COUNTING BPS STATES IN CONFORMAL GAUGE THEORIES

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Butti,Forcella,Hanany,Vegh, Zaffaroni, to appear

Counting problems in N=1 supersymmetric gauge theory are an old and vast subject:

There are various type of partition functions for BPS states:

½ BPS states = Chiral Ring
¼ BPS states
Supersymmetric Index

- Study of the moduli space; generators for the chiral ring and their relations

- Dependence of the partition function on the coupling

- Statistical properties of the BPS states and relation to black holes entropies

The Chiral Ring

Interest in chiral primary gauge invariant operators:

$$\bar{Q}_{\alpha} O = 0$$

$$O \sim O + \bar{Q}_{\alpha}(...)$$

- A product on chiral primaries is defined via OPE
- The set of chiral primaries form a ring
- Expectation values and correlation functions do not depend on position.

In supersymmetric gauge theories with chiral matter superfields X, vector multiplets W and superpotential $\mathcal{W}$

chiral gauge invariant operators are combinations of

$$\mathrm{Tr}\,(X^n), \quad \mathrm{Tr}\,(X^n W_\alpha), \quad \mathrm{Tr}\,(X^n W_\alpha W_\beta)$$

however constrained by

1. Finite N size effects

$$Tr\,(X^{N+1}) = \sum_{i=1}^{N} c_i\, Tr(X^i)$$

2. F term relations

$$\partial_X \mathcal{W} = \bar{\mathcal{D}}\bar{\mathcal{D}}\,(\bar{X}) \sim 0$$

- Classical relations may get quantum corrections
- Appearance of W included in a superfield structure

GENERAL PROBLEM: count the number of BPS operators According to their global charge:

$$g_N(a) = \sum n_k(N) a^k$$

$n_k(N)$ = number of BPS operators with charge k

- a is a chemical potential for global and R charges
- N is the number of colors

Note: the number of gauge invariant operators is infinite

However these typically have charges under the global symmetries and the number of operators with given charge is finite

For N=4 SYM the problem is very simple:

3 adjoint fields Φ_i F-terms $[\Phi_i, \Phi_j] = 0$

3 commuting adjoint matrices can be simultaneously diagonalized:

g(N) is the generating function for symmetric polynomials in the eigenvalues

For a generic N=1 gauge theory the problem is hard:

- non trivial F term relations for many fields
- finite N relations

How to compute gauge invariants and generating functions:

The problem of finding gauge invariants goes back to the nineteenth century. In mathematics this is invariant theory.

$\mathrm{N}\times\mathrm{N}$ matrices X_{ij}

$$R[X_{ij}] = \mathbb{C}[X_{ij}]/\{\partial W(X_{ij}) = 0\}$$

$$R^{INV} = R[X_{ij}] \,//\, G$$

- General methods due to Hilbert: free resolutions, syzygies…
- Now algorithmical (Groebner basis)
- With computers and computer algebra programs really computable (but for small values of N)
- Still very hard to get general formulae for generic N

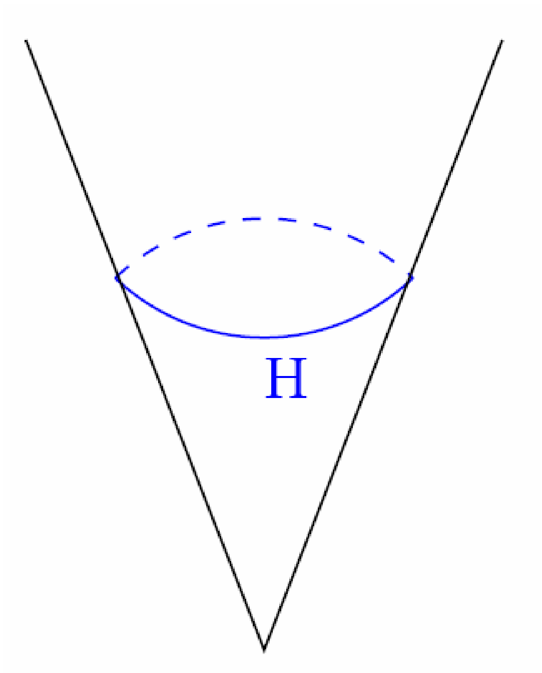
The problem drastically simplifies for the class of superconformal gauge theories with AdS dual:

$$AdS_5 \times H$$

- D3 branes probing Calabi-Yau conical singularities
- Four dimensional CFTs on the worldvolume

Connection provided by the AdS/CFT correspondence

D3 branes probing a conical Calabi-Yau with base H:



The near horizon geometry is $AdS_5 \times H$

The worldvolume theory is a 4d conformal gauge theory

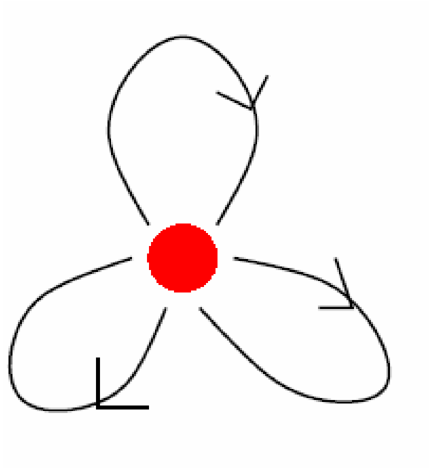
CY condition implies that H is Sasaki-Einstein.

Few metrics known ($H = S^5, T^{1,1}, Y^{p,q}, L^{p,q,r}$)

Many interesting question solved without knowledge of the metric

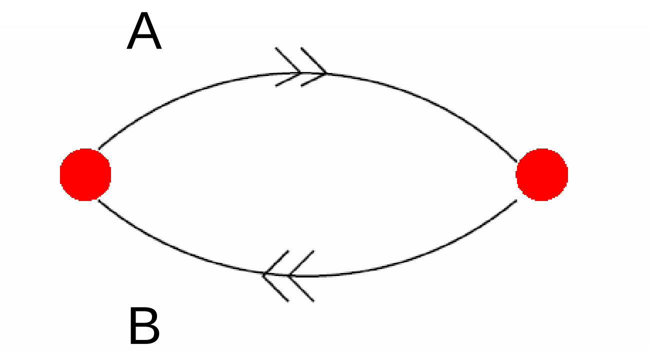
EXAMPLES:

$C^3 = C(S^5)$



N=4 SYM

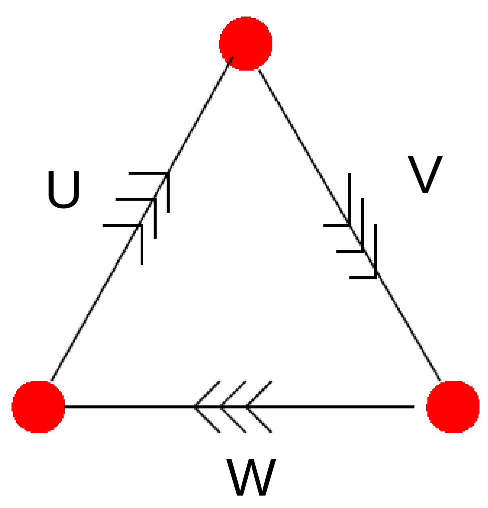
$C(T^{1,1})$



Conifold

$$W = \varepsilon_{ij}\varepsilon_{pq}A_i B_p A_j B_q$$

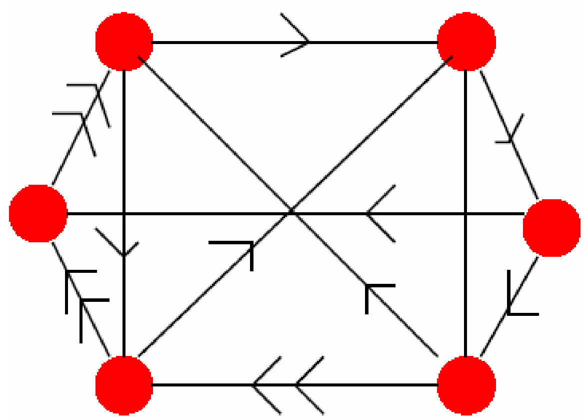
C^3 / Z_3



Orbifold
Projection
Of N=4 SYM

$$W = \varepsilon_{ijk} U_i V_j W_k$$

$C(L^{152})$



$$W = \ldots$$

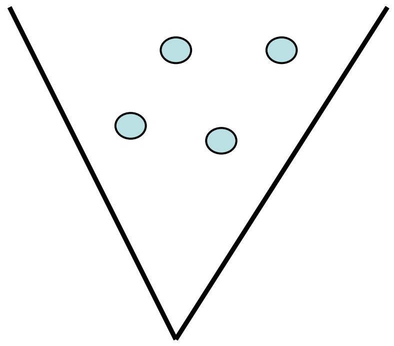
General properties:

- SU(N) gauge groups
- adjoints or bi-fundamental fields
- superpotential terms = closed loops in the quiver
- An infinite class of superconformal theories that generalize abelian orbifolds of N=4 SYM

General properties:

- The moduli space of the U(1) theory is the CY
- The moduli space of the U(N) theory is the symmetrized product of N copies of the CY:

N branes:



- The fact that the gauge group is SU(N) implies the existence of baryons in the spectrum

General properties:

Classification in terms of global charges:

Non anomalous abelian symmetries in the CFT:

- r flavors (R) symmetries
- s baryonic symmetries

- CY isometries
- RR fields

r = 1,2,3

r is at least 1 ⟹ R symmetry

r=3 ⟹ 3 isometries=toric CY

DIGRESSION:

Correspondence between CY and CFT

Comparison with predictions of the AdS/CFT

•Connections to dimers: (Okunkov,Nekrasov,Vafa – Hanany,Kennaway)

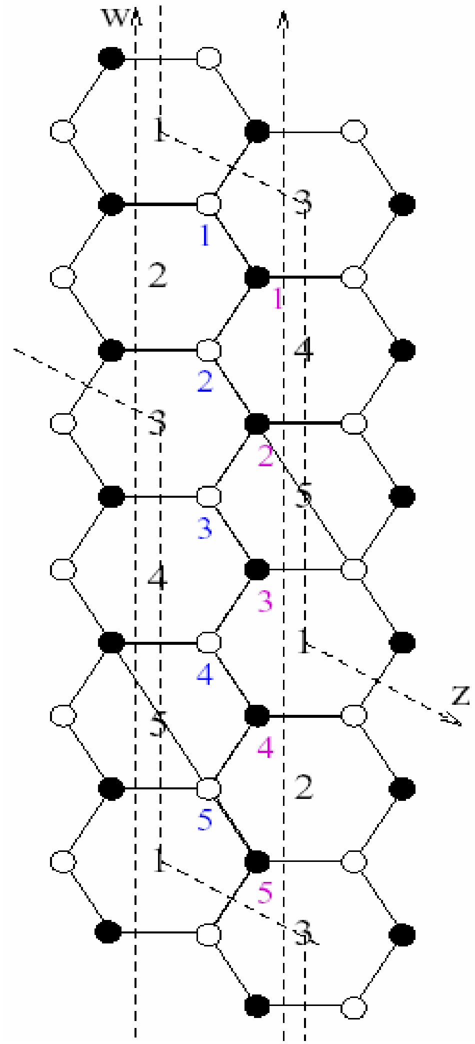
•Geometric computation without metric (Martelli,Sparks,Yau)

•AdS/CFT, combinatorics, a-maximization

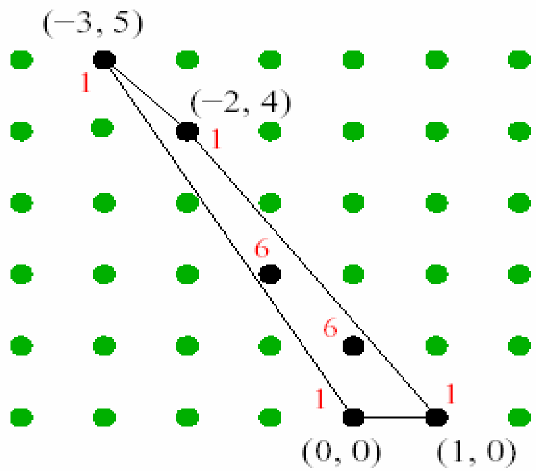
Connection to dimers:

Okounkov,Nekrasov,Vafa – Franco,Kennaway,Hanany,Vegh,Wecht

L^{152}

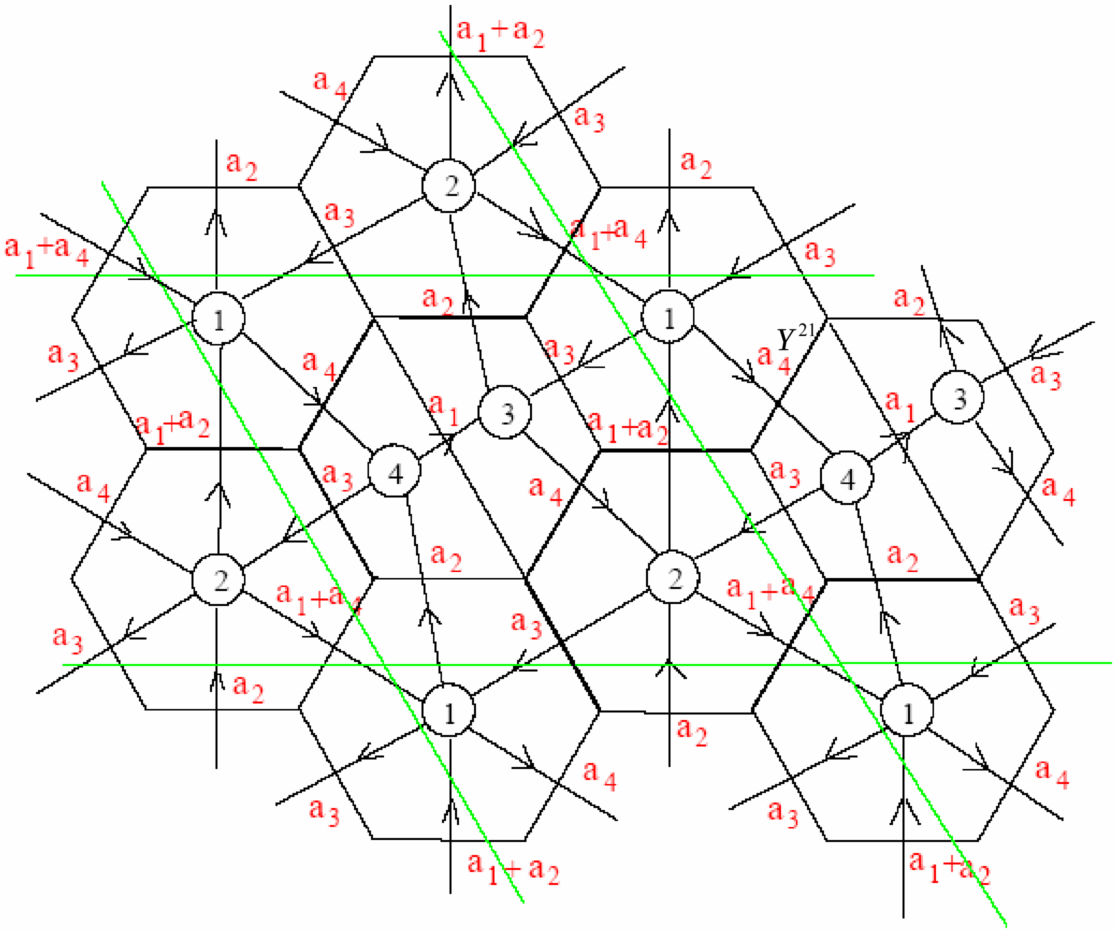


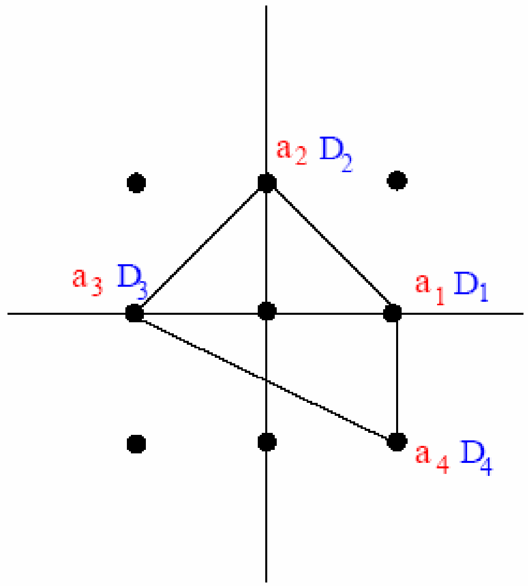
$$K = \left(\begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 1 & 0 & 0 & w & z \\ 2 & 1 & 1 & 0 & 0 & w \\ 3 & wz^{-1} & 1 & 1 & 0 & 0 \\ 4 & 0 & wz^{-1} & 1 & 1 & 0 \\ 5 & 0 & -wz^{-1} & wz^{-1} & 1 & 1 \end{array}\right)$$



Dimers, combinatorics and charges:

Hanany-Witten construction for local CY





delPezzo 1 = $Y^{2\,1}$

Connection to a-maximization:

Central charge of the CFT determined by combinatorial data:

$$a = \frac{9}{32} \mathrm{Tr}\, R^3 = \sum_{i,j,k} |< \mathrm{V}_\mathrm{i}, \mathrm{V}_\mathrm{j}, \mathrm{V}_\mathrm{k} >| \, a_i a_j a_k$$

$$\sum_{i=1}^{d} a_i = 2$$

Butti,Zaffaroni
Benvenuti,Pando-Zayas,Tachikawa
Lee,Rey

Thanks to a-maximization (Intriligator,Wecht) the exact R-charge of the CFT is obtained by maximizing a

The BPS spectrum of states: the N=1 chiral ring

Consider the cases:	N=4	Conifold
Mesonic operators:	$\mathrm{Tr}(\Phi^k)$	$\mathrm{Tr}(AB)^n$
Baryonic operators:		$\mathrm{Det}(A)\ \ \mathrm{Det}(B)$

All gauge invariant single and multi trace operators subject to F term conditions:

N=4 $\qquad [\Phi_i, \Phi_j] = 0$

Conifold $\qquad A_i\ B_p\ A_j = A_j\ B_p\ A_i$

Warming up: N4 SYM

Focus on single trace operators: $\Longrightarrow$ $\mathrm{Tr}(\Phi_1^n \Phi_2^m \Phi_3^p)$

$$[\Phi_i, \Phi_j] = 0$$

$$\Phi_1 \Longrightarrow q_1$$

$$\Phi_2 \Longrightarrow q_2$$

$$\Phi_3 \Longrightarrow q_3$$

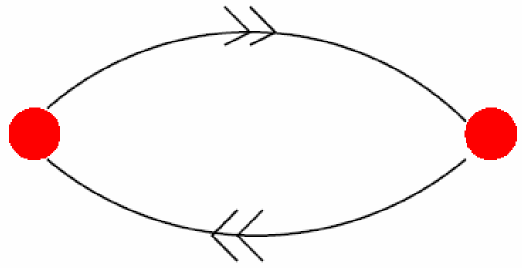
Generating function: $$g_1(q) = \frac{1}{(1-q_1)(1-q_2)(1-q_3)}$$

Warming up: the conifold

Focus on single trace operators: ⟹ $Tr(A_{i_1} B_{j_1} \ldots A_{i_k} B_{j_k})$

$A_i \longrightarrow t_1$

$A_i\, B_p\, A_j = A_j\, B_p\, A_i$



$B_i \longrightarrow t_2$

Generating function:

$$g_1(t_1, t_2) = \sum_{n=1}^{\infty} (n+1)^2 t_1^n t_2^n$$

Not the smart way of computing:

Moduli space → D3 branes → CY

Two equivalent moduli space parameterizations:

- VEV of elementary fields (modulo complexified gauge transformations)
- Chiral gauge invariant operators:

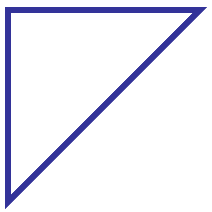
↓

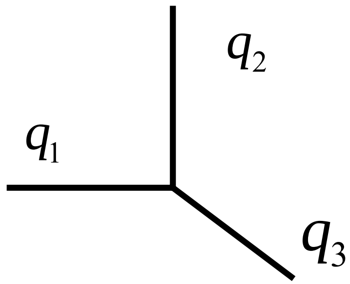
set of holomorphic functions on the moduli space (CY).

Example: N4 SYM

Holomorphic functions on $\mathbb{C}^3$: $z_1^n z_2^m z_3^p \Rightarrow \mathrm{Tr}(\Phi_1^n \Phi_2^m \Phi_3^p)$

Generating function: $$g_1(q) = \frac{1}{(1-q_1)(1-q_2)(1-q_3)}$$





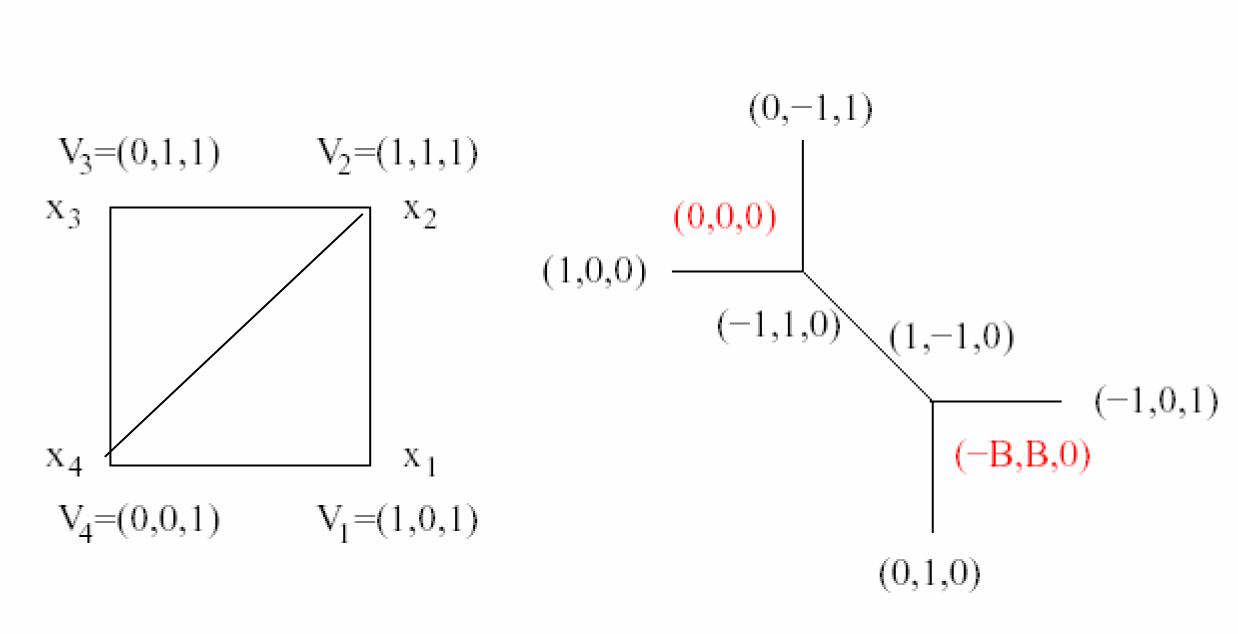
Mesonic operators have zero baryonic charge: only 3 independent charges q

Toric Calabi-Yau:

Obtained by gluing copies of C^3

$$g_1(q) = \sum_I \frac{1}{(1-q_1{}^{n^I})(1-q_2{}^{m^I})(1-q_3{}^{p^I})}$$

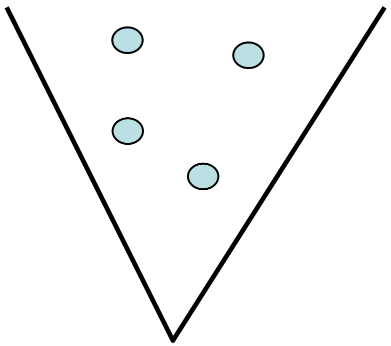
CONIFOLD



[Martelli,Sparks,Yau]

Full set of operators and their dual interpretation

Mesonic operators:



Single traces = holomorphic function on the CY
index theorem(Martelli,Sparks,Yau)

Multi traces = for N colors, holomorphic functions on the symmetric product $\mathrm{Sym}(\mathrm{CY})^N$
(Benvenuti,Feng,Hanany,He)

Bulk KK states in ADS: gravitons and multigravitons

Baryonic operators:

Determinants = operators with large dimension (N)

Solitonic states in AdS: D3 branes wrapped on 3 cycles

Subtelties also for mesons:

Mesonic operators in the bulk are KK states

Dimension of order N ⇨

- Giant gravitons
- D3 wrapping trivial 3 cycles
- stabilized by rotation

$$Tr\,(X^{N+1}) = \sum_{i=1}^{N} c_i\, Tr(X^i)$$

QUANTIZING SUPERSYMMETRIC D3 BRANE STATES

1. It contains at once all BPS states
2. It allows a strong coupling computation (AdS/CFT)

Consider a D3 brane wrapped on a 3 cycle in H, including excited states, possibly non static

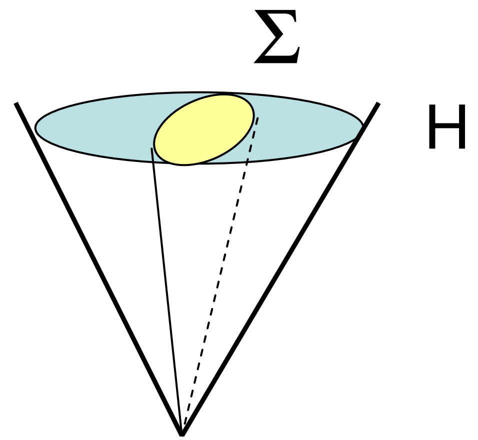
- Branes on trivial cycles stabilized by flux+rotation =Giant Gravitons → mesons
- Branes on non-trivial cycles, possibly excited and non static → baryons

SUPERSYMMETRIC CLASSICAL D3 BRANES CONFIGURATIONS

D3 brane on cycle Σ in H: Divisor C(Σ) in the CY

it wraps (Σ,t) → it wraps (Σ,r)

(rotating t to r in AdS)



SUPERSYMMETRIC D3 BRANE = holomorphic surface in CY

Translation to geometrical problem:

count all holomorphic surfaces in a given equivalence class of divisors:

An holomorphic surface is locally written as an equation in suitable complex variables

- Mesons: ⇒ trivial divisor
  zero locus of holomorphic functions

$$P(z_i) = \sum a_{nmp} z_1^n z_2^m z_3^p$$

- Baryons: ⇒ non trivial divisors
  sections of suitable line bundles
  baryonic charge B=degree of the line bundle

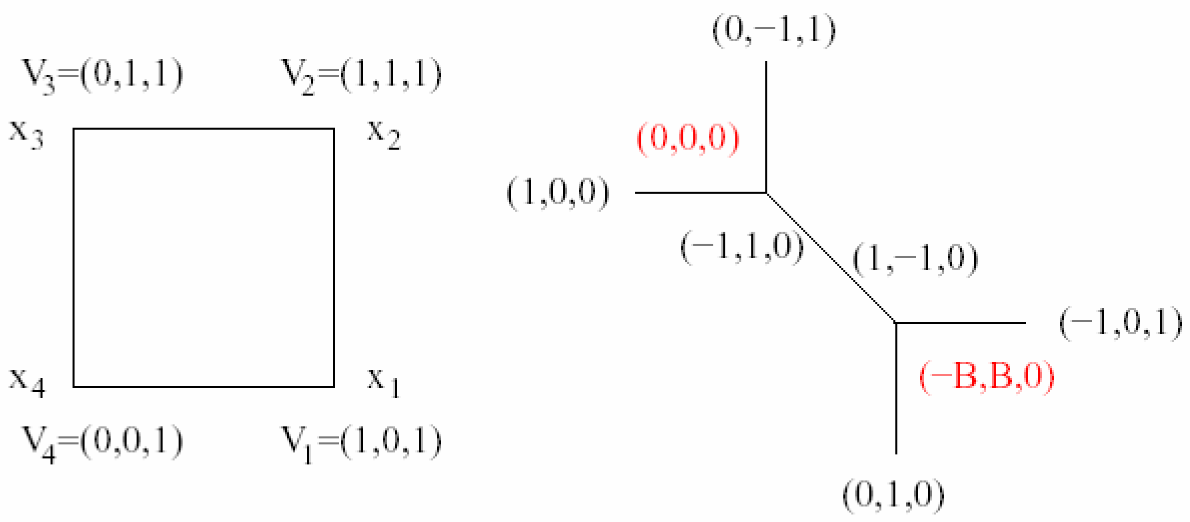
$$P(z_i) \in H^0(X, O(B))$$

Computing the generating function for BPS D3 brane states:

Classical BPS D3 brane states identified with holomorphic surfaces of class B (line bundle sections), computed using index theorem

$$g_{1,B}(q) = \sum_I \frac{q^{q^I}}{(1-q_1^{n^I})(1-q_2^{m^I})(1-q_3^{p^I})}$$

CONIFOLD



QUANTIZATION OF THE CLASSICAL CONFIGURATION SPACE OF SUPERSYMMETRIC D3 BRANES

Done with geometric quantization [Beasley]:

Full Hilbert space at fixed baryonic charge B obtained from N=1 result by taking N-fold symmetrized products of sections

$P_i \in H^0(X,O(B))$ $| P_1, P_2, \ldots, P_N >$

$$\sum \upsilon^N g_{B,N}(q) = Exp(\sum_{k=1} g_{1,B}(q^k)\upsilon^k / k)$$

(counts symmetrized products)

Also known as pletystic exponential

Count symmetrized products of elements P in a set S with generating function

$$g_1(q) = \sum_{n \in S} q^n$$

Introduce a new parameter: υ

$$g(q,\upsilon) = \frac{1}{\prod_{n \in S}(1-\upsilon q^n)} = \sum_{N=1} \upsilon^N g_N(q)$$

$$Exp(\sum_{n \in S} \log(1-\upsilon q^n)) = Exp(\sum_{k=1}^{\infty} \sum_{n \in S} \upsilon^k q^{kn}/k) = Exp(\sum_{k=1}^{\infty} g_1(q^k)\upsilon^k/k)$$

Example: the conifold

Geometry: the conifold can be written as a quotient: four complex variables modded by a complex rescaling

$$\lambda \in \mathbb{C}^*, \qquad (x_1 \sim x_1\lambda, x_2 \sim x_2/\lambda, x_3 \sim x_3\lambda, x_4 \sim x_4/\lambda)$$

charges (1,-1,1,-1)

Similar to projective space.
Homogeneous coordinates exist for all toric manifold
Holomorphic surfaces can be written as

$$P(x_1, \ldots, x_4) = 0$$

Example: the conifold

Field Theory: four charges, one R, two flavors, one baryonic

	$SU(2)_1$		$SU(2)_2$		$U(1)_R$	$U(1)_B$	
	j_1	m_1	j_2	m_2			
A_1	$\frac{1}{2}$	$+\frac{1}{2}$	0	0	$\frac{1}{2}$	1	x_1
A_2	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	$\frac{1}{2}$	1	x_3
B_1	0	0	$\frac{1}{2}$	$+\frac{1}{2}$	$\frac{1}{2}$	-1	x_2
B_2	0	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	-1	x_4

Homogeneous rescaling = baryon number

Mesonic operators (charge B=0)

$$P(x_i) = x_1 x_2 + x_3 x_4 + \ldots.$$

$g_{1,B=0}(q)$ counts operators $\mathrm{Tr}\,(A_{i_1} B_{j_1} \ldots . A_{i_m} B_{j_m})$

(m+1,m+1)

with charge t1 for A and t2 for B

$$g_{1,B=0}(t_1, t_2) = \sum_{m=0} (m+1)^2 t_1^m t_2^m$$

$g_{N,B=0}(q)$ counts multitraces

$$\mathrm{Tr}(AB)^k \ldots . \mathrm{Tr}(AB)^m$$

Baryonic operators at charge B=1

$$\mathrm{P}(x_i) = x_1 + x_3 + x_1^2 x_2 + \ldots.$$

$g_{1,B}(q) =$ counts operators $A_{I;J} = A_{i_1} B_{j_1} \ldots A_{i_m} B_{j_m} A_{i_{m+1}}$

(m+2,m+1)

with charge t1 for A and t2 for B

$$g_{1,B=1}(t_1, t_2) = \sum_{m=0} (m+1)(m+2) t_1^m t_2^m$$

$g_{N,B}(q)$ counts determinants

$$\epsilon^1_{p_1,\ldots,p_N} \epsilon_2^{k_1,\ldots,k_N} (A_{I_1;J_1})^{p_1}_{k_1} \ldots (A_{I_N;J_N})^{p_N}_{k_N}$$

Comments:

Not all baryonic operators are factorizables as:

$$\mathrm{Det}\,(A)\times\mathrm{Mesons}$$

For example there are 2(N-1) non factorizable components of:

$$\mathrm{Det}(ABA,A,....,A)$$

Full partition function for the conifold

N=1

$$g_1(t,b,x,y;\mathcal{C}) = \frac{1}{(1-tbx)(1-\frac{tb}{x})(1-\frac{ty}{b})(1-\frac{t}{by})}$$

$$g_1(t,b,x,y;\mathcal{C}) = \sum_{B=-\infty}^{\infty} b^B g_{1,B}(t,x,y;\mathcal{C})$$

Finite N:

$$\sum_{N=0}^{\infty} \nu^N g_{N,B}(\{t_i\};CY) \; = \sum_{B=-\infty}^{\infty} b^B \exp\left(\sum_{k=1}^{\infty} \frac{\nu^k}{k} g_{1,B}(\{t_i^k\};CY)\right)$$

Checked against explicit computation for N=2,3.

Structure of the partition function:

N=2: 10 generators (with relations):

$$\mathrm{Det}(A)=\varepsilon\varepsilon A_i A_j \qquad \mathrm{Det}(B) \qquad \mathrm{Tr}(AB)$$

3 3 4

General N:

- (N+1,1) generators Det(A)
- (1,N+1) generators Det(B)
- (n,n) generators $\mathrm{Tr}(AB)^n$ n=1,…,N-1
- 2(N-1) generators Det(ABA,A,....,A)
- ... other non factorizable baryons

Comments I

Similar analysis can be done for other CY:

A general lesson on the structure of BPS partition functions:

- N=1 result decomposes in sectors with fixed baryonic charge B
- The finite N result for baryonic charge B is obtained by PE (symmetrized products) from N=1 result
- The full partition function for finite N is obtained by resumming the contribution of fixed baryonic charge

Comments II

- ½ BPS partition functions seem independent of the coupling constant: strongly coupled AdS/CFT computation agrees with weakly coupling analysis

- Relation of baryonic charge sectors with discretized Kahler moduli of the geometry

- Similarity of results with topological strings/Nekrasov partition functions

CONCLUSION

Intriguing interplay between geometry and QFT

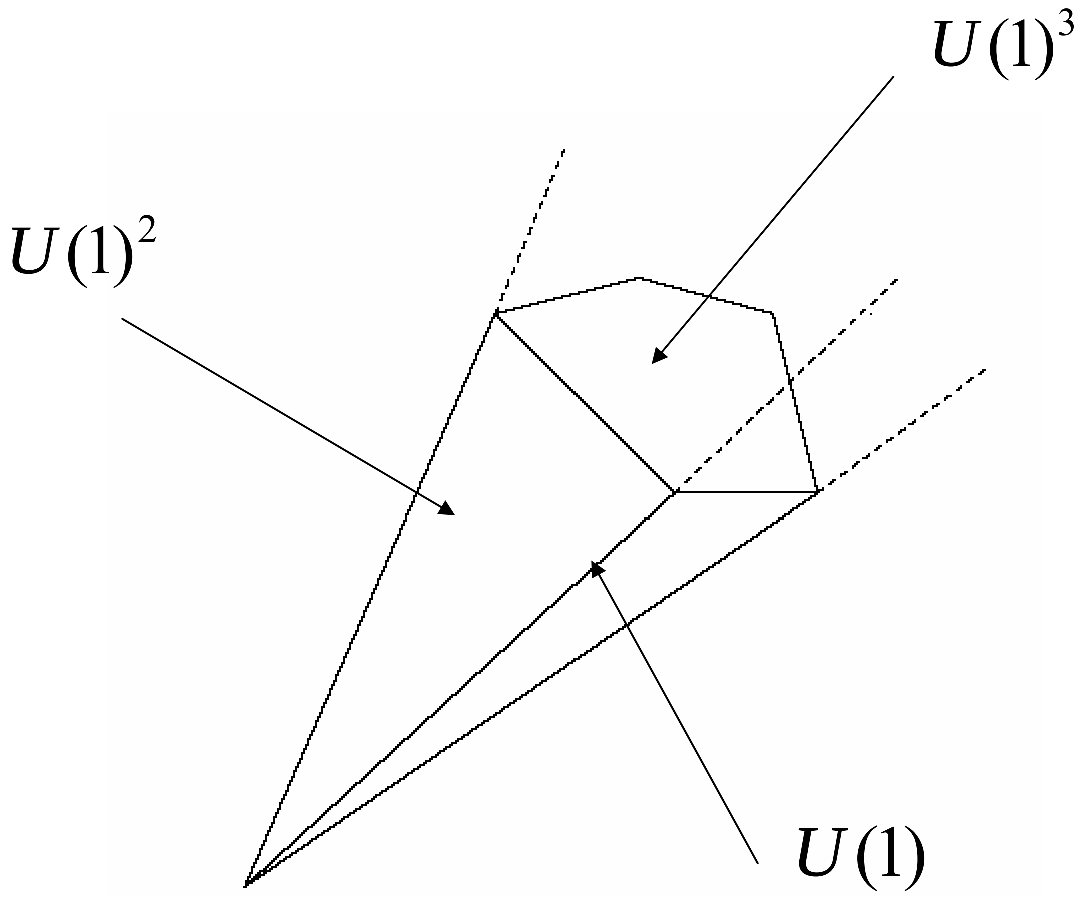
- AdS/CFT: index theorem and localization
- QFT computation: invariant theory

Other interesting questions:

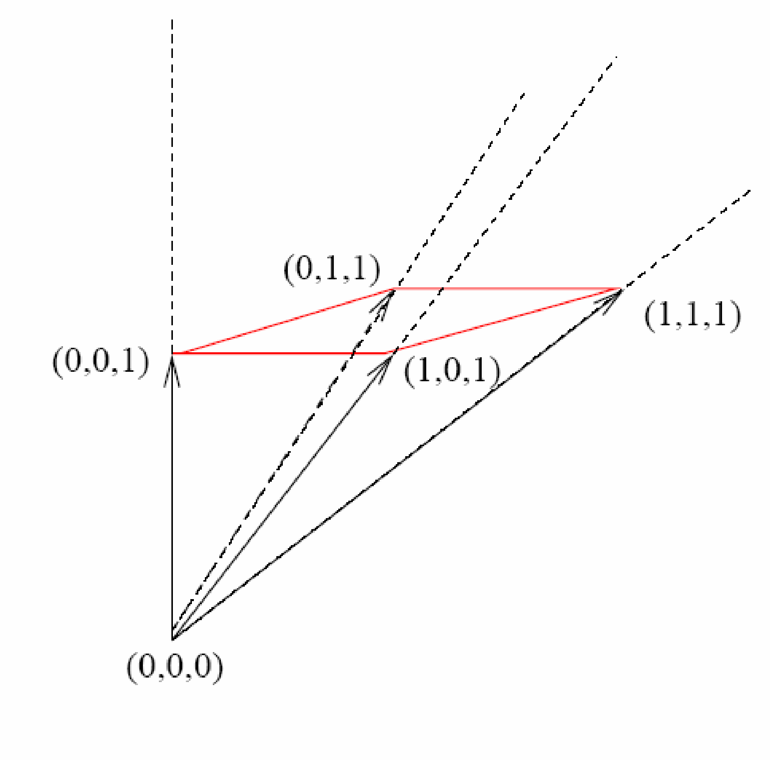
- Partition functions for general CY
- Index and ¼ BPS states
- Non CY vacua
- Termodinamic properties of partition functions

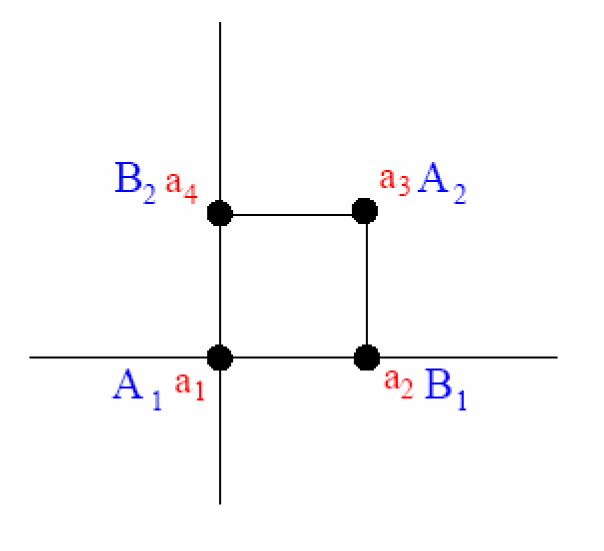
Toric case is simpler:

Geometrically: toric cones are torus fibrations over 3d cones.

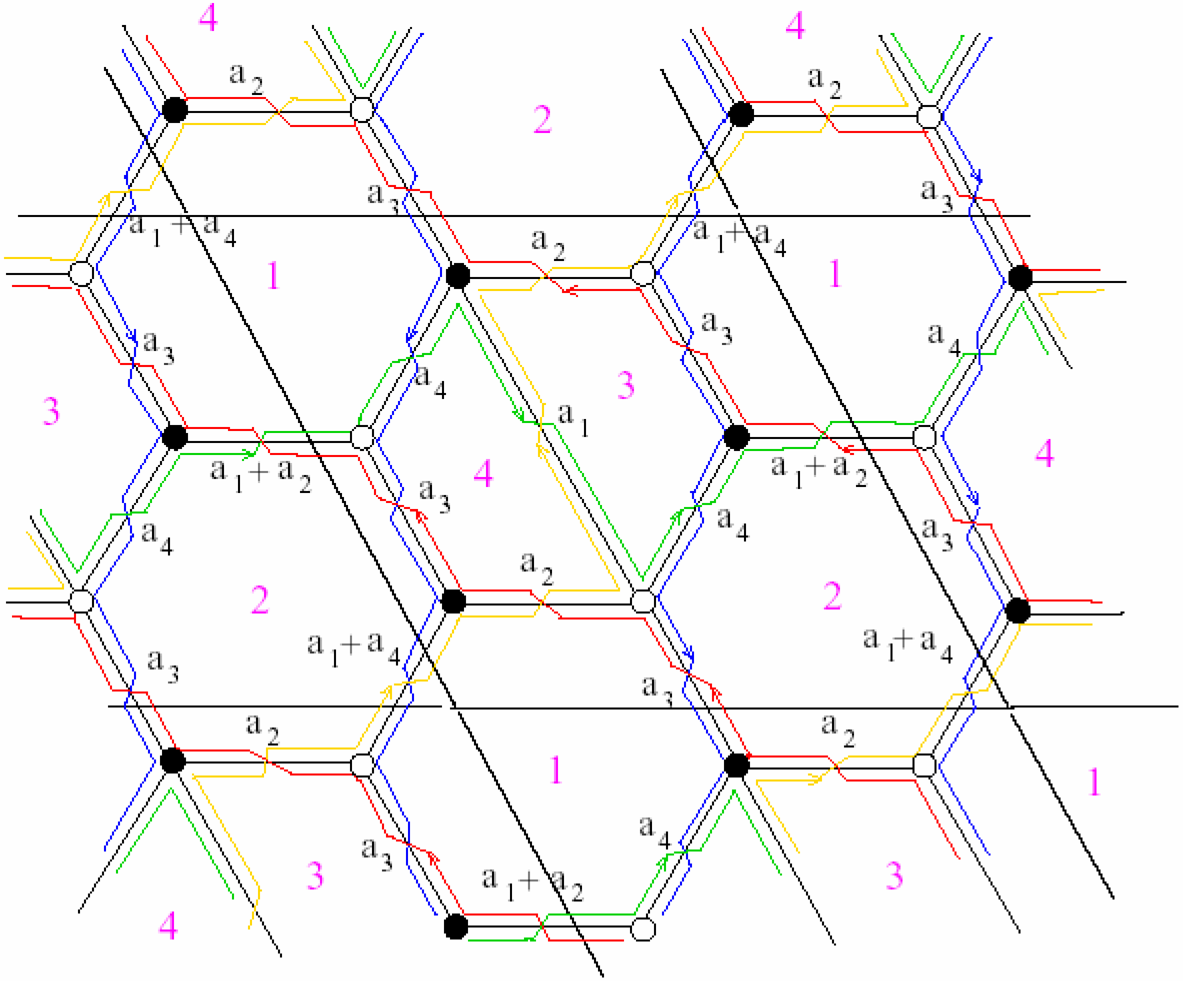


All information on toric CY: convex polygon with integer vertices.

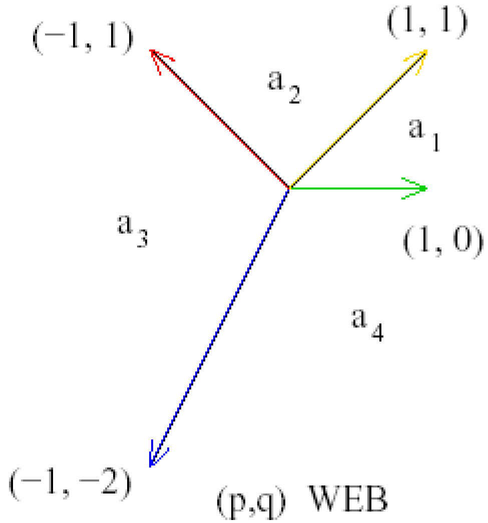




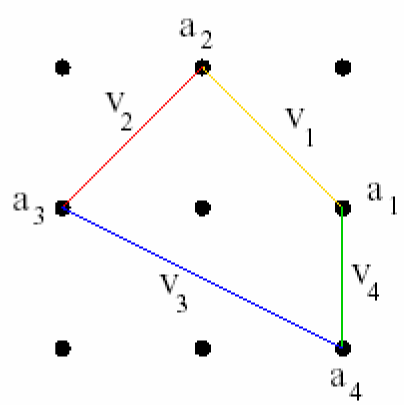
Conifold toric diagram



DIMER MODEL



(p,q) WEB



TORIC DIAGRAM

Y^{21}

Global charges and geometry:

There are d abelian symmetries in the CFT:

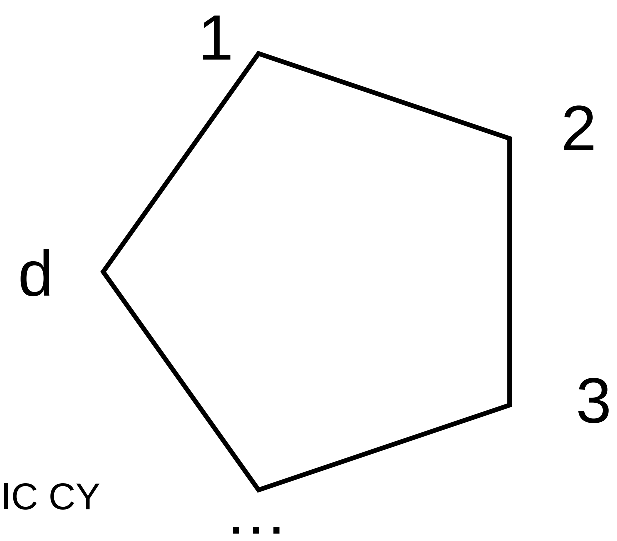
- 1 R symmetry
- 2 flavor symmetries

isometries

- d-3 baryonic symmetries

reduction of RR potentials
on the d-3 3-cycles

d = number of vertices
d-3 three cycles



Toric case is simpler:

Gauge theory: constraints on number of fields:

$$G + V = F$$

G = number gauge groups
F = number of fields
V = number of superpotential terms

Conformal invariance requires linear conditions on R-charges

G beta functions conditions
V superpotential conditions

for F=V+F fields

→ IR fixed point

Comment:

Conformal invariance conditions have d-1 independent solutions

d-1=number of global non anomalous abelian charges

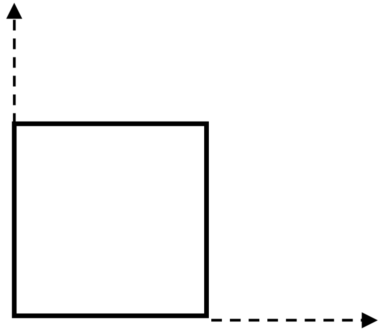
R charges of the F elementary fields can be expressed in terms of d charges with $\sum_{i=1}^{d} a_i = 2$

$$a_i \qquad i=1,\ldots,d$$

Global charges are associated with vertices

Example: conifold

d=4 vertices
d-3=1 three cycles

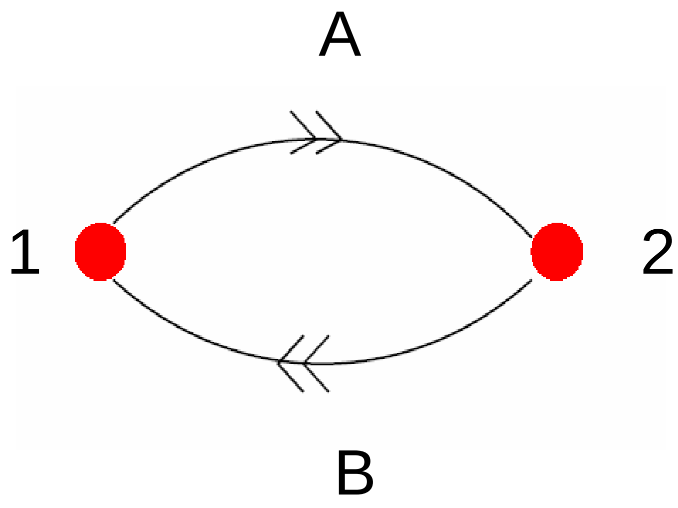


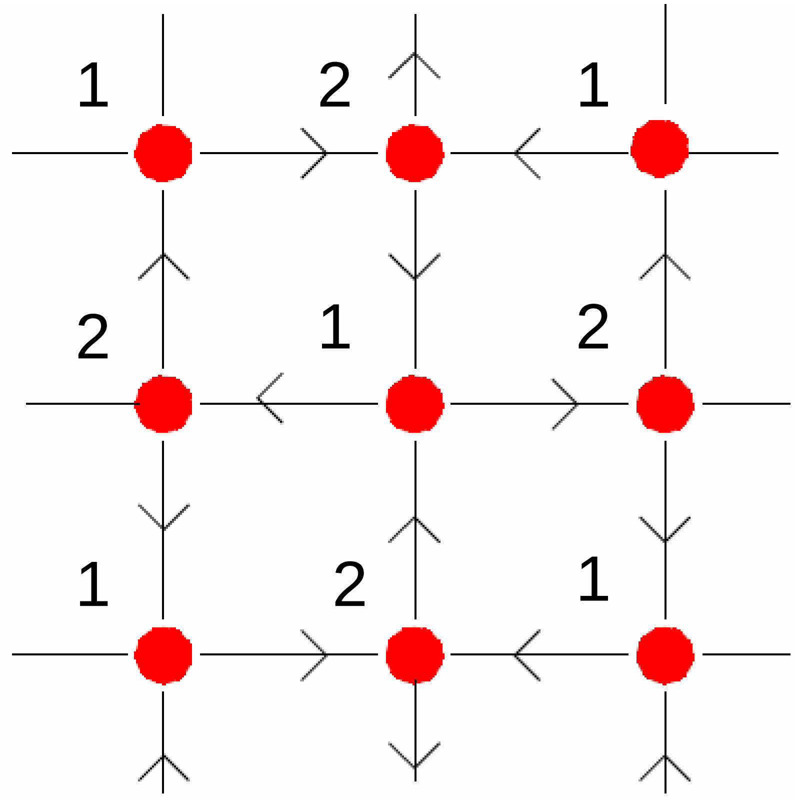
Four charges: one R, two flavors,
one baryonic

three isometries
RR 4-form on 3-cycle

	$SU(2)_1$		$SU(2)_2$		$U(1)_R$	$U(1)_B$
	j_1	m_1	j_2	m_2		
A_1	$\frac{1}{2}$	$+\frac{1}{2}$	0	0	$\frac{1}{2}$	1
A_2	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	$\frac{1}{2}$	1
B_1	0	0	$\frac{1}{2}$	$+\frac{1}{2}$	$\frac{1}{2}$	-1
B_2	0	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	-1

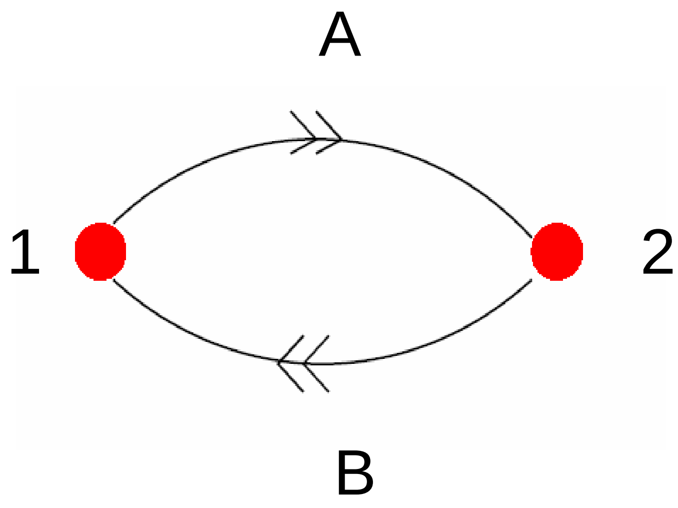
Example: conifold

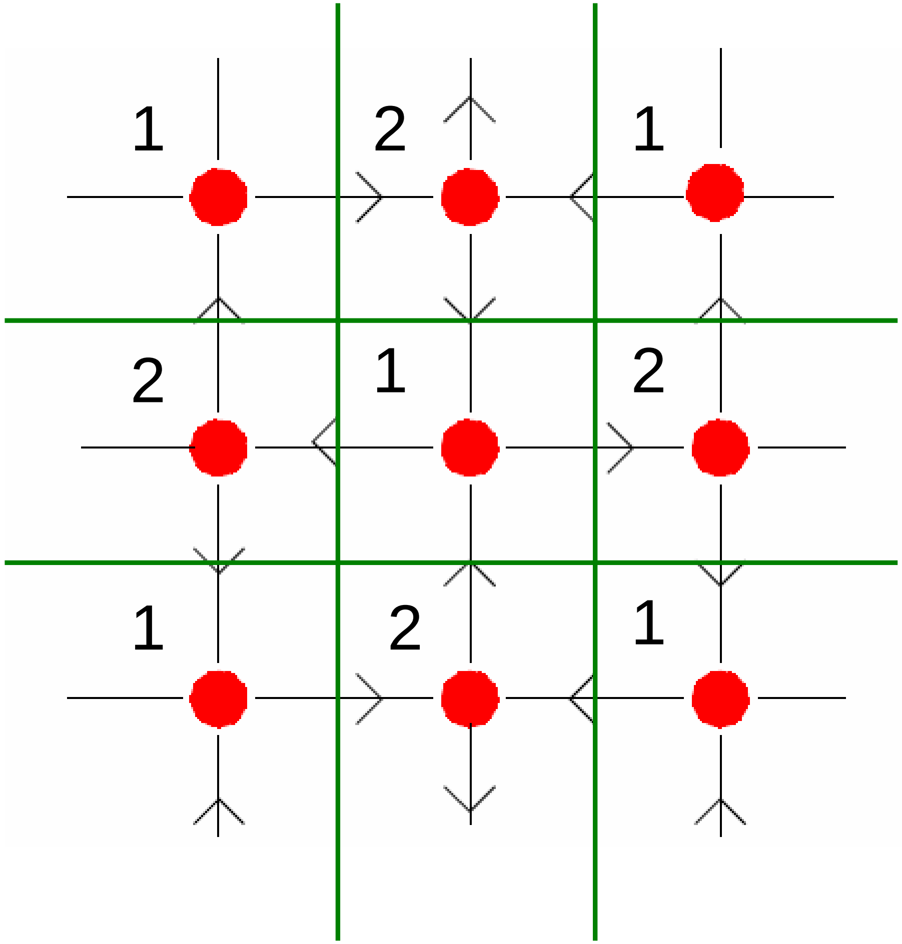




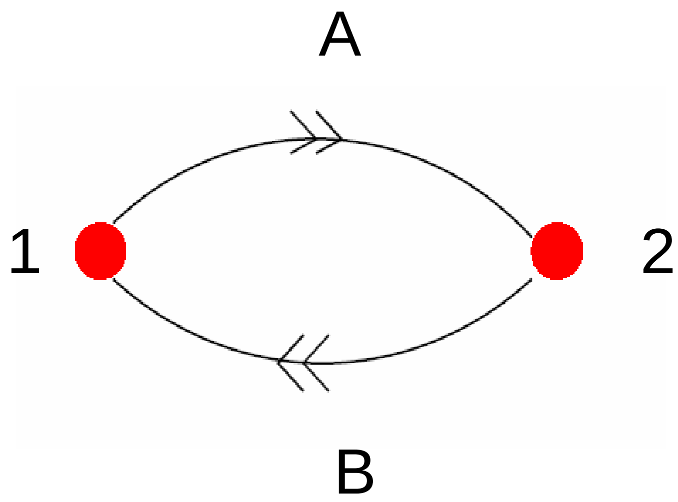
[Franco-Hanany-Kennaway-Vegh-Wecht]
[Feng-He-Kennaway-Vafa]

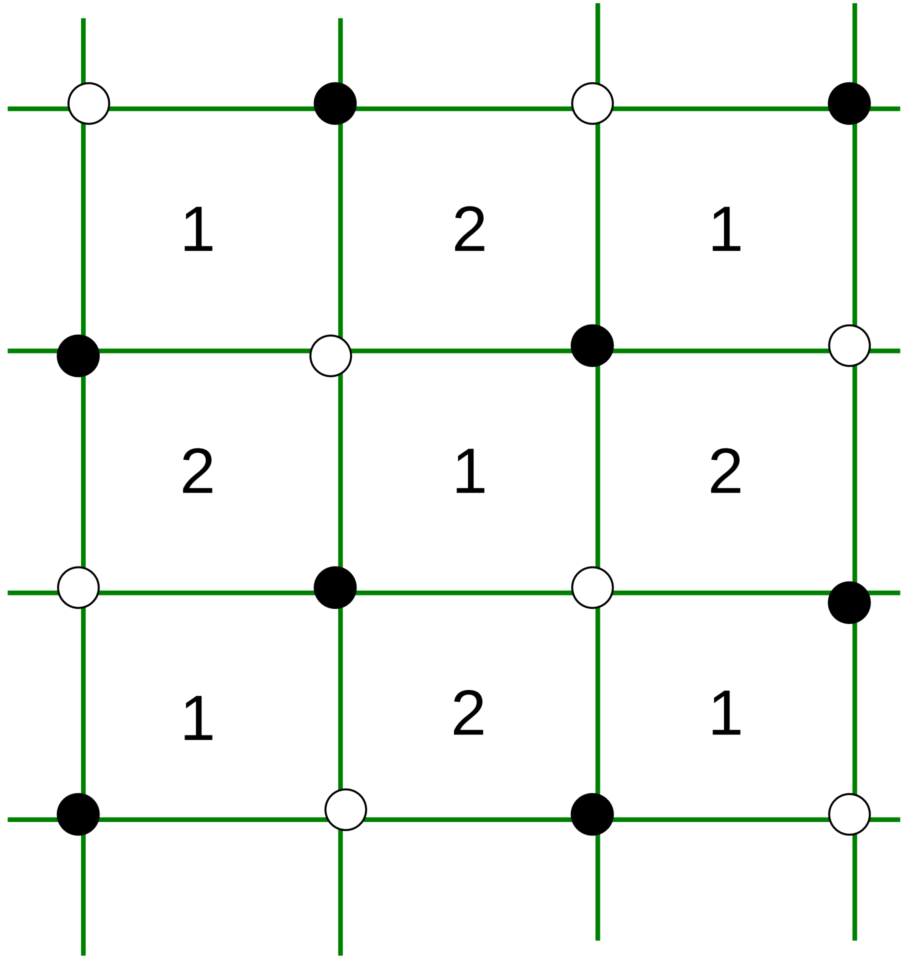
Example: conifold



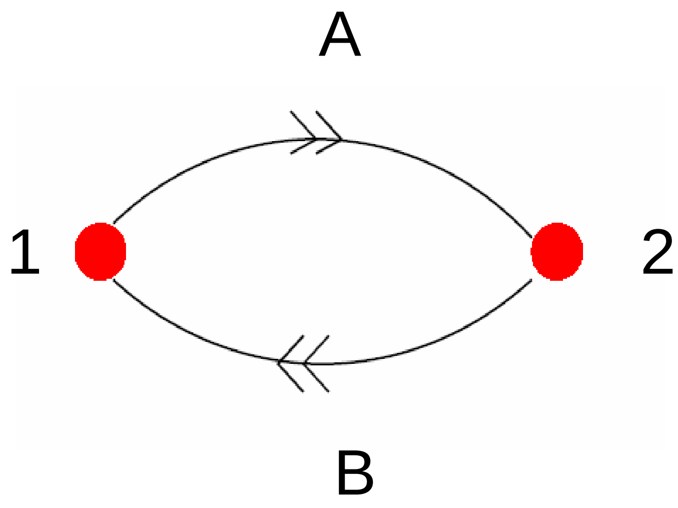


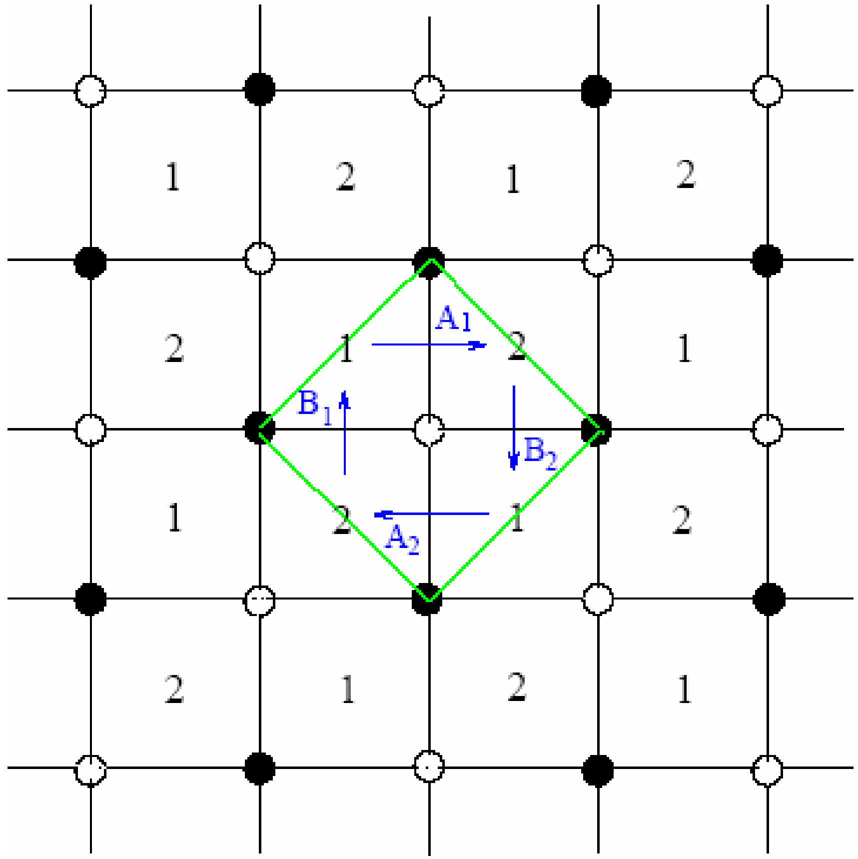
Example: conifold



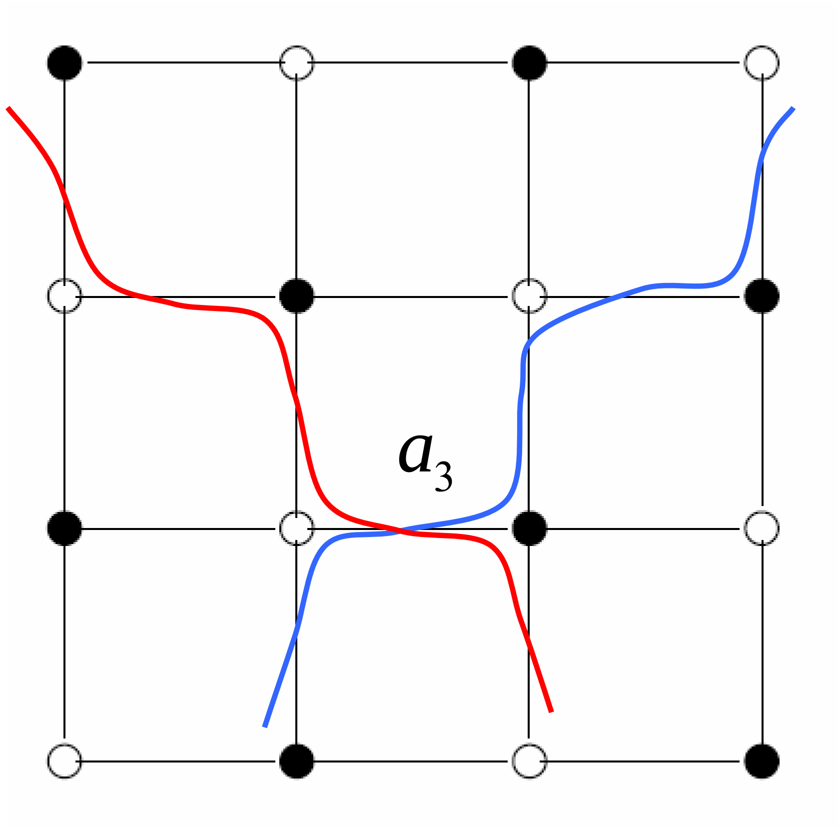


Example: conifold

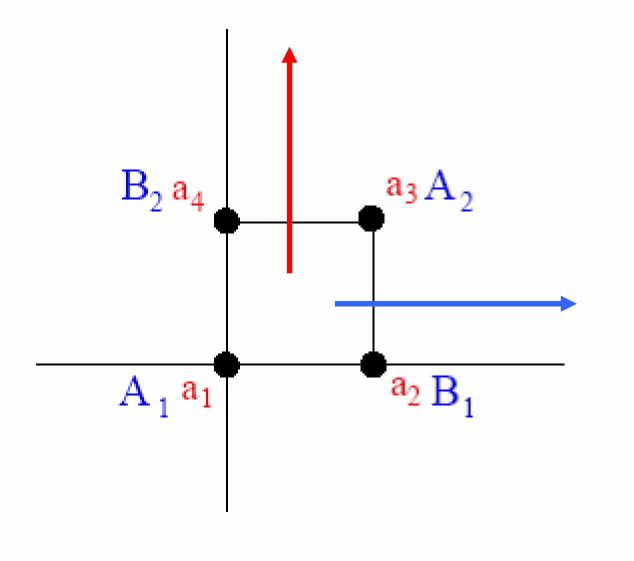




Example: conifold



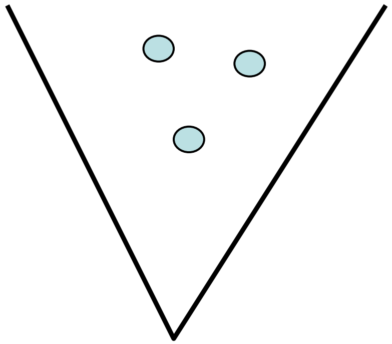
zig zag path



Global charges are associated with vertices

The full multitrace contribution for N colors is given in terms of the result for N=1:

N branes → Holomorphic Functions on $(\mathrm{Sym}(CY)^N)$



$$\sum \upsilon^N g_N(q) = Exp(\sum_{k=1} g_1(q^k)\upsilon^k / k)$$

(counts symmetrized products)

Known also as the pletystic exponential

$g_1(q)$ is the generating function for holomorphic functions

[Benvenuti,Feng,Hanany,He]