Singletons, Doubletons and HS Master Fields

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Why Higher Spins?

1. Crucial problem in Field Theory

2. Key role in String Theory
- Strings beyond low-energy SUGRA
- HSGT as symmetric phase of String Theory?

3. Positive results from AdS/CFT

Summary

- Field Theory: The Vasiliev equations

  Consistent non-linear equations for **all spins** (all symm tensors):
  - Diff invariant
  - SO(D+1,C)-invariant natural vacuum solutions (S^D,H_D,(A)dS_D)
  - Infinite dimensional (tangent-space) algebra
  - Correct free field limit → Fronsdal eqs
  - Arguments for uniqueness

- Group Theory: UIRs of SO(D-1,2)

- Link: Lorentz-covariant basis & Reflection map

- Subtleties

Focus on D=4 AdS bosonic model

The Vasiliev Equations

∞-dim. extension of AdS-gravity with gauge fields valued in HS tangent-space algebra hs(4) $\subset$ Env(so(3,2))/I(D)

so(3,2) : $[M_{ab}, M_{cd}]_\star = 4i\eta_{[c|[b}M_{a]|d]}\ , \quad [M_{ab}, P_c]_\star = 2i\eta_{c[b}P_{a]}\ , \quad [P_a, P_b]_\star = i\lambda^2 M_{ab}$

Generators of hs(4):
(symm. and TRACELESS!)

$$T_s \sim M_{a_1b_1}\cdots M_{a_tb_t}P_{a_{t+1}}\cdots P_{a_{s-1}}\ ,$$
$$t = 0, 1, ..., s-1,$$

$$[T_{s_1}, T_{s_2}] = T_{s_1+s_2-2} + T_{s_1+s_2-4} + \ldots + T_{|s_1-s_2|+2}$$

Gauge field $\in$Adj(hs(4)) (*master 1-form*):

$$A(x) = \sum_{s=0}^{\infty}\sum_{t=0}^{s-1}\frac{i}{2}dx^\mu A^{\{s-1,t\}}_{\mu,a_1\ldots a_{s-1},b_1\ldots b_t}(x)M^{a_1b_1}\ldots M^{a_tb_t}P^{a_{t+1}}\ldots P^{a_{s-1}}$$

But: representation theory of hs(4) needs more!

- **Massless UIRs** of all spins in AdS include **a scalar**!
- **"Unfolded" eq.**[ns] require a "**twisted adjoint**" rep.

The Vasiliev Equations

Introduce a *master 0-form* (contains a scalar, Weyl, HS Weyl and derivatives)

$$\Phi(x) = \sum_{s,k=0}^{\infty} \frac{1}{k!} \Phi^{\{s+k,s\}}_{a_1 \ldots a_{s+k}, b_1 \ldots b_s}(x)\, M^{a_1 b_1} \ldots M^{a_s b_s} P^{a_{s+1}} \ldots P^{a_{s+k}}$$

N.B.: spin-s sector spanned by all {s+k,s} tensors, k=0,1,2...
(upon constraints, all on-shell-nontrivial covariant derivatives of the physical fields,
i.e., all the dynamical information is in the 0-form at a point)

e.g. s=2: Ricci=0 ↔ Riemann = Weyl [**tracelessness ⇒ dynamics** !]
[Bianchi ⇒ infinite chain of ids.]

Unfolded full eqs:

$$\widehat{F} \equiv \widehat{d}\widehat{A} + \widehat{A} \star \widehat{A} = \frac{i}{4}(dz^{\alpha} \wedge dz_{\alpha} \widehat{\Phi} \star \kappa + d\bar{z}^{\dot{\alpha}} \wedge d\bar{z}_{\dot{\alpha}} \widehat{\Phi} \star \bar{\kappa})$$

$$\widehat{\mathcal{D}}\widehat{\Phi} \equiv \widehat{d}\widehat{\Phi} + \widehat{A} \star \widehat{\Phi} - \widehat{\Phi} \star \bar{\pi}(\widehat{A}) = 0$$

- Manifest HS-covariance
- Consistency ($d^2 = 0$) ⇒ gauge invariance
- **NOTE:** covariant constancy conditions, **but** infinitely many fields + trace constraints ⇒ **DYNAMICS**

(U)IRs of so(3,2)

- Noncompact algebra ⇒ ∞-dimensional UIRs
- Compact time translation ($E \sim P_0 \sim M_{04}$) ⇒ discrete energy spectrum

E induces the splitting: $\mathfrak{g} \;=\; \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+$

$$\begin{array}{ccccc} \mathfrak{g}_0 & = & \mathfrak{so}(3) & \oplus & \mathfrak{so}(2) \\ & & (M_{rs} & , & E) \end{array}$$ compact subalgebra, $\mathfrak{g}_\pm \;=\; \{L_r^\pm = M_{0r} \mp iM_{4r}\}$ ladder ops.

$$[L_r^-, L_s^+] = 2iM_{rs} + 2\delta_{rs}E \;, \quad [E, L_r^\pm] = \pm L_r^\pm \;, \quad [M_{rs}, M_{tu}] = 4i\delta_{[t|[s}M_{r]|u]}$$

l.w. IR → $D(E_0,s_0)$, built on l.w.s. $|E_0,s_0\rangle$:

$$L_r^-|E_0, s_0\rangle \;=\; 0$$ ⇒ E bounded from below

$$D(E_0, s_0) = \mathcal{V}(E_0, s_0)/I$$

$$\mathcal{V}(E_0, s_0) = \{L_{r_1}^+ \ldots L_{r_n}^+|E_0, s_0\rangle\}_{n=0}^\infty \;, \quad I = \{\mathcal{V}(E_m, s') : \; L_r^-|E_m, s'\rangle = 0\}$$

Saturation of unitarity bounds ⇒ multiplet shortening.

UIRs of so(3,2)

- Massless: $E_0 = s_0+1 \rightarrow D(s_0+1, s_0)$ (but: **two** scalars, D(1,0) & D(2,0))

- Singletons: scalar D(1/2,0), spinor D(1,1/2)

Massless particles = two-singletons composites! *(Flato-Fronsdal, '78)*

$$D(1/2,0) \otimes D(1/2,0) = \bigoplus_{s=0}^{\infty} D(s+1,s) \ ,$$
$$D(1,1/2) \otimes D(1,1/2) = D(2,0) \oplus \bigoplus_{s=1}^{\infty} D(s+1,s)$$

Composite l.w. states:

$$|s+1,s\rangle_{r_1 \dots r_s} = \psi_{\{s\}}^{r_1 \dots r_s} \sum_{k=0}^{s} \alpha_{k,s} (L^+_{r_1} \dots L^+_{r_k})(1)(L^+_{r_{k+1}} \dots L^+_{r_s})(2)|0\rangle_1 |0\rangle_2$$

UIRs of so(3,2)

SU(2)-doublet oscillators: $[a_i, a^{\dagger\, j}]_\star = \delta_i^j \ , \quad a_i|1/2,0\rangle = 0$

Oscillator realization:

$$E = \frac{1}{2}(a^{\dagger\, i} a_i + 1)\ , \quad M_{rs} = \frac{i}{2}(\sigma_{rs})_i{}^j a^{\dagger\, i} a_j\ , \quad L_r^+ = \frac{i}{2}(\sigma_r)_{ij} a^{\dagger\, i} a^{\dagger\, j}\ , \quad L_r^- = \frac{i}{2}(\sigma_r)^{ij} a_i a_j$$

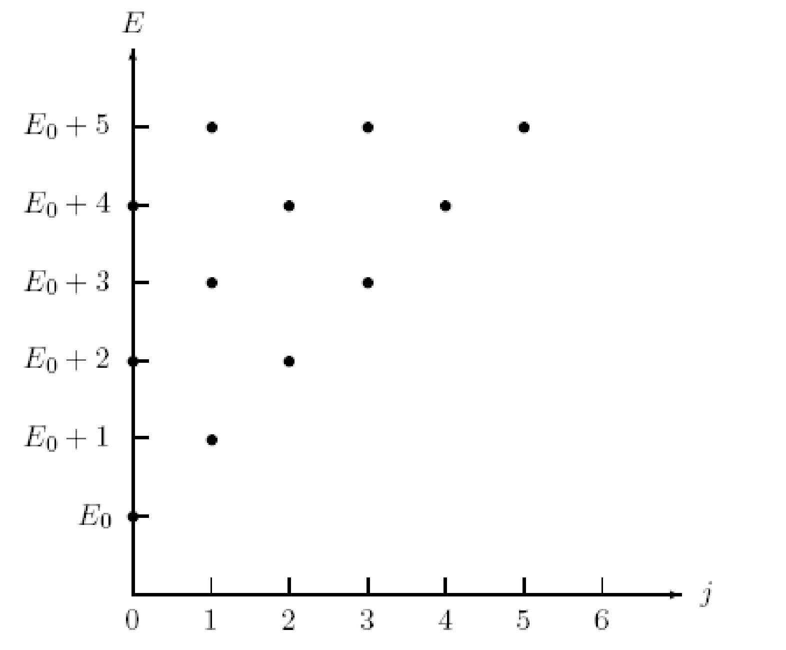
HS-algebra acts reducibly on Fock space, $\mathcal{F} = \mathcal{F}_{even} \oplus \mathcal{F}_{odd}$

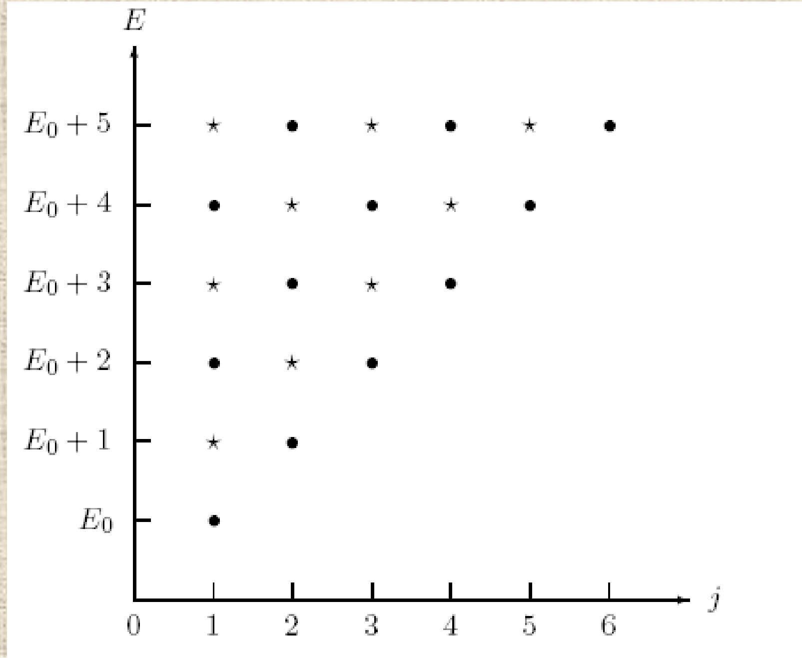
$$D(1/2,0) = \left\{(a^{\dagger\, i} a^{\dagger\, j})^n |1/2,0\rangle\right\}_{n=0}^{\infty}\ , \quad D(1,1/2) = \left\{(a^{\dagger\, i} a^{\dagger\, j})^n |1,1/2\rangle^k\right\}_{n=0}^{\infty}$$

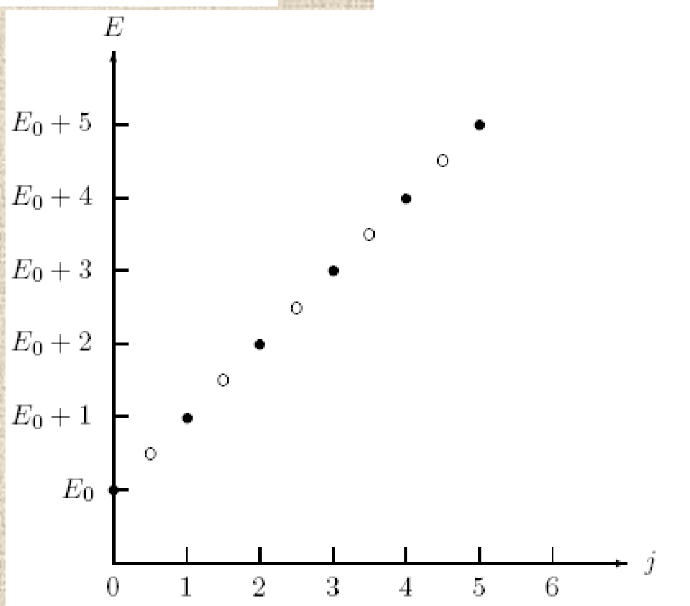
N.B.: $|1,1/2\rangle^i = a^{\dagger\, i}|1/2,0\rangle$

$$|1,0\rangle = |1/2,0\rangle_1 |1/2,0\rangle_2\ , \quad |2,0\rangle = a_i^\dagger(1) a^{\dagger\, i}(2)|1,0\rangle \equiv y|1,0\rangle$$

Weight diagrams







Mapping Doubletons to Master Fields

Admissibility criterion: spectrum of phys. fields matches doubletons

(*Konstein-Vasiliev, '89*)

Now: **map** doubletons (left module) to HS Master Fields (double-sided module)

- From compact to Lorentz-covariant basis of states
- Reflecting a LL into a LR-module, preserving rep. properties

$$D_0^{\otimes 2} \oplus D_{1/2}^{\otimes 2} \;\rightarrow\; |\Phi\rangle = \sum_{m,n} \phi_{m,n} |m\rangle_1 |n\rangle_2 \;\rightarrow \Phi(M_{ab}, P_a)$$

- s=0: find a Lorentz-scalar superposition $|\{0,0\}\rangle_0 = \psi(x)|1,0\rangle \in (D_0)^{\otimes 2}$:

$$x \equiv L_r^+ L_r^+ = y^2$$

$$M_{ab}|\{0,0\}\rangle_0 = 0 \;, \quad i.e. \; M_{0r}\psi(x)|1,0\rangle = 0$$

a harmonic eq. in y ⇒ $|\{0,0\}\rangle_0 = \cos(y)|1,0\rangle \;\in Env(so(3,2))$

Degeneracy! Also possible to expand on states in $D(2,0) \in (D_{1/2})^{\otimes 2}$.

Same procedure yields $|\{0,0\}\rangle_{1/2} = \frac{\sin(y)}{y}|2,0\rangle \;\in Env(so(3,2))$

Mapping Doubletons to Master Fields

Oscillator realization: $|\{0,0\}\rangle_{1/2} = \sin y|1,0\rangle \;\Rightarrow\; |\{0,0\}\rangle_{0+i(1/2)} = e^{iy}|1,0\rangle$

$$|1/2,0\rangle\langle 1/2,0| =: e^{-a^{\dagger\, i}a_i}:$$

Define Reflector: $R(|1/2,0\rangle) = \langle 1/2,0|\;, \quad R(a^{\dagger\, i}) = ia^i\;, \quad R(f\star g) = R(g)\star R(f)$

$$\Rightarrow \quad R_2(e^{iy}|1/2,0\rangle_1|1/2,0\rangle_2) =: e^{-a_i^\dagger a^i}|1/2,0\rangle\langle 1/2,0| := \mathrm{Id}$$

i.e., the {0,0} operator in Φ!

R gives **correct (tw. Adj.) tranformations**!

$$R_2 \quad : \quad \delta|\Phi\rangle = [\epsilon(1)+\epsilon(2)]\star|\Phi\rangle \;\longrightarrow\; \delta\Phi = \epsilon\star\Phi - \Phi\star\pi(\epsilon)$$

(since $R(\epsilon|n\rangle) = -\langle n^c|\,\pi(\epsilon)$)

By <u>HS-symmetry</u>, this extends to all {s+k,s}-monomials in tw. Adj.!

(For general {s+k,s}: 1) decompose 4d to 3d YD, |{s+k,s}⟩ →

|{s+k,s};{s+t,0}⟩, |{s+k,s};{s+t,1}⟩, t=0,...,k (M_{0r}~ step op.)

2) k=0 → bottom/top superpositions ~ trigonometric ψ(y) on lws |s+1,s⟩ ;

k>0 → descendants of k=0 via left-action of P^k)

Conclusions and Outlook

- General result: L-basis: $|\{s+k,s\}\rangle \sim$ **$e^{iy} \times Pol(a^i,a^{\dagger i})$** $|1/2,0\rangle_1 |1/2,0\rangle_2$
  Reflection : $R_2(Pol(a^i,a^{\dagger i})) = M^s P^k$

(N.B.: coeffs in Pol and composite lws are exactly those needed to turn the naturally normal-ordered $R_2(Pol_{s,t,j}(a^i,a^{\dagger i}))$ into the twisted adjoint Weyl-ordered monomials)

- As for scalar, doubling $(\mathbf{D_0})^{\otimes 2} \oplus (\mathbf{D_{1/2}})^{\otimes 2}$ needed to reconstruct exp(iy) (map before imposing b.c. on the fields) .
  Can combine degenerate spin-s so(3,2)-IR into Lorentz-irred. (anti)self-dual combinations.

- Adj ~ nonunitary, unbounded-E l.w. realization,

$$R_2: \ \delta|A\rangle = [\epsilon(1) + \pi(\epsilon(2))] \star |A\rangle \ \rightarrow \ \delta A = [\epsilon, A]_\star$$

- Extension to O(5;C), *i.e.* arbitrary signature O(p,5-p) (nonunitary if p≠3)

- Possible interesting extension to massive HS & D>4.

In components

$$|\{s+k,s\};\{s+t,j\}\rangle = e^{iy} \times \mathrm{Ply}_{s,t,j}(a^i, a^{\dagger\, i})|1/2,0\rangle_1|1/2,0\rangle_2$$

$$\xrightarrow{R_2} R_2(\mathrm{Ply}_{s,t,j}(a^i, a^{\dagger\, i})) = \begin{cases} M_{0r_1}...M_{0r_s}P_{r_{s+1}}...P_{r_{s+t}}(P_0)^{s+k-t}\ , \ j=0 \\ M_{qr_1}M_{0r_2}...M_{0r_s}P_{r_{s+1}}...P_{r_{s+t}}(P_0)^{s+k-t}\ , \ j=1 \end{cases}$$

The Vasiliev Equations

NC extension, x →(x,Z): $[z_\alpha, z_\beta]_\star = -2i\varepsilon_{\alpha\beta}\ , \quad [\bar{z}_{\dot\alpha}, \bar{z}_{\dot\beta}]_\star = -2i\varepsilon_{\dot\alpha\dot\beta}$

$$d \to \hat{d} = d + d_Z$$

$$A(x|Y) \to \hat{A}(x|Z,Y) \equiv (dx^\mu \hat{A}_\mu + dz^\alpha \hat{A}_\alpha + d\bar{z}^{\dot\alpha}\hat{A}_{\dot\alpha})(x|Z,Y)\ , \quad A_\mu(x|Y) = \hat{A}_\mu|_{Z=0}$$

$$\Phi(x|Y) \to \hat{\Phi}(x|Z,Y)\ , \quad \Phi(x|Y) = \hat{\Phi}(x|Z,Y)|_{Z=0}$$

$$\hat{F} \equiv \hat{d}\hat{A} + \hat{A}\star\hat{A} = \frac{i}{4}(dz^\alpha \wedge dz_\alpha \hat{\Phi}\star\kappa + d\bar{z}^{\dot\alpha}\wedge d\bar{z}_{\dot\alpha}\hat{\Phi}\star\bar{\kappa})$$

$$\hat{\mathcal{D}}\hat{\Phi}(x|Y,Z) \equiv \hat{d}\hat{\Phi} + \hat{A}\star\hat{\Phi} - \hat{\Phi}\star\bar{\pi}(\hat{A}) = 0$$

Local sym: $\delta\hat{A} = \hat{D}\hat{\epsilon}\ , \quad \delta\hat{\Phi} = -[\hat{\epsilon}, \hat{\Phi}]_\pi$

$$\hat{F}_{\mu\nu} = \hat{F}_{\alpha\mu} = \hat{F}_{\dot\alpha\mu} = \hat{F}_{\alpha\dot\alpha} = 0\ ,$$
$$\hat{F}_{\alpha\beta} = -\frac{i}{2}\epsilon_{\alpha\beta}\hat{\Phi}\star\kappa\ ,$$
$$\hat{F}_{\dot\alpha\dot\beta} = -\frac{i}{2}\epsilon_{\dot\alpha\dot\beta}\hat{\Phi}\star\bar{\kappa}\ ,$$
$$\hat{\mathcal{D}}_\mu\hat{\Phi} = \hat{\mathcal{D}}_\alpha\hat{\Phi} = \hat{\mathcal{D}}_{\dot\alpha}\hat{\Phi} = 0$$

Solving for Z-dependence yields consistent nonlinear corrections as an expansion in Φ.

For space-time components, projecting on phys. space {Z=0} → $\hat{F}_{\mu\nu}(x|A,\Phi;Y)|_{Z=0} = 0\ , \quad (\hat{\mathcal{D}}_\mu\hat{\Phi})(x|\Phi;Y)|_{Z=0} = 0$

Appendix II

Also the other way around! (base ↔ fiber evolution)

Locally give x-dep. via gauge functions (space-time ~ pure gauge!)...

$$\widehat{A}_\mu = \widehat{L}^{-1}\star\partial_\mu \widehat{L}\ , \quad \widehat{A}_\alpha = \widehat{L}^{-1}\star(\widehat{A}'_\alpha + \partial_\alpha)\star\widehat{L}\ , \quad \widehat{\Phi} = \widehat{L}^{-1}\star\widehat{\Phi}'\star\pi(\widehat{L})$$

$$\widehat{L} = \widehat{L}(x|Z,Y)\ , \quad \widehat{A}'_\alpha = \widehat{A}_\alpha(0|Z,Y)\ , \widehat{\Phi}' = \widehat{\Phi}(0|Z,Y)$$

...and substitute in Z-eq.[ns]:
(fiber evolution)

$$\widehat{F}'_{\alpha\beta} = -\frac{i}{2}\epsilon_{\alpha\beta}\widehat{\Phi}'\star\kappa\ ,\ \widehat{F}'_{\alpha\dot{\beta}} = 0\ ,\ \widehat{D}'_\alpha\widehat{\Phi}' = 0$$

Exact solution can be obtained with: *(Sezgin, Sundell – '05)*

1. $$A = L^{-1}\star dL \to AdS_4\ , \quad ds^2_{(0)} = \frac{4dx^2}{(1-x^2)^2}\ , (x^2 \leq 1)$$

2. SO(3,1)-invariance:

$$[\widehat{M}'_{\alpha\beta}, \widehat{\Phi}']_\pi = 0\ , \quad [\widehat{M}'_{\alpha\beta}, \widehat{A}'_\alpha] = 0 \Rightarrow \begin{array}{rcl} \widehat{\Phi}' & = & f(u,\bar{u})\ , \quad u \equiv y^\alpha z_\alpha \\ \widehat{A}'_\alpha & = & z_\alpha A(u,\bar{u}) \end{array}$$