$. That yields the weak-field condition which, with the speed of light c and the Planck constant $\hbar$ written explicitly, has the form

$$E \ll \frac{2\pi m^2 c^3}{e\hbar} \sim 10^{16}\,\mathrm{V/cm}.$$

Even the strongest experimentally accessible electric fields satisfy this condition well enough [11]. Thus, the saddle-point trajectories (world-line instantons) in the path integral are circles of radius $\frac{m}{eE}$ [9]:

$$x_3(u) = \frac{m}{eE}\cos(2\pi n u), \qquad x_4(u) = \frac{m}{eE}\sin(2\pi n u).$$

The integer n counts the number of times, which the closed trajectory is travelled. The corresponding action of the world-line instanton, Eq. (80), reads

$$S = \frac{m}{a}\left(\frac{2\pi n m}{eE}\right)^2 \int_0^1 du \sin^2(2\pi n u) = \frac{eE}{2\pi n} \cdot \left(\frac{2\pi n m}{eE}\right)^2 \cdot \frac{1}{2} = \frac{\pi n m^2}{eE}. \tag{84}$$

An important conclusion is that this expression coincides with the exponent in the Schwinger formulae, Eqs. (62) and (63). The action (84) at $n = 1$ is nothing but an elementary (i.e. minimal nontrivial) flux of the electric field through the circle of radius $\frac{m}{eE}$. Indeed, such an elementary flux reads $eE \cdot \pi(\frac{m}{eE})^2 = \frac{\pi m^2}{eE}$. Multiple fluxes can be obtained from the elementary one upon the multiplication by an integer n.

We will now demonstrate that such a circle of radius

$$R_c \equiv \frac{m}{eE} \tag{85}$$

can be viewed as a critical 2D bubble of the metastable vacuum phase. This phase, characterized by the electric field E, is decaying to a stable phase, filled with particle-antiparticle pairs. Whether a bubble of a given radius R, spontaneously created in the metastable phase, has a chance to drive the vacuum to the stable phase depends on the magnitude of

R. Namely, if R is smaller than the radius of a critical fluctuation, then the bubble will collapse back. Instead, if at least one bubble of the radius equal or larger than the critical one appears, it starts an unlimited expansion, unless the whole space is filled by its new, stable, vacuum phase. One can readily find the critical radius by extremizing the action of a bubble. In the case of pair-production, the created bubble is equivalent to a particle of mass m, evolving (in Euclidean space under consideration) along a circle of radius R. Such a bubble increases the action of the vacuum by an amount of $m \cdot 2\pi R$, and decreases this action by an amount of the corresponding flux $eE \cdot \pi R^2$ of the electric field, which is "eaten up" inside the circle. Thus, the action of the "bubble", $S_{\rm b}[R]$, should be defined as a difference of these two contributions to the vacuum action:

$$S_{\rm b}[R] = m \cdot 2\pi R - eE \cdot \pi R^2.$$

The extremality condition, $\frac{dS_{\rm b}}{dR} = 0$, indeed yields the value of the critical radius given by Eq. (85). The corresponding action of the critical bubble, $S_{\rm b}[R_c] = \frac{\pi m^2}{eE}$, coincides with Eq. (84) at $n = 1$.

Finally, we discuss the difference of the case where the produced particles are fermions, Eq. (63), from the case where they are bosons, Eq. (62). In the spinor case, one should insert into the path integral the following spin factor [cf. Eq. (49)]:

$$S[x_\mu(\tau), A_\mu] = {\rm tr}_{\rm L} \exp\left[\frac{ie}{2}\sigma_{\mu\nu}\int_0^T d\tau F_{\mu\nu}(x(\tau))\right],$$

where $\sigma_{\mu\nu} = \frac{1}{2}[\gamma_\mu, \gamma_\nu]$, and "L" stands for "Lorentz". For the vector-potential given by Eqs. (74) and (81), one has $F_{\mu 1} = F_{\mu 2} = 0$, while $F_{34} = iE$, so that

$$\exp\left[\frac{ie}{2}\sigma_{\mu\nu}\int_0^T d\tau F_{\mu\nu}(x(\tau))\right] = \exp\left[-\frac{eET}{2}(\sigma_{34} - \sigma_{43})\right] = \exp(-eET\sigma_{34}).$$

To calculate the trace over Lorentz indices, we use the anticommutation relation for the Euclidean γ-matrices, $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu} \cdot \hat{1}$. According to this relation, $\gamma_\mu^2 = \hat{1}$ for any $\mu = 1, 2, 3, 4$, and $\gamma_4\gamma_3 = -\gamma_3\gamma_4$. Therefore, $(\gamma_3\gamma_4)^2 \equiv \gamma_3\gamma_4\gamma_3\gamma_4 = -\hat{1}$, and, in general, one has

$$(\gamma_3\gamma_4)^{2k} = (-1)^k \cdot \hat{1}, \qquad (\gamma_3\gamma_4)^{2k+1} = (-1)^k \cdot \gamma_3\gamma_4,$$

where $k = 0, 1, 2, \ldots$. Thus, the exponent $\exp(-C \cdot \gamma_3\gamma_4)$, with a real-valued constant C, can be decomposed as

$$\exp(-C \cdot \gamma_3\gamma_4) = \hat{1} \cdot \sum_{k=0}^{\infty} \frac{(-1)^k C^{2k}}{(2k)!} + \gamma_3\gamma_4 \cdot \sum_{k=0}^{\infty} \frac{(-1)^k \cdot (-C)^{2k+1}}{(2k+1)!}.$$

One can now substitute $\sum\limits_{k=0}^{\infty}\frac{(-1)^k C^{2k}}{(2k)!}=\cos C$, and use the formula $\mathrm{tr}\,\gamma_3\gamma_4=\mathrm{tr}\,(-\gamma_4\gamma_3)$, so that, due to the cyclicity of the trace-operation, $\mathrm{tr}\,\gamma_3\gamma_4=0$. Finally, since $\sigma_{34}=\frac{1}{2}(\gamma_3\gamma_4-\gamma_4\gamma_3)=\gamma_3\gamma_4$, one gets

$$S[x_\mu(\tau),A_\mu]=\mathrm{tr}_{\mathrm{L}}\exp(-eET\gamma_3\gamma_4)=4\cos(eET). \tag{86}$$

Note that the cosine-function is the sum of two imaginary exponents. For this reason, the spin factor does not affect the saddle-point value of proper time, Eq. (69), nor does it modify the equation of motion (71). Rather, the spin factor itself should be calculated at the saddle-point value (69), equal to

$$T_*^{\mathrm{dimensionless}}=\frac{ma}{2}=\frac{m^2}{eE}\cdot\pi n.$$

The corresponding dimensionful value differs by a factor of $1/m^2$ [cf. Eq. (68)], and reads

$$T_*^{\mathrm{dimensionful}}=\frac{\pi n}{eE}.$$

Plugging it into Eq. (86), one obtains

$$S[x_\mu(\tau),A_\mu]=4\cdot(-1)^n.$$

Additionally, one should account for the overall prefactor of $(-1/2)$ from Eq. (49). Thus, the imaginary part of the effective action in the spinor case should differ from its counterpart in the scalar case by a factor of

$$2\cdot(-1)^{n+1}.$$

Comparing Eq. (63) with Eq. (62), we see that this is indeed the case. This fact demonstrates the consistency of the above calculation, where the T-integration is done before the path integrations over $x_\mu(\tau)$ and $\psi_\mu(\tau)$, with the calculation of the previous Subsection, where the path integrations were done first.

Note finally that the factor $(-1)^{n+1}$, which alters sign at every winding of an electron around the circle of radius $\frac{m}{eE}$, resembles the spin factor of a free fermion in 2D. This spin factor [12] is equal to $\mathrm{e}^{i\pi Q}$, where

$$Q=\frac{1}{2\pi}\int_0^T d\tau\varepsilon_{\mu\nu}\ddot{x}_\mu\dot{x}_\nu$$

is an (integer) algebraic number of self-intersections of the trajectory. At every counter-clockwise winding of the trajectory, Q increases by 1, while at every clockwise winding, Q

decreases by 1. One can consider a representation for the propagator of a fermion as a path integral over all the trajectories connecting the initial and the final points. The sum of contributions to this path integral produced by two such trajectories, whose Q_1 and Q_2 differ from each other by 1, cancel each other, because

$$\mathrm{e}^{i\pi Q_1} + \mathrm{e}^{i\pi Q_2} = (-1)^{Q_1}\left[1 + (-1)^{Q_2 - Q_1}\right] = 0.$$

For this reason, fermionic random walks are smooth compared to the crumpled bosonic random walks. Formally, this fact is expressed by the different values, for these two types of random walks, of the critical exponent ν in the formula $R \sim L^\nu$, where R is the distance between the initial and the final points, and L is the length of the walk. This critical exponent is equal to 1/2 in the bosonic case, and to 1 in the fermionic case. The number $1/\nu$ is called Hausdorff dimension of the random walk.

5. More applications of path integrals: Polyakov's derivation of the one-loop running coupling in 2D nonlinear O(N) sigma-model and in 4D Yang-Mills theory. Fujikawa's derivation of chiral (Adler-Bell-Jackiw) anomaly in QED.

Polyakov's derivation of the one-loop running coupling in 2D nonlinear O(N) sigma-model and in 4D Yang-Mills theory.

In this chapter, we discuss the method of renormalization [12, 13] based on the integration over fields with large relative momenta. We start with the perturbative renormalization of 2D nonlinear O(3) sigma-model, whose bare action has the form

$$S = \frac{1}{2}\int d^2\xi (\partial_\mu \mathbf{n})^2,$$

where $\mathbf{n} = (n_1, n_2, n_3)$ is a vector of a fixed length, $\mathbf{n}^2 = \frac{1}{g^2}$. In what follows, we denote N-dimensional vectors as $\mathbf{n}$, and 2D-vectors as $\vec{\xi}$. One can introduce spherical coordinates in the field space, so that

$$n_1 = \sqrt{\frac{1}{g^2} - n_3^2}\cdot\cos\varphi, \qquad n_2 = \sqrt{\frac{1}{g^2} - n_3^2}\cdot\sin\varphi,$$

and choose n_3 to be a rapidly varying field component, which will be integrated over. We have

$$\partial_\mu n_1 = -\frac{n_3\cdot\partial_\mu n_3}{\sqrt{\frac{1}{g^2} - n_3^2}}\cdot\cos\varphi - \sqrt{\frac{1}{g^2} - n_3^2}\sin\varphi\cdot\partial_\mu\varphi,$$

$$\partial_\mu n_2 = -\frac{n_3 \cdot \partial_\mu n_3}{\sqrt{\frac{1}{g^2} - n_3^2}} \cdot \sin\varphi + \sqrt{\frac{1}{g^2} - n_3^2}\cos\varphi \cdot \partial_\mu\varphi.$$

Accordingly, the squares of the derivatives read

$$(\partial_\mu n_1)^2 = \frac{(n_3 \cdot \partial_\mu n_3)^2}{\frac{1}{g^2} - n_3^2}\cos^2\varphi + \left(\frac{1}{g^2} - n_3^2\right)\sin^2\varphi \cdot (\partial_\mu\varphi)^2 + \sin(2\varphi) \cdot n_3 \cdot \partial_\mu n_3 \cdot \partial_\mu\varphi,$$

$$(\partial_\mu n_2)^2 = \frac{(n_3 \cdot \partial_\mu n_3)^2}{\frac{1}{g^2} - n_3^2}\sin^2\varphi + \left(\frac{1}{g^2} - n_3^2\right)\cos^2\varphi \cdot (\partial_\mu\varphi)^2 - \sin(2\varphi) \cdot n_3 \cdot \partial_\mu n_3 \cdot \partial_\mu\varphi.$$

Furthermore, under the assumption $|n_3| \ll \frac{1}{g}$, one can approximate the first terms on the right-hand sides of the last two equations as

$$\frac{(n_3 \cdot \partial_\mu n_3)^2}{\frac{1}{g^2} - n_3^2}\cos^2\varphi \simeq g^2(n_3 \cdot \partial_\mu n_3)^2\cos^2\varphi, \quad \frac{(n_3 \cdot \partial_\mu n_3)^2}{\frac{1}{g^2} - n_3^2}\sin^2\varphi \simeq g^2(n_3 \cdot \partial_\mu n_3)^2\sin^2\varphi.$$

Using this approximation, one has

$$(\partial_\mu \mathbf{n})^2 = (\partial_\mu n_1)^2 + (\partial_\mu n_2)^2 + (\partial_\mu n_3)^2 \simeq (\partial_\mu n_3)^2 + \left(\frac{1}{g^2} - n_3^2\right) \cdot (\partial_\mu\varphi)^2 + g^2 n_3^2 (\partial_\mu n_3)^2.$$

With the use of this expression, the partition function splits into the parts with low and high momenta:

$$\mathcal{Z} = \int\limits_{0<p<\Lambda'} \mathcal{D}\varphi(\vec{p})\mathrm{e}^{-\frac{1}{2g^2}\int d^2\xi(\partial_\mu\varphi)^2} \times$$

$$\times \int\limits_{\Lambda'<p<\Lambda} \mathcal{D}n_3(\vec{p})\mathrm{e}^{-\frac{1}{2}\int d^2\xi(\partial_\mu n_3)^2} \exp\left[\frac{1}{2}\int d^2\xi n_3^2(\partial_\mu\varphi)^2 - \frac{g^2}{2}\int d^2\xi n_3^2(\partial_\mu n_3)^2\right].$$

In the weak-coupling regime, $g \ll 1$, the term of the order of g^2 in the last exponent can be omitted. The n_3-integration yields, up to an inessential multiplicative constant, a factor of

$$\exp\left[\frac{1}{2}\int d^2\xi \langle n_3^2\rangle \cdot (\partial_\mu\varphi)^2\right], \quad \text{where} \quad \langle n_3^2\rangle = \int\limits_{\Lambda'<p<\Lambda} \mathcal{D}n_3(\vec{p}) n_3^2(\vec{\xi})\mathrm{e}^{-\frac{1}{2}\int d^2\xi(\partial_\mu n_3)^2}.$$

The partition function takes the form

$$\mathcal{Z} = \int\limits_{0<p<\Lambda'} \mathcal{D}\varphi(\vec{p})\mathrm{e}^{-\frac{1}{2g'^2}\int d^2\xi(\partial_\mu\varphi)^2},$$

where

$$\frac{1}{g'^2} = \frac{1}{g^2} - \langle n_3^2\rangle. \tag{87}$$

The mean value $\langle n_3^2 \rangle$ is represented by a tadpole diagram:

$$\langle n_3^2 \rangle = \int\limits_{\Lambda'<p<\Lambda} \frac{d^2p}{(2\pi)^2} \frac{\mathrm{e}^{i\vec{p}(\vec{\xi}-\vec{\xi})}}{p^2} = \frac{1}{4\pi} \int_{\Lambda'}^{\Lambda} \frac{dp^2}{p^2} = \frac{1}{2\pi} \ln \frac{\Lambda}{\Lambda'}.$$

To derive the renormalization-group equation for g in a differential form, one chooses $\Lambda' = \Lambda - d\Lambda$, that yields

$$\ln \frac{\Lambda}{\Lambda'} = \ln \frac{1}{1 - \frac{d\Lambda}{\Lambda}} \simeq \ln \left(1 + \frac{d\Lambda}{\Lambda}\right) \simeq \frac{d\Lambda}{\Lambda}.$$

Equation (87) takes the form

$$g^{-2}(\Lambda') - g^{-2}(\Lambda) = -\frac{1}{2\pi} \cdot \frac{d\Lambda}{\Lambda}.$$

The left-hand side of this equation, once expanded up to the term linear in $d\Lambda$, is

$$g^{-2}(\Lambda') - g^{-2}(\Lambda) \simeq (\Lambda' - \Lambda) \cdot \frac{dg^{-2}}{d\Lambda} = -d\Lambda \cdot (-2) \cdot g^{-3} \cdot \frac{dg}{d\Lambda} = 2\frac{dg}{g^3}.$$

Thus, the differential renormalization-group equation reads

$$\Lambda \cdot \frac{dg}{d\Lambda} = -\frac{g^3}{4\pi}.$$

Alternatively, it can be written as an equation for $g^2(\Lambda)$. The reason is that, by means of the constraint $\mathbf{n}^2 = \frac{1}{g^2}$, the (bare) partition function of the model is expressed through the square of the (bare) coupling, g^2, and not through g itself. One has

$$\Lambda \cdot \frac{dg^2}{d\Lambda} = 2g \cdot \Lambda \cdot \frac{dg}{d\Lambda} = 2g \cdot \left(-\frac{g^3}{4\pi}\right) = -\frac{g^4}{2\pi}.$$

The fact that the corresponding β-function,

$$\beta(g^2)\Big|_{N=3} = -\frac{g^4}{2\pi}, \tag{88}$$

is negative-definite means that 2D nonlinear O(3) sigma-model is asymptotically free, similarly to the Yang-Mills theory.

The perturbative renormalization of 2D nonlinear O(N) sigma-model, presented above for $N = 3$, has a disadvantage that, at the very beginning of the procedure, one reduces the number of dynamical degrees of freedom to $(N - 1)$, thereby breaking the O(N)-symmetry down to O($N-1$). This problem can be circumvented if one renormalizes the model without recourse to perturbation theory. A way to perform such a nonperturbative renormalization is to reduce the partition function of the model to a saddle-point path-integral. This can be done by implementing the constraint $\mathbf{n}^2 = \frac{1}{g^2}$ through the functional δ-function

$$\delta\left(\mathbf{n}^2 - \frac{1}{g^2}\right),$$

and further representing this δ-function by means of a Lagrange multiplier $\lambda(\vec{\xi})$. The partition function then reads

$$\mathcal{Z} = \int_{-i\infty}^{+i\infty} \mathcal{D}\lambda \int \mathcal{D}\mathbf{n} \exp\left\{-\frac{1}{2}\int d^2\xi \left[(\partial_\mu \mathbf{n})^2 + \lambda\left(\mathbf{n}^2 - \frac{1}{g^2}\right)\right]\right\}.$$

Integration over each of N components of the field $\mathbf{n}$ yields

$$\int \mathcal{D}n_i \exp\left\{-\frac{1}{2}\int d^2\xi \left[(\partial_\mu n_i)^2 + \lambda n_i^2\right]\right\} = \left[\det(-\partial_\mu^2 + \lambda(\vec{\xi}))\right]^{-1/2} =$$

$$= \exp\left[-\frac{1}{2}\,\mathrm{tr}\,\ln\,(-\partial_\mu^2 + \lambda)\right].$$

Therefore, the partition function has the form

$$\mathcal{Z} = \int_{-i\infty}^{+i\infty} \mathcal{D}\lambda \exp\left[-\frac{N}{2}\,\mathrm{tr}\,\ln\,(-\partial_\mu^2 + \lambda) + \frac{1}{2g^2}\int d^2\xi\,\lambda\right].$$

Now, if $\lambda = \mathcal{O}(1)$ at $N \to \infty$, then this is a typical saddle-point integral, since the action is of the order of $\mathcal{O}(N)$, while the entropy, represented by the integration measure, is of the order of $\mathcal{O}(1)$. One can check this *a posteriori*, by calculating the saddle-point value $\lambda_{\mathrm{s.p.}}$. The saddle-point equation, which one obtains by varying the action with respect to λ, has the form

$$-\frac{N}{2}\,\mathrm{tr}\,\frac{1}{-\partial_\mu^2 + \lambda_{\mathrm{s.p.}}} + \frac{1}{2g^2} = 0.$$

In terms of the Green function of the $\mathbf{n}$-field, $G(x, y|\lambda) = \langle x|(-\partial_\mu^2 + \lambda)^{-1}|y\rangle$, the saddle-point equation can be written as $G(x, x|\lambda_{\mathrm{s.p.}}) = \frac{1}{g^2 N}$. Seeking a translationally-invariant solution to this equation, $\lambda_{\mathrm{s.p.}}(\vec{\xi}) = M^2$, one has

$$\frac{1}{g^2} = N \int\limits_{0<p<\Lambda} \frac{d^2p}{(2\pi)^2}\,\frac{1}{\vec{p}^{\,2} + M^2} = \frac{N}{4\pi}\int_0^{\Lambda^2} \frac{dp^2}{p^2 + M^2} \simeq \frac{N}{4\pi}\ln\frac{\Lambda^2}{M^2}. \tag{89}$$

Here, at the last step, we have disregarded M^2 compared to the square of the UV-cutoff, Λ^2, and approximated $\ln\frac{\Lambda^2+M^2}{M^2}$ by $\ln\frac{\Lambda^2}{M^2}$. Equation (89) yields the desired saddle-point value of the Lagrange multiplier:

$$M^2 = \Lambda^2 \cdot \mathrm{e}^{-\frac{4\pi}{g^2 N}}. \tag{90}$$

Such an appearance of the quantity M, with the dimensionality of mass, represents the phenomenon of dimensional transmutation. This quantity is manifestly nonperturbative, i.e. nonanalytic in g. Note that all N components of the **n**-field acquire the same mass, and thus the O(N)-symmetry is not broken within this method of renormalization. Furthermore, the obtained $\lambda_{\rm s.p.}$ is indeed of the order of $\mathcal{O}(1)$ at $N\to\infty$, that justifies our use of the saddle-point approximation in this limit.

One can now use the expression

$$g^2=\frac{4\pi}{N}\frac{1}{\ln\frac{\Lambda^2}{M^2}}=\frac{2\pi}{N}\frac{1}{\ln\frac{\Lambda}{M}}, \tag{91}$$

to derive the β-function in the leading large-N approximation. One has

$$\beta(g^2)=\Lambda\cdot\frac{dg^2}{d\Lambda}=\frac{2\pi}{N}\cdot\Lambda\cdot\frac{(-1)}{\ln^2\frac{\Lambda}{M}}\cdot\frac{1}{\Lambda}=-\frac{2\pi}{N}\cdot\left(\frac{Ng^2}{2\pi}\right)^2.$$

Thus,

$$\beta(g^2)=-\frac{Ng^4}{2\pi}\quad\text{at}\quad N\to\infty.$$

The corresponding perturbative result for the one-loop β-function, valid at any N, reads [13]

$$\beta_{\rm pert}(g^2)=-\frac{(N-2)g^4}{2\pi}. \tag{92}$$

In particular, it reproduces correctly the above-derived ($N=3$)-result, Eq. (88). Thus, 2D nonlinear O(3) sigma-model is asymptotically free at $N>2$, that is the origin of the dimensional transmutation. For $N=2$, there is no asymptotic freedom, since the model with the symmetry group O(2)$\sim$U(1) is Abelian.

We apply now the above-illustrated method of perturbative renormalization to a derivation of the one-loop running coupling in 4D Yang-Mills theory [12]. The bare action of the theory has the form

$$S_0=\frac{1}{4g_0^2}\int d^4x(F_{\mu\nu}^a[A])^2, \tag{93}$$

where $F_{\mu\nu}^a[A]=\partial_\mu A_\nu^a-\partial_\nu A_\mu^a+f^{abc}A_\mu^bA_\nu^c$ is the non-Abelian field-strength tensor, and g_0 is the bare coupling. One splits the total Yang-Mills field A_μ^a into the low- and high-momentum parts,

$$A_\mu^a=B_\mu^a+a_\mu^a. \tag{94}$$

In the so-obtained partition function, one integrates over the high-momentum fields a_μ^a, and arrives at an effective action S for the low-momentum fields B_μ^a. As in any renormalizable field theory, like 2D nonlinear O(3) sigma-model considered above, the functional dependence of this effective action on B_μ^a and on the renormalized coupling g should be the same as that of S_0 on, respectively, A_μ^a and on the bare coupling g_0. We are interested in the momentum-dependence, which the running strong-coupling g acquires by means of the renormalization procedure.

Plugging decomposition (94) into Eq. (93), we have

$$S_0 = \frac{1}{4g_0^2}\int d^4x \left[\partial_\mu(B_\nu^a + a_\nu^a) - \partial_\nu(B_\mu^a + a_\mu^a) + f^{abc}(B_\mu^b + a_\mu^b)(B_\nu^c + a_\nu^c)\right] \times$$
$$\times \left[\partial_\mu(B_\nu^a + a_\nu^a) - \partial_\nu(B_\mu^a + a_\mu^a) + f^{ade}(B_\mu^d + a_\mu^d)(B_\nu^e + a_\nu^e)\right].$$

In terms of the adjoint covariant derivative,

$$(D_\mu a_\nu)^a = \partial_\mu a_\nu^a + f^{abc}B_\mu^b a_\nu^c, \tag{95}$$

this expression can be written as

$$S_0 = \frac{1}{4g_0^2}\times$$
$$\times \int d^4x \left[(D_\mu a_\nu - D_\nu a_\mu)^a + F_{\mu\nu}^a[B] + f^{abc}a_\mu^b a_\nu^c\right]\left[(D_\mu a_\nu - D_\nu a_\mu)^a + F_{\mu\nu}^a[B] + f^{ade}a_\mu^d a_\nu^e\right] =$$
$$= \frac{1}{4g_0^2}\int d^4x \left\{(F_{\mu\nu}^a[B])^2 + [(D_\mu a_\nu - D_\nu a_\mu)^a]^2 + 2F_{\mu\nu}^a[B]\cdot\left[(D_\mu a_\nu - D_\nu a_\mu)^a + f^{abc}a_\mu^b a_\nu^c\right] + \right.$$
$$\left. + \mathcal{O}(|a_\mu^a|^3)\right\}. \tag{96}$$

From now on, we denote for simplicity $F_{\mu\nu}^a[B]$ as $F_{\mu\nu}^a$. Owing to the antisymmetry of this tensor, the term $\frac{1}{2g_0^2}\int d^4x F_{\mu\nu}^a(D_\mu a_\nu - D_\nu a_\mu)^a$ on the right-hand side of Eq. (96) can be written as

$$\frac{1}{2g_0^2}\int d^4x F_{\mu\nu}^a(D_\mu a_\nu - D_\nu a_\mu)^a = \frac{1}{g_0^2}\int d^4x F_{\mu\nu}^a(D_\mu a_\nu)^a \equiv \frac{1}{g_0^2}\int d^4x F_{\mu\nu}^a(\partial_\mu a_\nu^a + f^{abc}B_\mu^b a_\nu^c).$$

Integrating further by parts, and changing the order of indices in f^{abc}, we continue this chain of equalities by writing

$$\frac{1}{g_0^2}\int d^4x F_{\mu\nu}^a(D_\mu a_\nu)^a = \frac{1}{g_0^2}\int d^4x a_\nu^c(-\delta^{ac}\partial_\mu - f^{cba}B_\mu^b)F_{\mu\nu}^a = -\frac{1}{g_0^2}\int d^4x a_\nu^a(D_\mu F_{\mu\nu})^a. \tag{97}$$

Thus, we have just confirmed the known mnemonic rule of the validity of integration by parts for the (adjoint) covariant derivative. Using this rule, we have for the term $\frac{1}{4g_0^2}\int d^4x\,[(D_\mu a_\nu - D_\nu a_\mu)^a]^2$ in Eq. (96):

$$\frac{1}{4g_0^2}\int d^4x\,[(D_\mu a_\nu - D_\nu a_\mu)^a]^2 = \frac{1}{2g_0^2}\int d^4x\left\{[(D_\mu a_\nu)^a]^2 - (D_\mu a_\nu)^a(D_\nu a_\mu)^a\right\} = \frac{1}{2g_0^2}\times$$

$$\times \int d^4x \left[-a^a_\nu D^{ac}_\mu D^{cb}_\mu a^b_\nu + a^a_\nu D^{ac}_\mu D^{cb}_\nu a^b_\mu\right] = \frac{1}{2g_0^2}\int d^4x a^a_\nu \left[-\delta_{\mu\nu} D^{ac}_\lambda D^{cb}_\lambda + D^{ac}_\mu D^{cb}_\nu\right] a^b_\mu. \quad (98)$$

We fix now the so-called background Feynman gauge $(D_\mu a_\mu)^a = 0$, that amounts to including to the action the following gauge-fixing term:

$$\frac{1}{2g_0^2}\int d^4x\,[(D_\mu a_\mu)^a]^2 = -\frac{1}{2g_0^2}\int d^4x a^a_\nu D^{ac}_\nu D^{cb}_\mu a^b_\mu.$$

Adding it to the term $\frac{1}{2g_0^2}\int d^4x a^a_\nu D^{ac}_\mu D^{cb}_\nu a^b_\mu$ on the right-hand side of Eq. (98), we have

$$\frac{1}{2g_0^2}\int d^4x a^a_\nu \left(D^{ac}_\mu D^{cb}_\nu - D^{ac}_\nu D^{cb}_\mu\right) a^b_\mu = \frac{1}{2g_0^2}\int d^4x a^a_\nu \left(-2f^{abc}F^c_{\mu\nu}\right) a^b_\mu =$$

$$= \frac{1}{2g_0^2}\int d^4x a^a_\mu \left(-2f^{abc}F^b_{\mu\nu}\right) a^c_\nu, \quad (99)$$

where at the last step we have renamed the indices as $b \leftrightarrow c$, $\mu \leftrightarrow \nu$. Equations (97)-(99) yield for the bare action (96) the following expression:

$$S_0 = \frac{1}{4g_0^2}\int d^4x \left\{(F^a_{\mu\nu})^2 - 4a^a_\nu (D_\mu F_{\mu\nu})^a - 2a^a_\mu \left[\delta_{\mu\nu}(D^2_\lambda)^{ac} + 2f^{abc}F^b_{\mu\nu}\right] a^c_\nu + \mathcal{O}(|a^a_\mu|^3)\right\}. \quad (100)$$

Before doing the integration over a^a_μ's, we use the expression for the SU(N)-generators in the adjoint representation, $(t^b)^{ac} = -if^{bac}$, yielding $f^{abc} = -i(t^b)^{ac}$, to write the covariant derivative (95) as $(D_\mu a_\nu)^a = \partial_\mu a^a_\nu - i(t^b)^{ac}B^b_\mu a^c_\nu$, or in short $D_\mu = \partial_\mu - iB^a_\mu t^a$. Using this formula, we have for the square of the covariant derivative, acting on some function $f(x)$:

$$D^2_\mu f = (\partial_\mu - iB^a_\mu t^a)(\partial_\mu - iB^b_\mu t^b)f = \partial^2 f - i(\partial_\mu B^a_\mu)t^a f - 2iB^a_\mu t^a \partial_\mu f - B^a_\mu B^b_\mu t^a t^b f,$$

or simply

$$-D^2_\mu = -\partial^2 + it^a(\partial_\mu B^a_\mu + 2B^a_\mu \partial_\mu) + (B^a_\mu t^a)^2 \equiv -\partial^2 + f_1(x) + f_2(x).$$

We proceed now to the integration over a^a_μ's, starting with the following contribution to the one-loop effective action:

$$S_{\rm I} = 2\cdot\frac{1}{2}\,{\rm tr}\,\ln(-2D^2_\mu). \quad (101)$$

Here, the overall factor of 2 is due to the number of physical polarizations of the a^a_μ-gluons, while the factor of 2 inside the logarithm yields just an inessential additive constant. We have

$$\frac{1}{2}\,{\rm tr}\,\ln(-D^2_\mu) = {\rm const} + \frac{1}{2}\,{\rm tr}\,\ln\left[1 + (-\partial^2)^{-1}(f_1 + f_2)\right],$$

where "const" is again some inessential additive constant. Taylor expansion of the logarithm in this formula yields in the one-loop approximation under study:

$$\frac{1}{2}\,\mathrm{tr}\left[(-\partial^2)^{-1}_{xx}f_2(x)-\frac{1}{2}(-\partial^2)^{-1}_{xy}f_1(y)(-\partial^2)^{-1}_{yx}f_1(x)\right].$$

The first and the second terms in the brackets describe respectively a tadpole diagram and a one-loop diagram. Both these diagrams have two external lines of the B^a_μ-field (and the a^a_μ-field propagating inside the loops). Fourier-transforming the B^a_μ-field and the a^a_μ-propagator as

$$B^a_\mu(x)=\int\frac{d^4p}{(2\pi)^4}\mathrm{e}^{ipx}B^a_\mu(p),\qquad (-\partial^2)^{-1}_{xy}=\int\frac{d^4q}{(2\pi)^4}\frac{\mathrm{e}^{iq(x-y)}}{q^2},$$

we have for the contribution of the tadpole diagram:

$$\mathrm{tr}\left[(-\partial^2)^{-1}_{xx}f_2(x)\right]=\mathrm{tr}\,(t^at^b)\cdot\int d^4xB^a_\mu(x)B^b_\mu(x)\int\frac{d^4q}{(2\pi)^4}\frac{1}{q^2}=$$

$$=\mathrm{tr}\,(t^at^b)\cdot\int\frac{d^4q}{(2\pi)^4}\frac{d^4p}{(2\pi)^4}B^a_\mu(p)B^b_\mu(-p)\frac{1}{q^2}.$$

For simplicity, we denote the Fourier image $B^a_\mu(p)$ in the same way as the field $B^a_\mu(x)$ itself, distinguishing the two by their arguments. Furthermore, using the cyclic permutation under the trace, we can write the contribution of the other one-loop diagram as

$$-\frac{1}{2}\,\mathrm{tr}\left[f_1(x)(-\partial^2)^{-1}_{xy}f_1(y)(-\partial^2)^{-1}_{yx}\right]=\frac{1}{2}\,\mathrm{tr}\,(t^at^b)\cdot\int d^4xd^4y\int\frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\times$$

$$\times\left\{\left[(\partial_\mu B^a_\mu(x))+2B^a_\mu(x)\partial^x_\mu\right]\frac{\mathrm{e}^{ip(x-y)}}{p^2}\right\}\cdot\left\{\left[(\partial_\nu B^b_\nu(y))+2B^b_\nu(y)\partial^y_\nu\right]\frac{\mathrm{e}^{iq(y-x)}}{q^2}\right\}.\qquad(102)$$

One can further use the Fourier transforms,

$$\left[(\partial_\mu B^a_\mu(x))+2B^a_\mu(x)\partial^x_\mu\right]\frac{\mathrm{e}^{ip(x-y)}}{p^2}=\int\frac{d^4k}{(2\pi)^4}\mathrm{e}^{ikx}B^a_\mu(k)(ik_\mu+2ip_\mu)\frac{\mathrm{e}^{ip(x-y)}}{p^2}$$

and

$$\left[(\partial_\nu B^b_\nu(y))+2B^b_\nu(y)\partial^y_\nu\right]\frac{\mathrm{e}^{iq(y-x)}}{q^2}=\int\frac{d^4t}{(2\pi)^4}\mathrm{e}^{ity}B^b_\nu(t)(it_\nu+2iq_\nu)\frac{\mathrm{e}^{iq(y-x)}}{q^2},$$

to integrate over d^4x and d^4y:

$$\int d^4x\mathrm{e}^{ix(k+p-q)}=(2\pi)^4\delta(k+p-q),\qquad\int d^4y\mathrm{e}^{iy(t+q-p)}=(2\pi)^4\delta(t+q-p).$$

This yields for Eq. (102):

$$-\frac{1}{2}\,\mathrm{tr}\,(t^at^b)\times$$

$$\times \int \frac{d^4p}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} d^4k d^4t \, B^a_\mu(k) B^b_\nu(t)(k+2p)_\mu(t+2q)_\nu \, \delta(k+p-q)\delta(t-p+q) \, \frac{1}{p^2q^2}.$$

Integration over d^4q leads to $q = k + p$ and $\delta(t - p + q) = \delta(t + k)$. Further integration over d^4t yields $t = -k$ and $t + 2q = k + 2p$. Accordingly, Eq. (102) takes the form

$$-\frac{1}{2} \operatorname{tr}(t^a t^b) \cdot \int \frac{d^4k}{(2\pi)^4} \frac{d^4p}{(2\pi)^4} B^a_\mu(k) B^b_\nu(-k) \frac{(k+2p)_\mu (k+2p)_\nu}{p^2(k+p)^2}.$$

Renaming the momenta as $p \to q$ and $k \to p$, we finally obtain for Eq. (102):

$$-\frac{1}{2} \operatorname{tr}(t^a t^b) \cdot \int \frac{d^4q}{(2\pi)^4} \frac{d^4p}{(2\pi)^4} B^a_\mu(p) B^b_\nu(-p) \frac{(p+2q)_\mu (p+2q)_\nu}{q^2(p+q)^2}.$$

The sum of the tadpole and the other one-loop diagram reads

$$\frac{1}{2} \operatorname{tr} \left[(-\partial^2)^{-1}_{xx} f_2(x) - \frac{1}{2} (-\partial^2)^{-1}_{xy} f_1(y) (-\partial^2)^{-1}_{yx} f_1(x) \right] = \operatorname{tr}(t^a t^b) \cdot \int \frac{d^4p}{(2\pi)^4} B^a_\mu(p) B^b_\nu(-p) \times$$

$$\times \frac{1}{2} \int \frac{d^4q}{(2\pi)^4} \left[\delta_{\mu\nu} \cdot \frac{1}{q^2} - \frac{1}{2} \cdot \frac{(2q+p)_\mu (2q+p)_\nu}{q^2(q+p)^2} \right]. \tag{103}$$

Owing to the conservation of the electric current in massless scalar QED, the polarization operator represented by the q-integral should obey the condition

$$p_\mu \int \frac{d^4q}{(2\pi)^4} \left[\delta_{\mu\nu} \cdot \frac{1}{q^2} - \frac{1}{2} \cdot \frac{(2q+p)_\mu (2q+p)_\nu}{q^2(q+p)^2} \right] = 0.$$

This condition fixes the tensor structure of the q-integral:

$$\frac{1}{2} \int \frac{d^4q}{(2\pi)^4} \left[\delta_{\mu\nu} \cdot \frac{1}{q^2} - \frac{1}{2} \cdot \frac{(2q+p)_\mu (2q+p)_\nu}{q^2(q+p)^2} \right] = (p^2 \delta_{\mu\nu} - p_\mu p_\nu) \bar{\Pi}(p^2). \tag{104}$$

The scalar function $\bar{\Pi}(p^2)$ defined by this equation can be found by contracting the indices and expanding the integral up to the terms quadratic in p_μ. Contracting the indices, we have

$$3p^2 \bar{\Pi}(p^2) = \frac{1}{2} \int \frac{d^4q}{(2\pi)^4 |q|^4} \left[4q^2 - \frac{1}{2} \cdot \frac{p^2 + 4qp + 4q^2}{1 + \frac{2qp}{q^2} + \frac{p^2}{q^2}} \right].$$

Furthermore, Taylor expansion of the square bracket in this formula in p_μ yields

$$4q^2 - \frac{1}{2} \cdot \frac{p^2 + 4qp + 4q^2}{1 + \frac{2qp}{q^2} + \frac{p^2}{q^2}} \simeq 4q^2 - \frac{1}{2}(p^2 + 4qp + 4q^2) \left[1 - \frac{2qp}{q^2} - \frac{p^2}{q^2} + \frac{4(qp)^2}{|q|^4} \right] \simeq$$

$$\simeq 4q^2 - \frac{1}{2} \left[p^2 + 4qp + 4q^2 - \frac{8(qp)^2}{q^2} - 8qp - 4p^2 + \frac{16(qp)^2}{q^2} \right].$$

The q^2-terms in this expression produce an inessential additive p_μ-independent constant, and the terms linear in q_μ yield 0 upon the integration. The remaining terms, $\frac{3}{2}p^2 - \frac{4(qp)^2}{q^2}$, yield

$$3p^2\bar{\Pi}(p^2) = \frac{1}{4}\int\frac{d^4q}{(2\pi)^4|q|^4}\left[3p^2 - \frac{8(qp)^2}{q^2}\right] \quad \Rightarrow \quad \bar{\Pi}(p^2) = \frac{1}{64\pi^4}\int\frac{d^4q}{|q|^4}\left[1 - \frac{8}{3}\cdot\frac{(qp)^2}{q^2p^2}\right].$$

We recall now that q_μ is the momentum of the a_μ^a-gluons, which propagate over the loop of the polarization operator, while p_μ is the momentum of the B_μ^a-gluons, which interact with the polarization operator by means of two external lines. Therefore, the q_μ-integration extends in the range $|p| < |q| < \Lambda_0$, where Λ_0 is a bare ultraviolet cut-off. For the two q_μ-integrals entering $\bar{\Pi}(p^2)$ we have

$$\int\frac{d^4q}{|q|^4} = 2\pi^2\ln\frac{\Lambda_0}{|p|} = \pi^2\ln\frac{\Lambda_0^2}{p^2} \quad \text{and} \quad \int\frac{d^4q}{|q|^4}\frac{q_\mu q_\nu}{q^2} = \frac{\delta_{\mu\nu}}{4}\cdot\pi^2\ln\frac{\Lambda_0^2}{p^2}.$$

Therefore, we obtain

$$\bar{\Pi}(p^2) = \frac{1}{64\pi^2}\left(1 - \frac{2}{3}\right)\ln\frac{\Lambda_0^2}{p^2} = \frac{1}{192\pi^2}\ln\frac{\Lambda_0^2}{p^2}.$$

Plugging this expression into Eq. (104) and further into Eq. (103), we obtain for the corresponding contribution to the one-loop effective action, Eq. (101), the following result:

$$S_{\rm I} = 2\cdot\,{\rm tr}\,(t^at^b)\cdot\int\frac{d^4p}{(2\pi)^4}B_\mu^a(p)B_\nu^b(-p)\cdot(p^2\delta_{\mu\nu} - p_\mu p_\nu)\cdot\frac{1}{192\pi^2}\ln\frac{\Lambda_0^2}{p^2}. \tag{105}$$

Consider now the linear (i.e. Abelian) part of the field-strength tensor in the momentum representation:

$$F_{\mu\nu}^{a\,({\rm lin})}(p) = \int d^4x{\rm e}^{-ipx}[\partial_\mu B_\nu^a(x) - \partial_\nu B_\mu^a(x)] = \int d^4x{\rm e}^{-ipx}[ip_\mu B_\nu^a(x) - ip_\nu B_\mu^a(x)],$$

where at the last step we have integrated by parts. The product of two such linear contributions to the field-strength tensor reads

$$F_{\mu\nu}^{a\,({\rm lin})}(p)F_{\mu\nu}^{b\,({\rm lin})}(-p) = \int d^4xd^4y{\rm e}^{-ipx+ipy}[ip_\mu B_\nu^a(x) - ip_\nu B_\mu^a(x)][-ip_\mu B_\nu^b(y) + ip_\nu B_\mu^b(y)] =$$

$$= p^2B_\nu^a(p)B_\nu^b(-p) - p_\mu p_\nu B_\nu^a(p)B_\mu^b(-p) - p_\mu p_\nu B_\mu^a(p)B_\nu^b(-p) + p^2B_\mu^a(p)B_\mu^b(-p).$$

Renaming the indices $\mu\leftrightarrow\nu$ in the term $-p_\mu p_\nu B_\nu^a(p)B_\mu^b(-p)$, we get

$$F_{\mu\nu}^{a\,({\rm lin})}(p)F_{\mu\nu}^{b\,({\rm lin})}(-p) = 2(p^2\delta_{\mu\nu} - p_\mu p_\nu)B_\mu^a(p)B_\nu^b(-p).$$

In the one-loop approximation under study, it is legitimate to use this expression in Eq. (105). Beyond this approximation, one should promote $F^{a\,(\mathrm{lin})}_{\mu\nu}(p)$ to the full non-Abelian field-strength tensor. In fact, it can be shown that the cubic, $\sim (B^a_\mu)^3$, and the quartic, $\sim (B^a_\mu)^4$, terms in $F^a_{\mu\nu}$ are recovered correctly, so that the full renormalized action remains gauge-invariant to every given order of perturbation theory. Therefore, implying the one-loop approximation, we can substitute $B^a_\mu(p)B^b_\nu(-p)\cdot(p^2\delta_{\mu\nu}-p_\mu p_\nu)$ in Eq. (105) by $\frac{1}{2}F^a_{\mu\nu}(p)F^b_{\mu\nu}(-p)$. Noticing also that

$$\mathrm{tr}\,(t^a t^b) = N\delta^{ab},$$

we can write

$$S_{\mathrm{I}} = N\int\frac{d^4p}{(2\pi)^4}F^a_{\mu\nu}(p)F^a_{\mu\nu}(-p)\cdot\frac{1}{192\pi^2}\ln\frac{\Lambda_0^2}{p^2} \equiv \frac{1}{4}\int\frac{d^4p}{(2\pi)^4}F^a_{\mu\nu}(p)F^a_{\mu\nu}(-p)\cdot\Pi^{\mathrm{dia}}(p^2), \quad (106)$$

where

$$\Pi^{\mathrm{dia}}(p^2) = \frac{N}{48\pi^2}\ln\frac{\Lambda_0^2}{p^2}.$$

The superscript "dia" is because this contribution to the vacuum polarization comes from the term $\sim a^a_\mu(D^2_\lambda)^{ac}a^c_\mu$ in Eq. (100), which describes the Landau diamagnetic interaction of the B^a_μ-field with the orbital motion of the a^a_μ-gluons. This interaction is present in the Abelian case as well, and leads to the screening of a test (color) charge in the vacuum.

A specific property of non-Abelian theories, distinguishing them from the Abelian ones, is that they additionally possess the Pauli paramagnetic interaction of the B^a_μ-field with the spin of the a^a_μ-gluons. Being opposite in sign to the diamagnetic interaction, the paramagnetic one turns out to be stronger, that leads to the antiscreening of a test color charge in the non-Abelian Yang-Mills vacuum. We will now demonstrate this statement quantitatively, by calculating the paramagnetic contribution to the one-loop effective action. This contribution stems from the term $\sim a^a_\mu f^{abc}F^b_{\mu\nu}a^c_\nu$ in Eq. (100), and reads

$$S_{\mathrm{II}} = \frac{1}{2}\,\mathrm{tr}\,\ln\left[1+(-\partial^2)^{-1}(2f^{abc}F^b_{\mu\nu})\right] \simeq -\frac{1}{4}(-\partial^2)^{-1}_{xy}(2f^{abc}F^b_{\mu\nu}(y))(-\partial^2)^{-1}_{yx}(2f^{adc}F^d_{\mu\nu}(x)),$$

where at the last step we have used the one-loop approximation. Since $f^{abc}f^{adc} = N\delta^{bd}$, this expression can be written as

$$S_{\mathrm{II}} = -N\int d^4x d^4y F^a_{\mu\nu}(x)F^a_{\mu\nu}(y)\int\frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\frac{\mathrm{e}^{ip(x-y)}}{p^2}\frac{\mathrm{e}^{iq(y-x)}}{q^2}.$$

Fourier transforming the product of the field-strength tensors as

$$F^a_{\mu\nu}(x)F^a_{\mu\nu}(y) = \int\frac{d^4k}{(2\pi)^4}\frac{d^4t}{(2\pi)^4}\mathrm{e}^{ikx+ity}F^a_{\mu\nu}(k)F^a_{\mu\nu}(t),$$

we integrate over d^4x and d^4y:

$$\int d^4x e^{ix(k+p-q)} = (2\pi)^4\delta(k+p-q), \qquad \int d^4y e^{iy(t+q-p)} = (2\pi)^4\delta(t+q-p).$$

This yields

$$S_{\rm II} = -N\int \frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}d^4kd^4tF^a_{\mu\nu}(k)F^a_{\mu\nu}(t)\delta(k+p-q)\delta(t+q-p)\frac{1}{p^2q^2}.$$

Further integration over d^4q yields $q = k+p$, and integration over d^4t yields $t = p-q = -k$, so that

$$S_{\rm II} = -N\int \frac{d^4k}{(2\pi)^4}\frac{d^4p}{(2\pi)^4}F^a_{\mu\nu}(k)F^a_{\mu\nu}(-k)\frac{1}{p^2(k+p)^2}.$$

Renaming the momenta $k \to p$, $p \to q$, we arrive at the expression similar to Eq. (106):

$$S_{\rm II} = -N\int \frac{d^4p}{(2\pi)^4}F^a_{\mu\nu}(p)F^a_{\mu\nu}(-p)\int \frac{d^4q}{(2\pi)^4}\frac{1}{q^2(q+p)^2}.$$

Furthermore, within the approximation where one is interested in the leading logarithmic contribution to the vacuum polarization, it is possible to neglect the momentum p_μ of the B^a_μ-gluons compared to the momentum q_μ of the a^a_μ-gluons in $(q+p)^2$, i.e.

$$\int \frac{d^4q}{(2\pi)^4}\frac{1}{q^2(q+p)^2} \simeq \int \frac{d^4q}{(2\pi)^4}\frac{1}{|q|^4} = \frac{1}{16\pi^2}\ln\frac{\Lambda_0^2}{p^2}.$$

Therefore, the paramagnetic contribution to the one-loop effective action reads

$$S_{\rm II} = \frac{1}{4}\int \frac{d^4p}{(2\pi)^4}F^a_{\mu\nu}(p)F^a_{\mu\nu}(-p)\cdot\Pi^{\rm para}(p^2), \tag{107}$$

where

$$\Pi^{\rm para}(p^2) = -\frac{N}{4\pi^2}\ln\frac{\Lambda_0^2}{p^2}.$$

Equations (106) and (107), along with the bare Lagrangian $\frac{1}{4g_0^2}(F^a_{\mu\nu})^2$, yield the following renormalized one-loop effective action:

$$S_{1-{\rm loop}} = \frac{1}{4}\int \frac{d^4p}{(2\pi)^4}F^a_{\mu\nu}(p)F^a_{\mu\nu}(-p)\cdot\left[\frac{1}{g_0^2}+\Pi^{\rm dia}(p^2)+\Pi^{\rm para}(p^2)\right] =$$

$$= \frac{1}{4}\int \frac{d^4p}{(2\pi)^4}F^a_{\mu\nu}(p)F^a_{\mu\nu}(-p)\cdot\left[\frac{1}{g_0^2}+\frac{N}{4\pi^2}\left(\frac{1}{12}-1\right)\ln\frac{\Lambda_0^2}{p^2}\right].$$

The fact that the absolute value of the paramagnetic contribution to the one-loop vacuum polarization is 12 times larger than the diamagnetic contribution leads to the antiscreening of a test color charge and to the asymptotic freedom of the running strong coupling $g(p)$. To obtain the latter, we write the effective action as

$$S_{1-\text{loop}} = \frac{1}{4}\int \frac{d^4p}{(2\pi)^4}\frac{1}{g^2(p)}F^a_{\mu\nu}(p)F^a_{\mu\nu}(-p),$$

where

$$\frac{1}{g^2(p)} = \frac{1}{g_0^2} - \frac{b}{16\pi^2}\ln\frac{\Lambda_0^2}{p^2} \quad \text{and} \quad b = \frac{11}{3}N. \tag{108}$$

Note that, in this formula, the bare coupling g_0 is just the running coupling $g(p)$ defined at the bare cut-off Λ_0, i.e. $g_0 = g(\Lambda_0)$. One can also define the renormalized cut-off Λ related to the bare one as

$$\Lambda = \Lambda_0 \exp\left(-\frac{8\pi^2}{bg_0^2}\right).$$

[Note the following correspondence of this formula with Eq. (90) of 2D nonlinear O(N) sigma-model: $M|_{\mathrm{O}(N)} \to \Lambda|_{\mathrm{YM}}$, $\Lambda|_{\mathrm{O}(N)} \to \Lambda_0|_{\mathrm{YM}}$.] Thus, substituting to Eq. (108)

$$\frac{\Lambda_0^2}{p^2} = \frac{\Lambda^2}{p^2}\exp\left(\frac{16\pi^2}{bg_0^2}\right),$$

one has

$$\frac{1}{g^2(p)} - \frac{1}{g_0^2} = -\frac{b}{16\pi^2}\left(\ln\frac{\Lambda^2}{p^2} + \frac{16\pi^2}{bg_0^2}\right) = -\frac{b}{16\pi^2}\ln\frac{\Lambda^2}{p^2} - \frac{1}{g_0^2}.$$

Defining $\alpha_s(p) \equiv \frac{g^2(p)}{4\pi}$, we arrive at the known result [14]:

$$\alpha_s(p) = \frac{4\pi}{b\cdot\ln\frac{p^2}{\Lambda^2}}.$$

[Note again the following correspondence with Eq. (91) of 2D nonlinear O(N) sigma-model: $\Lambda|_{\mathrm{O}(N)} \to |p|_{\mathrm{YM}}$.] The Yang-Mills β-function $\beta(g) = |p|\frac{dg}{d|p|}$ can readily be obtained by differentiating the formula

$$g(p) = \frac{4\pi}{\sqrt{2b\ln\frac{|p|}{\Lambda}}},$$

that yields

$$\beta(g) = -\frac{1}{2}\cdot g(p)\cdot\frac{1}{\ln\frac{|p|}{\Lambda}}.$$

Noticing that

$$\frac{1}{\ln\frac{|p|}{\Lambda}} = \frac{2b}{(4\pi)^2}\cdot g^2,$$

we arrive at the one-loop Yang-Mills β-function

$$\beta(g) = -\frac{b}{(4\pi)^2}\cdot g^3.$$

It can be compared with the β-function of 2D nonlinear O(N) sigma-model, Eq. (92).

Fujikawa's derivation of chiral (Adler-Bell-Jackiw) anomaly in QED.

Consider the Euclidean action of QED,

$$S[A_\mu,\bar{\psi},\psi]=\int d^4x\left[\frac{1}{4}F_{\mu\nu}^2+\bar{\psi}\left(\hat{D}+m\right)\psi\right],$$

where $D_\mu=\partial_\mu-ieA_\mu$ is the covariant derivative, $\hat{D}\equiv\gamma_\mu D_\mu$, and $\bar{\psi}=\psi^\dagger\gamma_4$. The Euclidean γ-matrices satisfy the anticommutation relation $\gamma_\mu\gamma_\nu+\gamma_\nu\gamma_\mu=2\delta_{\mu\nu}\mathbf{1}_{4\times4}$, and the matrix γ_5 reads $\gamma_5=\gamma_1\gamma_2\gamma_3\gamma_4$. The local chiral transformation is defined as

$$\psi'(x)=\mathrm{e}^{i\alpha(x)\gamma_5}\psi(x),\qquad \bar{\psi}'(x)=\bar{\psi}(x)\mathrm{e}^{i\alpha(x)\gamma_5}, \tag{109}$$

where $\alpha(x)\equiv\alpha$ is the parameter of the transformation. Consider the variation of the action under an infinitesimal chiral transformation $\delta\psi=i\alpha\gamma_5\psi$, $\delta\bar{\psi}=i\alpha\bar{\psi}\gamma_5$. The variation of the mass term of the electron reads $\delta(\bar{\psi}\psi)\simeq 2i\alpha\bar{\psi}\gamma_5\psi$, where "$\simeq$" means "disregarding the $\mathcal{O}(\alpha^2)$-terms". The variation of the kinetic term has the form

$$\delta\left(\bar{\psi}\hat{D}\psi\right)=\bar{\psi}(1+i\alpha\gamma_5)\hat{D}(1+i\alpha\gamma_5)\psi-\bar{\psi}\hat{D}\psi\simeq i\left[\alpha\bar{\psi}\gamma_5\hat{D}\psi+\bar{\psi}\hat{D}(\alpha\gamma_5\psi)\right].$$

This expression can be simplified by noticing that

$$\hat{D}(\alpha\gamma_5\psi)=\left(\hat{\partial}-ie\hat{A}\right)(\alpha\gamma_5\psi)=-\gamma_5\left(\hat{\partial}-ie\hat{A}\right)(\alpha\psi)=-\gamma_5(\hat{\partial}\alpha)\psi-\gamma_5\alpha\hat{D}\psi.$$

Anticommuting γ_μ with γ_5 in the term $-\gamma_5(\hat{\partial}\alpha)\psi$, we obtain

$$\delta\left(\bar{\psi}\hat{D}\psi\right)=(\partial_\mu\alpha)\cdot i\bar{\psi}\gamma_\mu\gamma_5\psi\equiv(\partial_\mu\alpha)\cdot J_\mu^A,$$

where $J_\mu^A=i\bar{\psi}\gamma_\mu\gamma_5\psi$ is the axial current. Thus, the variation of the action reads

$$\delta S[A_\mu,\bar{\psi},\psi]=\int d^4x\left[(\partial_\mu\alpha)\cdot J_\mu^A+2im\alpha\bar{\psi}\gamma_5\psi\right]. \tag{110}$$

Integrating in the first term on the right-hand side by parts, we conclude that the invariance of the action under chiral transformations, expressed by the equation $\delta S=0$, leads to the following formula for the divergence of the axial current:

$$\partial_\mu J_\mu^A=2im\bar{\psi}\gamma_5\psi.$$

Therefore, in the massless case, the axial current is conserved on the classical level, i.e. $\partial_\mu J_\mu^A=0$ for $m=0$.

Let us now consider the change of the Euclidean path-integral measure under the chiral transformation, Eq. (109). It reads

$$\mathcal{D}\bar{\psi}\mathcal{D}\psi = \mathcal{D}\bar{\psi}'\mathcal{D}\psi' \cdot \det\left[\mathrm{e}^{2i\alpha(x)\gamma_5}\delta(x-y)\right],$$

where the determinant is just the Jacobian of the chiral transformation. Since ψ and $\bar{\psi}$ in the path integral are Grassmann variables, the determinant appears to the power $+1$, rather than -1 (as it would be the case for bosonic integration variables). The determinant is taken over the spinor indices and over the continuous space-time coordinates x and y. Following the method of Ref. [15], one can furthermore regularize the δ-function in the formula above, by expanding $\psi(x)$ and $\bar{\psi}(x)$ over a complete set $\{\phi_n(x)\}$ of orthonormal functions. The property of orthonormality means

$$\int d^4x \phi_n^{j\,\dagger}(x)\phi_m^i(x) = \delta_{nm}\delta^{ij}, \tag{111}$$

where i and j are spinor indices. The expansions have the form

$$\psi^i(x) = \sum_{n=1}^{\infty} c_n^i \phi_n^i(x) \quad \text{and} \quad \bar{\psi}^i(x) = \sum_{n=1}^{\infty} \bar{c}_n^i \phi_n^{i\,\dagger}(x),$$

where c_n^i and $\bar{c}_n^i$ are Grassmann variables, and in the both formulae no summation over i is implied. The corresponding path-integral measure reads [cf. the quantum-mechanical case, Eq. (5), and the text preceding it]

$$\mathcal{D}\bar{\psi}\mathcal{D}\psi = \prod_{n=1}^{\infty}\prod_i d\bar{c}_n^i \cdot \prod_{m=1}^{\infty}\prod_j dc_m^j.$$

The regularization of the measure implies a restriction to a large, but finite, set of basis functions ϕ_n. The regularized measure is accordingly defined as

$$(\mathcal{D}\bar{\psi})_R(\mathcal{D}\psi)_R = \prod_{n=1}^{N}\prod_i d\bar{c}_n^i \cdot \prod_{m=1}^{N}\prod_j dc_m^j, \quad \text{where} \quad N \gg 1.$$

It changes under the chiral transformation as

$$(\mathcal{D}\bar{\psi})_R(\mathcal{D}\psi)_R = (\mathcal{D}\bar{\psi}')_R(\mathcal{D}\psi')_R \cdot \det_{\substack{n,m\\k,j}}\left[\int d^4x \phi_n^{k\,\dagger}(x)\left(\mathrm{e}^{2i\alpha(x)\gamma_5}\right)^{kj}\phi_m^j(x)\right].$$

In particular, for an infinitesimal chiral transformation, i.e. $\alpha \to 0$, the determinant can be expanded up to the term linear in α as

$$\det_{\substack{n,m\\k,j}}[\cdots] \simeq 1 + 2i\sum_{n=1}^{N}\int d^4x \phi_n^\dagger(x)\alpha(x)\gamma_5\phi_n(x),$$

where "1" on the right-hand side is due to the orthonormality of ϕ_n's, Eq. (111).

Let us consider for a while the unregularized case, by taking the $(N \to \infty)$-limit. In that case, the orthonormal functions $\phi_n(x)$ satisfy the completeness relation

$$\sum_{n=1}^{\infty} \phi_n^i(x)\phi_n^{j\,\dagger}(y) = \delta^{ij}\delta(x-y), \tag{112}$$

that yields

$$\ln \det_{\substack{n,m\\k,j}} \left[\int d^4x \phi_n^{k\,\dagger}(x) \left(e^{2i\alpha(x)\gamma_5}\right)^{kj} \phi_m^j(x)\right] \to \mathrm{tr}\,\ln\left[1 + 2i\sum_{n=1}^{\infty}\int d^4x \phi_n^\dagger(x)\alpha(x)\gamma_5\phi_n(x)\right] =$$

$$= \ln\left[1 + 2i\delta(0)\cdot \mathrm{sp}\,\gamma_5 \cdot \int d^4x\alpha(x)\right] \to 2i\delta(0)\cdot\mathrm{sp}\,\gamma_5\cdot\int d^4x\alpha(x) \quad \text{at} \quad \alpha(x)\to 0, \tag{113}$$

where "sp" denotes the trace over the spinor indices. Since $\mathrm{sp}\,\gamma_5 = 0$, while $\delta(0) = \infty$, Eq. (113) represents an uncertainty of the form $\infty \cdot 0$. The regularization, which we are now returning to, is intended to remove this uncertainty.

Specifically, the infinity produced by $\delta(x-y)$ in Eq. (112) can be removed by assuming that the completeness relation is changed in the regularized case as

$$\sum_{n=1}^{N} \phi_n^i(x)\phi_n^{j\,\dagger}(y) = R^{ij}(x,y).$$

The right-hand side of this formula is the (i,j)-element of some matrix-valued operator $\mathbf{R}$, which can be chosen, e.g., in the form

$$\mathbf{R} = \frac{1}{\mathbf{1} - (a\hat{D})^2}, \tag{114}$$

where $\mathbf{1} \equiv \mathbf{1}_{4\times 4}$, and $1/a$ is the ultraviolet cut-off. Other possible forms of the regularizing operator are $\mathbf{R} = e^{a^2\hat{D}^2}$, $\mathbf{R} = \frac{1}{\mathbf{1}+a\hat{D}}$, etc. The regularization is removed in the $(a \to 0)$-limit, where $\mathbf{R}(x,y) \to \mathbf{1}\cdot\delta(x-y)$. Upon the calculation of the divergence of the axial current, we will formulate the condition, which should be obeyed by the operator $\mathbf{R}$, in order for the result to be independent of the particular functional form of $\mathbf{R}$. It will be shown that the above-given examples of the regularizing operator satisfy that condition.

The partition function associated with the integration over the fermionic fields,

$$\mathcal{Z}[A_\mu] \equiv \int \mathcal{D}\bar{\psi}\mathcal{D}\psi e^{-S[A_\mu,\bar{\psi},\psi]},$$

changes under the chiral transformation to

$$\mathcal{Z}'[A_\mu] = \int \mathcal{D}\bar{\psi}'\mathcal{D}\psi' \mathrm{e}^{-(S[A_\mu,\bar{\psi}',\psi']+\delta S[A_\mu,\bar{\psi}',\psi'])}.$$

Here, the variation of the action is given by Eq. (110), i.e. in the massless case of interest $\delta S = -\int d^4x\alpha\cdot\partial_\mu J_\mu^A$, while the change of the integration measure is given by the regularized version of Eq. (113), that is

$$\exp\left\{2i\int d^4x\alpha(x)\cdot \mathrm{sp}\ [\gamma_5\cdot \mathbf{R}(x,x)]\right\}.$$

Chiral invariance on the quantum level is imposed by the condition $\mathcal{Z}'[A_\mu] = \mathcal{Z}[A_\mu]$, which thus reads

$$-\delta S = \int d^4x\alpha(x)\cdot\partial_\mu J_\mu^A = 2i\int d^4x\alpha(x)\cdot \mathrm{sp}\ [\gamma_5\cdot\mathbf{R}(x,x)]\,. \tag{115}$$

We will calculate $\mathrm{sp}\ [\gamma_5\cdot\mathbf{R}(x,x)]$ by choosing the regularizing operator in the form (114). To this end, we use the fact that

$$\gamma_\mu\gamma_\nu = \frac{1}{2}(\gamma_\mu\gamma_\nu+\gamma_\nu\gamma_\mu+\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu) = \delta_{\mu\nu}+\frac{1}{2}[\gamma_\mu,\gamma_\nu],$$

to represent $\hat{D}^2$ as

$$\hat{D}^2 = D_\mu^2+\frac{1}{2}[\gamma_\mu,\gamma_\nu]D_\mu D_\nu.$$

Furthermore, the product $D_\mu D_\nu$ can also be written as a sum of a symmetric and an anti-symmetric terms,

$$D_\mu D_\nu = \frac{1}{2}(D_\mu D_\nu + D_\nu D_\mu + [D_\mu, D_\nu]),$$

so that, upon the multiplication by $[\gamma_\mu,\gamma_\nu]$, only the antisymmetric term contributes:

$$\hat{D}^2 = D_\mu^2+\frac{1}{4}[\gamma_\mu,\gamma_\nu][D_\mu,D_\nu].$$

The commutator $[D_\mu, D_\nu]$ reads

$$[D_\mu,D_\nu] \equiv (\partial_\mu - ieA_\mu)(\partial_\nu - ieA_\nu) - (\partial_\nu - ieA_\nu)(\partial_\mu - ieA_\mu) =$$

$$= -ie((\partial_\mu A_\nu)+A_\nu\partial_\mu) - ieA_\mu\partial_\nu + ie((\partial_\nu A_\mu)+A_\mu\partial_\nu)+ieA_\nu\partial_\mu = -ieF_{\mu\nu},$$

that leads to the expression of the form [cf. the non-Abelian case after Eq. (47)]

$$\hat{D}^2 = D_\mu^2 - \frac{ie}{4}[\gamma_\mu,\gamma_\nu]F_{\mu\nu} = D_\mu^2+\frac{e}{2}\Sigma_{\mu\nu}F_{\mu\nu}, \quad \text{where} \quad \Sigma_{\mu\nu}\equiv\frac{1}{2i}[\gamma_\mu,\gamma_\nu].$$

Plugging it into Eq. (114), and expanding the result in e, we obtain the following leading, order-$\mathcal{O}(e^2)$, contribution to the right-hand side of Eq. (115):

$$2ia^4\int d^4x\alpha(x)\cdot\mathbf{R}_0\left(\frac{e}{2}F_{\mu\nu}\right)\mathbf{R}_0\left(\frac{e}{2}F_{\lambda\rho}\right)\cdot\mathrm{sp}\left(\gamma_5\Sigma_{\mu\nu}\Sigma_{\lambda\rho}\right), \quad \text{where} \quad \mathbf{R}_0=\frac{\mathbf{1}}{1-(a\partial_\mu)^2}. \tag{116}$$

The trace $\mathrm{sp}\left(\gamma_5\Sigma_{\mu\nu}\Sigma_{\lambda\rho}\right)$ in this formula should be proportional to $\varepsilon_{\mu\nu\lambda\rho}$, that is the only tensor in 4D space with the appropriate transformation properties. Thus, $\mathrm{sp}\left(\gamma_5\Sigma_{\mu\nu}\Sigma_{\lambda\rho}\right)= c\cdot\varepsilon_{\mu\nu\lambda\rho}$, where the constant c can be found by fixing $\mu=1$, $\nu=2$, $\lambda=3$, $\rho=4$. Given the definition of $\Sigma_{\mu\nu}$, this yields

$$c=-\frac{1}{4}\,\mathrm{sp}\left(\gamma_5[\gamma_1,\gamma_2][\gamma_3,\gamma_4]\right).$$

Noting that $-\gamma_2\gamma_1=\gamma_1\gamma_2$, so that $[\gamma_1,\gamma_2]=2\gamma_1\gamma_2$, and similarly $[\gamma_3,\gamma_4]=2\gamma_3\gamma_4$, we can further write $c=-\mathrm{sp}\left(\gamma_5\gamma_1\gamma_2\gamma_3\gamma_4\right)$. Due to the explicit form of the matrix γ_5, one has

$$c=-\mathrm{sp}\left(\gamma_1\gamma_2\gamma_3\gamma_4\gamma_1\gamma_2\gamma_3\gamma_4\right).$$

Next, anticommuting the second of the two matrices γ_1 in this formula to the left, up to the first matrix γ_1, and using the fact that $\gamma_\mu^2=1$ for any $\mu=1,2,3,4$, we have $c=\mathrm{sp}\left(\gamma_2\gamma_3\gamma_4\gamma_2\gamma_3\gamma_4\right)$. In the same way, we can anticommute to the left the second of the two matrices γ_2, up to the first matrix γ_2, that yields $c=\mathrm{sp}\left(\gamma_3\gamma_4\gamma_3\gamma_4\right)$. Finally, anticommuting the matrices γ_4 and γ_3, we obtain $c=-\mathrm{sp}\,\mathbf{1}=-4$. Thus,

$$\mathrm{sp}\left(\gamma_5\Sigma_{\mu\nu}\Sigma_{\lambda\rho}\right)=-4\varepsilon_{\mu\nu\lambda\rho},$$

and Eq. (116) takes the form

$$-2ie^2a^4\varepsilon_{\mu\nu\lambda\rho}\int d^4x\alpha(x)\cdot\mathbf{R}_0F_{\mu\nu}\mathbf{R}_0F_{\lambda\rho}=$$

$$=-4ie^2a^4\int d^4x\alpha(x)\int d^4yd^4zR_0(x,y)F_{\mu\nu}(y)R_0(y,z)\tilde{F}_{\mu\nu}(z)R_0(z,x),$$

where

$$R_0(x,y)\equiv\left(\frac{1}{1-(a\partial_\mu)^2}\right)_{xy} \quad \text{and} \quad \tilde{F}_{\mu\nu}\equiv\frac{1}{2}\varepsilon_{\mu\nu\lambda\rho}F_{\lambda\rho}.$$

We notice that this integral can be viewed as a triangular diagram, with R_0's playing the role of the propagators. It can most easily be calculated in the momentum representation, by substituting

$$R_0(x,y)=\int\frac{d^4q}{(2\pi)^4}\frac{\mathrm{e}^{iq(x-y)}}{1+a^2q^2},\ R_0(y,z)=\int\frac{d^4p}{(2\pi)^4}\frac{\mathrm{e}^{ip(y-z)}}{1+a^2p^2},\ R_0(z,x)=\int\frac{d^4r}{(2\pi)^4}\frac{\mathrm{e}^{ir(z-x)}}{1+a^2r^2},$$

that yields

$$-4ie^2a^4\int d^4x\alpha(x)\int d^4yd^4z\int\frac{d^4qd^4pd^4r}{(2\pi)^{12}}\frac{\mathrm{e}^{i[x(q-r)+y(p-q)+z(r-p)]}}{(1+a^2q^2)(1+a^2p^2)(1+a^2r^2)}F_{\mu\nu}(y)\tilde{F}_{\mu\nu}(z). \tag{117}$$

Integrating over d^4y and d^4z, we have

$$\int d^4yF_{\mu\nu}(y)\mathrm{e}^{iy(p-q)}=F_{\mu\nu}(q-p),\qquad \int d^4z\tilde{F}_{\mu\nu}(z)\mathrm{e}^{iz(r-p)}=\tilde{F}_{\mu\nu}(p-r),$$

where, to simplify notations, we denote the field-strength tensor and its Fourier image by the same symbol. One can introduce, instead of p and r, new momenta $t=q-p$ and $k=q-r$. Equation (117) takes then the form

$$-4ie^2a^4\int d^4x\alpha(x)\int\frac{d^4k}{(2\pi)^4}\mathrm{e}^{ikx}\int\frac{d^4t}{(2\pi)^4}F_{\mu\nu}(t)\tilde{F}_{\mu\nu}(k-t)\times$$
$$\times\int\frac{d^4q}{(2\pi)^4}\frac{1}{(1+a^2q^2)[1+a^2(q-t)^2][1+a^2(q-k)^2]}. \tag{118}$$

We should now seek the lowest-in-a term of this expression, which is expected to remain finite in the $(a\to 0)$-limit. To this end, one changes the momentum q to the rescaled one, $\omega=aq$, in terms of which

$$d^4q=2\pi^2q^3dq=\frac{\pi^2}{a^4}\omega^2d\omega^2\ \ \text{and}\ \ a^2(q-t)^2\simeq a^2(q-k)^2\simeq\omega^2.$$

Equation (118) then reads

$$-\frac{ie^2}{4\pi^2}\int d^4x\alpha(x)\int\frac{d^4k}{(2\pi)^4}\mathrm{e}^{ikx}\int\frac{d^4t}{(2\pi)^4}F_{\mu\nu}(t)\tilde{F}_{\mu\nu}(k-t)\int_0^\infty\frac{\omega^2d\omega^2}{(1+\omega^2)^3}=$$
$$=-\frac{ie^2}{8\pi^2}\int d^4x\alpha(x)\int\frac{d^4k}{(2\pi)^4}\mathrm{e}^{ikx}\int\frac{d^4t}{(2\pi)^4}F_{\mu\nu}(t)\tilde{F}_{\mu\nu}(k-t). \tag{119}$$

Introducing, instead of k, a new momentum $\lambda=k-t$, we further have

$$\int\frac{d^4k}{(2\pi)^4}\mathrm{e}^{ikx}\tilde{F}_{\mu\nu}(k-t)=\int\frac{d^4\lambda}{(2\pi)^4}\mathrm{e}^{i(\lambda+t)x}\tilde{F}_{\mu\nu}(\lambda)=\mathrm{e}^{itx}\tilde{F}_{\mu\nu}(x),$$

where we continue distinguishing the field-strength from its Fourier image, by writing explicitly the corresponding argument. The d^4t-integration now yields

$$\int\frac{d^4t}{(2\pi)^4}\mathrm{e}^{itx}F_{\mu\nu}(t)=F_{\mu\nu}(x),$$

and thus Eq. (119) takes the following simple form:

$$-\frac{ie^2}{8\pi^2}\int d^4x\alpha(x)F_{\mu\nu}(x)\tilde{F}_{\mu\nu}(x).$$

Finally, this expression, once plugged into Eq. (115), yields the following divergence of the axial current:

$$\partial_\mu J_\mu^A = -\frac{ie^2}{8\pi^2} F_{\mu\nu}\tilde{F}_{\mu\nu}. \tag{120}$$

The obtained right-hand side of Eq. (120) does not depend on a, thus remaining finite in the $(a \to 0)$-limit. Instead, higher-in-e contributions are proportional to higher powers of a, and therefore vanish at $a \to 0$. Equation (120), called chiral or Adler-Bell-Jackiw anomaly [16], was originally obtained by using the diagrammatic approach. Here, we have followed Ref. [15] to show that the anomaly stems from the non-invariance of the measure in the path integral under chiral transformations. This is a general situation, which holds for all known anomalies in quantum field theory, e.g. for the so-called conformal or scale anomaly in the v.e.v. of the trace of the energy-momentum tensor. Namely, a *quantum* anomaly appears when the action of a theory respects a certain symmetry (i.e. the theory is invariant under the corresponding symmetry-transformations on the classical level), but the integration measure in the path integral is not invariant under these transformations. We note also that every anomaly is expressed by an equality, which, from the mathematical viewpoint, holds in the weak sense. For instance, Eq. (120) holds in the sense of Eq. (115), i.e. when the divergence of the corresponding current is integrated, together with some function $\alpha(x)$, over d^4x. In general, such a function, as well as the d^4x-integration in Eq. (115), can be promoted respectively to a gauge-invariant functional of the fields and to an average over some gauge-invariant state.

Finally, one can illustrate that the obtained Eq. (120) is independent of a particular form of the regularizing operator $\mathbf{R}$. For this purpose, denoting the Fourier image of R_0 as $R_0(p) \equiv r(a^2p^2)$, one notices that the integral $\int_0^\infty \frac{\omega^2 d\omega^2}{(1+\omega^2)^3}$ in Eq. (119) is, up to a constant multiplicative factor, equal to $\int_0^\infty \omega^2 d\omega^2 \cdot r''(\omega^2)$. Integrating in this formula by parts, one concludes that, for any function $r(\omega^2)$ such that $r(\infty) = 0$, this integral is equal to $r(0)$. Thus, for any regularizing operator $\mathbf{R}$, such that $R_0(p \to \infty) \to 0$ and $R_0(p = 0) = 1$, the result obtained, Eq. (120), is independent of a particular functional form of $\mathbf{R}$.

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[1] R. P. Feynman and A. R. Hibbs, "Quantum mechanics and path integrals" (McGraw-Hill, 1965).

[2] L. D. Landau and E. M. Lifshitz, "Quantum mechanics. Nonrelativistic theory" (3rd edition, Pergamon, 1991).

[3] A. M. Polyakov, Nucl. Phys. B **120**, 429 (1977).

[4] G. 't Hooft, Phys. Rev. D **14**, 3432 (1976) [Erratum-ibid. D **18**, 2199 (1978)].

[5] N. O. Agasian and Yu. A. Simonov, Mod. Phys. Lett. A **10**, 1755 (1995).

[6] Yu. A. Simonov, Phys. Atom. Nucl. **58**, 107 (1995); D. Antonov and H. J. Pirner, Eur. Phys. J. C **51**, 633 (2007).

[7] M. Reuter, M. G. Schmidt and C. Schubert, Annals Phys. **259**, 313 (1997).

[8] A. I. Nikishov, Sov. Phys. JETP **30**, 660 (1970); N. B. Narozhnyi and A. I. Nikishov, Sov. J. Nucl. Phys. **11**, 596 (1970).

[9] I. K. Affleck, O. Alvarez and N. S. Manton, Nucl. Phys. B **197**, 509 (1982).

[10] In our presentation of the subject, we mostly follow G. V. Dunne and C. Schubert, Phys. Rev. D **72**, 105004 (2005).

[11] A. Ringwald, "Fundamental physics at an X-ray free electron laser", arXiv:hep-ph/0112254.

[12] A. M. Polyakov, "Gauge fields and strings" (Harwood Academic Publishers, 1987).

[13] A. M. Polyakov, Phys. Lett. B **59**, 79 (1975).

[14] D. J. Gross and F. Wilczek, Phys. Rev. Lett. **30**, 1343 (1973); H. D. Politzer, Phys. Rev. Lett. **30**, 1346 (1973).

[15] K. Fujikawa, Phys. Rev. Lett. **42**, 1195 (1979); see also: S. N. Vergeles, Ph.D. thesis (L. D. Landau Institute for theoretical physics, 1979); A. A. Migdal, Phys. Lett. B **81**, 37 (1979).

[16] J. Schwinger, Phys. Rev. **82**, 664 (1951); S. L. Adler, Phys. Rev. **177**, 2426 (1969); J. S. Bell and R. Jackiw, Nuovo Cim. A **60**, 47 (1969).