Phase Transitions: Scaling and Universality

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Plan of the presentation

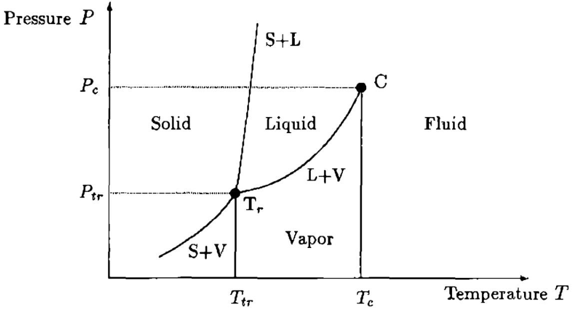
1. Phase transitions and critical phenomena

2. Renormalization Group theory

3. Field-theoretical approach

Phase diagram of a pure substance

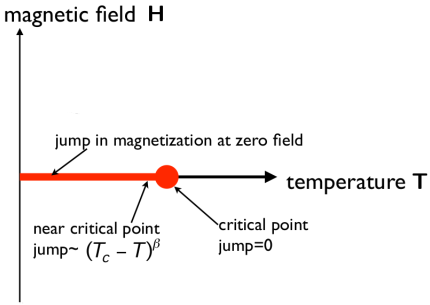
Phase of a system ↔ Collective behavior depending on external conditions



At the critical point we observe a continuous phase transition.

Curie point

Order parameter : A macroscopic quantity that reveals the transition.



Spontaneous symmetry breaking

Symmetry properties of the underlying Hamiltonian are not fully respected by the equilibrium thermodynamic state.

Thermal fluctuations

Simple model of an uniaxial ferromagnet (Ising)

$$\mathcal{H}_{\mathcal{I}} = -J \sum_{<ij>} s_i s_j - H \sum_i s_i \qquad s_i = \pm 1$$

- Magnetization per unit volume $m = \langle s_i \rangle \equiv \frac{\text{Tr } s_i \; e^{-\beta \mathcal{H}_{\mathcal{I}}}}{\text{Tr } e^{-\beta \mathcal{H}_{\mathcal{I}}}}$
- Correlation function of the fluctuations of s_i

$$G(r_i - r_j) = \langle s_i s_j \rangle - \langle s_i \rangle \langle s_j \rangle \sim e^{-\frac{|r_i - r_j|}{\xi}}$$

- ξ correlation length : distance over which the fluctuations of the microscopic degrees of freedom are significantly correlated with each other.

$\longrightarrow$ ξ will diverge at the critical point

Scaling laws

Near critical points thermodynamic functions are homogeneous functions.

Homogeneous function of degree n

$$f(\lambda x, \lambda y, \lambda z, \dots) = \lambda^n f(x, y, z, \dots)$$

$$f(x, y, z, \dots) = x^n f(1, \frac{y}{x}, \frac{z}{x}, \dots) \equiv x^n \phi(\frac{y}{x}, \frac{z}{x}, \dots)$$

Equation of state near the Curie point of a ferromagnet

$$H = M^\delta \, \phi\left(\frac{t}{M^{1/\beta}}\right)$$

H is a homogeneous function of $t \equiv \frac{T-T_c}{T_c}$ and $M^{1/\beta}$ of degree $\beta\delta$.

Critical exponents

Power laws near the critical point ($|t| \equiv |\frac{T-T_c}{T_c}| \to 0,\ H=0$)

- α Heat capacity: $C \sim |t|^{-\alpha}$
- β Order parameter: $M \sim |t|^{\beta}$
- γ Susceptibility: $\chi = \frac{\partial M}{\partial H} \sim |t|^{-\gamma}$
- δ Equation of state ($t=0,\ H \sim 0$): $M \sim H^{\frac{1}{\delta}}$
- ν Correlation length: $\xi \sim |t|^{-\nu}$
- η Correlation function ($t=0$):
  $G(r_1 - r_2) = \langle s(r_1)s(r_2)\rangle - \langle s(r_1)\rangle\langle s(r_2)\rangle \sim \frac{1}{|r_1-r_2|^{d-2+\eta}}$

Universality

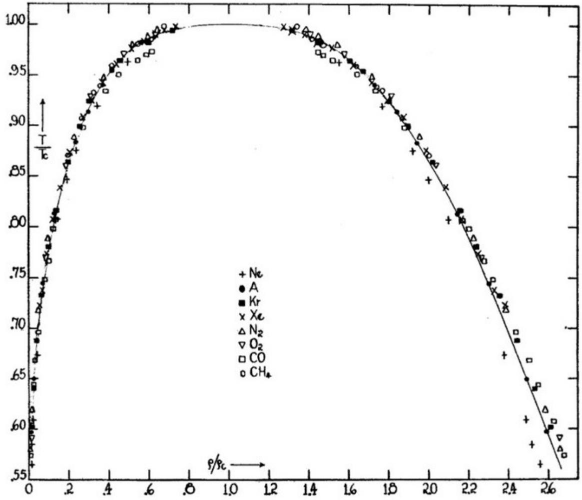
The six exponents are related to each other by four equations so that only two of them are independent:

$$
\begin{aligned}
\gamma &= \nu(2-\eta) \\
\alpha + 2\beta + \gamma &= 2 \\
\gamma &= \beta(\delta - 1) \\
\nu d &= 2 - \alpha
\end{aligned}
$$

Universality classes

It is experimentally observed that many different systems have the same critical exponents.

Coexistence curves of different fluids



E.A.Guggenheim, *J.Chem.Phys.* **13**, 253 (1945)

$$\left(\frac{\rho}{\rho_c} - 1\right) \sim \left|\frac{T}{T_c} - 1\right|^{\beta}$$

$$\beta \simeq \frac{1}{3}$$

Theory

Van der Waals, Weiss, Landau,... $\longrightarrow$ Classical theories unable to account for fluctuations

α	β	γ	δ	ν	η
0	1/2	1	3	1/2	0

..., K.G. Wilson (1971) $\longrightarrow$ New way of looking at physics: Renormalization Group (RG) theory

	α	β	ν	η
liquid-vapour	0.111(1)	0.324(2)	0.6297(4)	0.042(6)
uniaxial magnets	0.110(5)	0.325(2)	0.630(2)	
Field Theory*	0.110(5)	0.326(2)	0.630(2)	0.035(4)

* R.Guida, J.Zinn-Justin, *J.Phys.* **A 31**, 8103 (1998)

Renormalization Group (RG) idea

- The RG idea is to re-express the parameters which define a problem in terms of some other set, while keeping unchanged those physical aspects of interest.
- In critical phenomena the objective is to study the long distance behavior of the system near the critical point.
  $\rightarrow$ RG transformation of parameters is obtained through some kind of coarse-graining of the microscopic degrees of freedom.
- The RG transformation $\mathcal{H}(K) \mapsto \mathcal{H}'(K')$ generates a map R in the parameter space:
$$K' = R(K)$$
- What the flow in the parameter space can tell us about the physical problem is the essence of the RG theory.

Fixed points

A fixed point K^* of the map R is such that

$$K^* = R(K^*)$$

Near a fixed point is possible to linearize the map

$$K' = K^* + R'(K^*)(K - K^*) + \dots$$

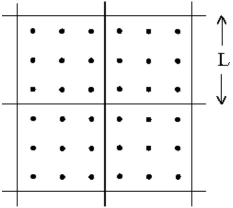
What happens to the flow of parameters near a fixed point is determined by the eigenvalues L^{y_i} of $R'(K^*)$

$$u_i' = L^{y_i} u_i$$

$y_i > 0 \longrightarrow$ u_i is a relevant variable

$y_i < 0 \longrightarrow$ u_i is an irrelevant variable

Block-spin picture



Ising model in zero magnetic field

$$-\frac{\mathcal{H}_{\mathcal{I}}}{k_B T} = K \sum_{<ij>} s_i s_j$$

Coarse-grained microscopic degrees of freedom s'_i

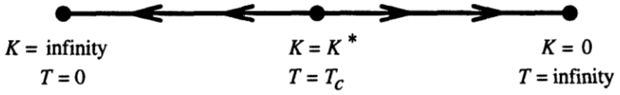
$$s'_i = sgn\left(\sum_{j=1}^{L^d} s_j^{(i)}\right)$$

RG transformation

$$e^{-\mathcal{H}'(s')} \equiv \mathrm{Tr}_s \prod_{blocks} P(s'_i, s^{(i)})\; e^{-\mathcal{H}(s)}$$

Large distance physics is preserved.

RG eigenvalues ↔ Critical exponents



Near the critical point the singular part of the free energy per site transforms according to

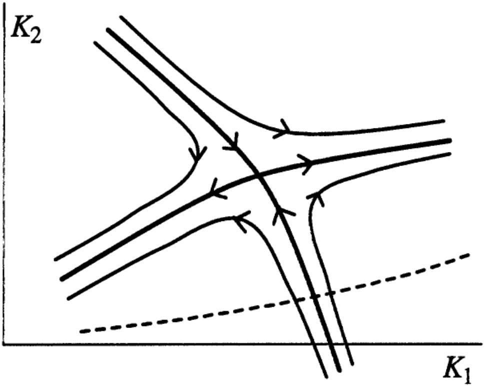
$$f_{sing}(L^{y_t}t, L^{y_h}H) = L^d f_{sing}(t, H)$$

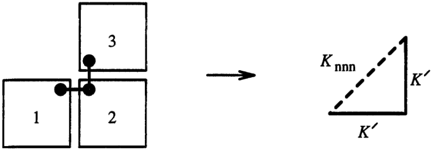
$\longrightarrow$ $f_{sing}(t, H)$ is a homogeneous function of $t^{\frac{y_h}{y_t}}$ and H of degree $\frac{d}{y_h}$

$$f_{sing}(t, H) = |t|^{\frac{d}{y_t}} \; \Phi\left(\frac{H}{|t|^{\frac{y_h}{y_t}}}\right)$$

$$\Rightarrow y_t = \frac{1}{\nu} \qquad y_h = \frac{\beta+\gamma}{\nu}$$

A more realistic picture





Fixed points and universality

Explanation of universality

All Hamiltonians which flow after RG transformations into the same fixed point have the same critical behavior.

The critical behavior is essentially determined by a few global properties

- Space dimensionality
- Nature and symmetry of the order parameter
- Pattern of symmetry breaking
- Range of the effective interactions

Landau-Ginzburg-Wilson (LGW) theory

LGW effective action

1. Order parameter $\Phi(x) \sim$ average spin in a block
2. Symmetry $\Phi(x) \mapsto -\Phi(x)$
3. Symmetry breaking pattern $Z_2 \to \mathcal{I}$
4. Short-range interactions $\to (\nabla\Phi(x))^2$

$$\mathcal{S}_{\mathcal{I}} = \int d^d x \left[\frac{1}{2}(\nabla\Phi(x))^2 + \frac{t}{2}\Phi(x)^2 + \frac{u}{4!}\Phi(x)^4\right]$$

Renormalized action

Original action

$$\mathcal{S}_o(\Phi) = \int d^d x \left[\frac{1}{2}(\nabla\Phi(x))^2 + \frac{t_o}{2}\Phi(x)^2 + \frac{u_o}{4!}\Phi(x)^4 \right]$$

Renormalized action

$$\mathcal{S}(\Phi) = \int d^d x \left[\frac{1}{2} Z (\nabla\Phi)^2 + \frac{t}{2}\ Z_t \Phi^2 + \frac{u}{4!}\ t^{\frac{4-d}{2}} Z_u \Phi^4 \right]$$

$$t_o = \frac{Z_t}{Z}\ t \qquad\qquad u_o = \frac{Z_u}{Z^2}\ u\ t^{\frac{4-d}{2}}$$

Z functions are computed with perturbative QFT techniques (MZM or MS scheme).

Fixed points and critical exponents

$$\beta(u) = t^{\frac{1}{2}} \frac{\partial u}{\partial t^{\frac{1}{2}}}_{|u_o}$$

A fixed point is defined by the solution to the equation: $\beta(u^*) = 0$
A fixed point is stable if $\beta'(u^*) > 0$

$$\eta(u) = \frac{\partial \ln Z}{\partial \ln t^{\frac{1}{2}}}_{|u_o} \qquad \eta_2(u) = \frac{\partial \ln(Z_t/Z)}{\partial \ln t^{\frac{1}{2}}}_{|u_o}$$

$$\eta \equiv \eta(u^*) \qquad \nu \equiv \frac{1}{\eta_2(u^*) + 2}$$

CP^{N-1} models

A class of models describing physical systems characterized by a global $U(N)$ symmetry and a local gauge $U(1)$ symmetry with antiferromagnetic interactions. They can undergo a phase transition with symmetry breaking pattern $U(N) \mapsto U(N-1) \otimes U(1)$. On a lattice:

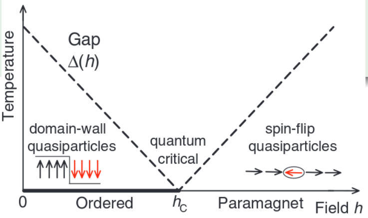
$$H_{CP^{N-1}} = J \sum_{\langle ij \rangle} |\bar{z}_i \cdot z_j|^2$$

- For $N = 2$ there is a continuous transition in the $O(3)$ universality class: $\xi \sim |t|^{-0.706}$.
- For $N = 3$ there is a continuous transition in the $O(8)$ universality class: $\xi \sim |t|^{-0.828}$. Monte Carlo simulations of the lattice model support this statement.
- For $N \geq 4$ the transition is discontinuous.

- Quantum fluctuations may induce a phase transition at $T = 0$.
- The ground state energy has a point of nonanalyticity as a function of a parameter g in the Hamiltonian.

Quantum Ising transition
($CoNb_2O_6$, Coldea et al. (2010) Science 327, 177)

$$\mathcal{H}_{\mathcal{I}} = -Jg \sum_i \sigma_i^x - J \sum_{\langle ij \rangle} \sigma_i^z \sigma_j^z$$



Quantum-classical mapping ($\mathcal{QC}$ mapping)

Continuous quantum phase transition

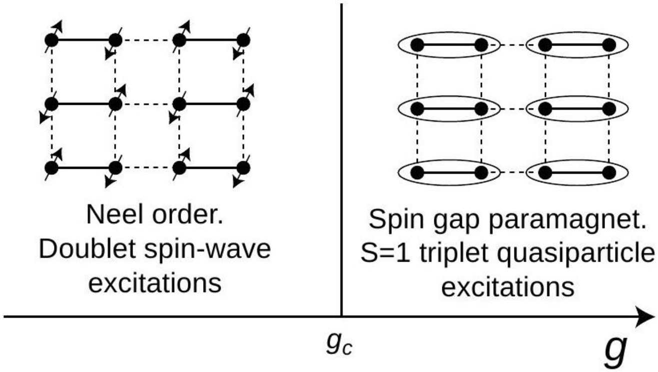
- Diverging correlation length: $\xi \sim |g - g_c|^{-\nu}$
- Vanishing energy gap: $\Delta \sim |g - g_c|^{z\nu}$

$\mathcal{QC}$ mapping

$$Z = Tr\ e^{-\frac{H(\hat{S})}{T}} = \sum_n \sum_{m_1,\dots,m_N} \langle n|e^{-\delta\tau H}|m_1\rangle \dots \langle m_N|e^{-\delta\tau H}|n\rangle$$

with $N\delta\tau = \frac{1}{T}$.
A quantum system in d spatial dimensions at zero temperature is described by a corresponding classical model in $D = d + 1$ spatial dimensions.

$SU(N)$ antiferromagnets



$$H = \frac{J}{n}\sum_{\langle i,j\rangle} S^{\beta}_{\alpha}(i)S^{\alpha}_{\beta}(j) + \frac{K}{n}\sum_{[i,j]} S^{\beta}_{\alpha}(i)S^{\alpha}_{\beta}(j) \qquad g = \frac{K}{J}$$

Experimental realization

Ultracold fermionic alkaline-earth atoms

In these atoms, such as ^{87}Sr, an almost perfect decoupling of the nuclear spin I from the electronic angular momentum J occurs, since $J = 0$ in the ground state $^1S_0$.
$\longrightarrow$ Independence of the interaction strength from the nuclear spin state and absence of spin-changing collisions.
$\longrightarrow$ Realization of different $SU(N)$ symmetries with $N \leq 2I + 1$.

Antiferromagnetic $SU(N)$ spin system

It can be realized by loading on a bipartite optical lattice one atom on each site of sublattice A (fundamental representation of $SU(N)$) and $N - 1$ atoms on each site of sublattice B (conjugate to fundamental representation).

Mapping to the CP^{N-1} model

Coherent states representations:

- A sublattice: $\langle q|S_\alpha^\beta|q\rangle = \frac{n_c}{2}\ Q_\alpha^\beta$
- B sublattice: $\langle q|S_\alpha^\beta|q\rangle = -\frac{n_c}{2}\ Q_\alpha^\beta$

with n_c the number of columns in the Young tableau representation.

Semiclassical limit

Performing the $\mathcal{QC}$ mapping with the coherent states representations and considering the semiclassical limit ($n_c \to \infty$), the action becomes:

$$\mathcal{S}_{CP^{N-1}} = \int d^d x \frac{1}{2\tilde{g}} Tr(\nabla Q)^2$$

where Q is a $N \times N$ traceless Hermitian matrix obeying the constraint $Q^2 = \mathbf{1}$.

LGW effective action

$$\mathcal{S}_{LGW} = \int d^d x \left[Tr(\nabla Q)^2 + t \; TrQ^2 + g \; TrQ^3 + \lambda \; TrQ^4 + \lambda' \; (TrQ^2)^2 \right]$$

with Q an arbitrary $n \times n$ traceless Hermitian matrix.

- The action has been derived only considering the $SU(n)$ symmetry: $Q \longmapsto UQU^\dagger$. It describes the critical modes of the ferromagnetic models.
- In the antiferromagnetic models the critical modes are described by the same action without the cubic term.

LGW effective action

The LGW action can be rewritten in terms of $n^2 - 1$ real fields:

$$Q = \sum q_i T_i \qquad d_{abc} = 2\ Tr\left[\{T_a, T_b\} T_c\right]$$

where $\{T_i\}$ are the $n^2 - 1$ generators of $SU(n)$ in the fundamental representation.

$$\mathcal{S}_{LGW} =$$

$$\int d^d x \left[\frac{1}{2}(\nabla q)^2 + \frac{t}{2}\ q^2 + \frac{g}{4}\ d_{ijk} q_i q_j q_k + \frac{1}{4} v\ (q^2)^2 + \frac{\lambda}{4!}\ d_{ijkl} q_i q_j q_k q_l\right]$$

$$v \equiv \left(\tfrac{\lambda}{n} + \lambda'\right) \qquad d_{ijkl} \equiv d_{ijr} d_{klr} + d_{ilr} d_{kjr} + d_{ikr} d_{jlr}$$

CP^1 and $O(3)$ models

The CP^1 model can be exactly mapped into the $O(3)$ model.

CP^1

$$d_{ijk} = 0 \qquad \longrightarrow \qquad \mathcal{S}_{LGW} = \int d^d x \left[\frac{1}{2}(\nabla q)^2 + \frac{t}{2}\, q^2 + \frac{v}{4}\, (q^2)^2 \right]$$

Critical exponents

$$\nu = 0.706 \qquad \eta = 0.038 \qquad \gamma = 1.386 \qquad \alpha = -0.117$$

$$\longrightarrow \qquad \xi \sim |t|^{-0.706}$$

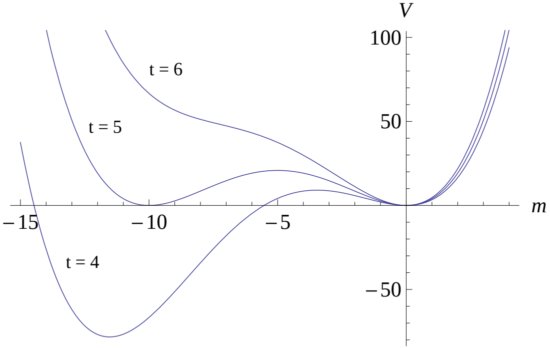
Mean field approximation

Uniform Q configurations which minimize the potential

$$V(Q) = t\ TrQ^2 + g\ TrQ^3 + \lambda\ TrQ^4 + \lambda'\ (TrQ^2)^2$$

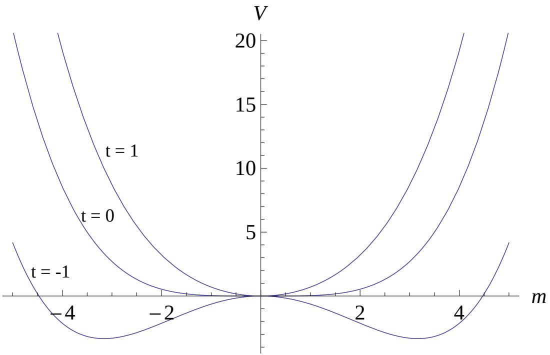
Solution: $\bar{Q}^{\alpha\beta} = m(\delta^{\alpha 1}\delta^{\beta 1} - \frac{1}{n}\delta^{\alpha\beta})$

Symmetry breaking pattern: $SU(n) \mapsto U(1) \otimes SU(n-1)$



$n = 3$
$g = 3$
$\lambda = \lambda' = \frac{1}{20}$

Mean field approximation



$n = 3$
$g = 0$
$\lambda = \lambda' = \frac{1}{20}$

CP^2

$$\mathcal{S}_{LGW} = \int d^d x \left[\frac{1}{2}(\nabla q)^2 + \frac{t}{2}\ q^2 + \frac{g}{4}\ d_{ijk} q_i q_j q_k + \frac{u}{4}\ (q^2)^2\right]$$

If $g = 0$: Symmetry breaking pattern $O(8) \mapsto O(7)$

ε-expansion at 1-loop order

RG flow in $d = 4 - \varepsilon$ dimensions.

Bare action

$$\mathcal{S}_o(q) = \int d^d x \left[\frac{1}{2}(\nabla q)^2 + \frac{t_o}{2}\, q^2 + \frac{v_o}{4}\, (q^2)^2 + \frac{\lambda_o}{4!}\, d_{ijkl} q_i q_j q_k q_l \right]$$

Renormalized action

$\mathcal{S}(q) =$

$$\int d^d x \left[\frac{1}{2} Z (\nabla q)^2 + \frac{t}{2}\, Z_t q^2 + \frac{v}{4}\, t^{\frac{4-d}{2}} Z_v (q^2)^2 + \frac{\lambda}{4!}\, t^{\frac{4-d}{2}} Z_\lambda d_{ijkl} q_i q_j q_k q_l \right]$$

$$t_o = \frac{Z_t}{Z}\, t \qquad v_o = \frac{Z_v}{Z^2}\, v\; t^{\frac{4-d}{2}} \qquad \lambda_o = \frac{Z_\lambda}{Z^2}\, \lambda\; t^{\frac{4-d}{2}}$$

ε-expansion at 1-loop order

Divergent contributions to the thermodynamic potential Γ come from the two- and four-point 1PI correlation functions:

$$\begin{aligned}
\Gamma_1^{div}(\varphi) &= \frac{1}{2} Tr \left[ln \frac{\delta^2 \mathcal{S}(\varphi)}{\delta\varphi_i(x_1)\delta\varphi_j(x_2)} - ln \frac{\delta^2 \mathcal{S}(0)}{\delta\varphi_i(x_1)\delta\varphi_j(x_2)} \right]^{div} = \\
&= \frac{1}{2} \left[\int d^d x \; V''_{ii}(x) \; \Delta(0) \right. \\
&- \left. \frac{1}{2} \int d^d x_1 \; d^d x_2 \; V''_{ij}(x_1) \; \Delta(x_1 - x_2) \; V''_{ji}(x_2) \; \Delta(x_2 - x_1) \right]
\end{aligned}$$

where

$$\Delta(x_1 - x_2) = \int \frac{d^d p}{(2\pi)^d} \frac{e^{ip(x_1 - x_2)}}{p^2 + t}$$

is the propagator and

$$V''_{ij}(x) = v \; t^{\frac{4-d}{2}} \; (\varphi^2 \; \delta_{ij} + 2 \; \varphi_i \; \varphi_j) + \frac{\lambda}{2} \; t^{\frac{4-d}{2}} \; d_{ij\alpha\beta} \; \varphi_\alpha \; \varphi_\beta$$

Renormalization constants at 1-loop order

In the MS-scheme the renormalization constants are defined by simply subtracting the divergent contributions:

$$
\begin{aligned}
Z &= 1 \\
Z_t &= 1 + \frac{1}{8\pi^2\varepsilon}\left(v(n^2+1) + \lambda\frac{n^2-4}{n}\right) \\
Z_v &= 1 + \frac{1}{8\pi^2\varepsilon}\left(v(n^2+7) + 2\lambda\frac{n^2-4}{n} + 2\frac{\lambda^2}{v}\frac{n^2-4}{n^2}\right) \\
Z_\lambda &= 1 + \frac{3}{4\pi^2\varepsilon}\left(2v + \lambda\frac{n^2-15}{3n}\right)
\end{aligned}
$$

Beta functions

$$\beta_\lambda(\lambda, v) = t^{\frac{1}{2}} \frac{\partial \lambda}{\partial t^{\frac{1}{2}}}_{|\lambda_o, v_o}$$

$$\beta_v(\lambda, v) = t^{\frac{1}{2}} \frac{\partial v}{\partial t^{\frac{1}{2}}}_{|\lambda_o, v_o}$$

1-loop order

From the relations between bare parameters and renormalized ones:

$$\beta_\lambda = -\varepsilon\lambda + \frac{n^2 - 15}{4\pi^2 n}\lambda^2 + \frac{3}{2\pi^2} v\lambda$$

$$\beta_v = -\varepsilon v + \frac{1}{8\pi^2}\left[(n^2+7)\ v^2 + \frac{2(n^2-4)}{n}\lambda v + \frac{2(n^2-4)}{n^2}\lambda^2\right]$$

Fixed points

Fixed points are given by the solutions to the system:

$$\begin{cases} \beta_\lambda = -\varepsilon\lambda + \frac{n^2-15}{4\pi^2 n}\lambda^2 + \frac{3}{2\pi^2} v\lambda = 0 \\ \\ \beta_v = -\varepsilon v + \frac{1}{8\pi^2}\left[\left(n^2+7\right)\, v^2 + \frac{2(n^2-4)}{n}\lambda v + \frac{2(n^2-4)}{n^2}\lambda^2\right] = 0 \end{cases}$$

Their stability is controlled by the eigenvalues of the matrix

$$\Omega = \begin{pmatrix} \frac{\partial\beta_\lambda}{\partial\lambda} & \frac{\partial\beta_\lambda}{\partial v} \\ \\ \frac{\partial\beta_v}{\partial\lambda} & \frac{\partial\beta_v}{\partial v} \end{pmatrix}$$

A fixed point is stable if all the eigenvalues of its stability matrix have positive real parts.

Stable fixed points

- $n = 2$: $O(3)$ fixed point.
- $n = 3$: $O(8)$ fixed point.
- $n \geq 4$: no stable fixed points in $4 - \varepsilon$ dimensions.

Enlarged symmetry in the CP^2 model

The antiferromagnetic CP^2 model acquires an enlarged $O(8)$ symmetry.

The symmetry group of the coarse-grained fixed point action is larger than that of the microscopic Hamiltonian because higher order terms are assumed to be irrelevant.

RG flow in three dimensions

There may be fixed points that exist in three dimensions, but cannot be analytically continued in $4 - \varepsilon$ dimensions.

$3d - MS$ scheme

The RG functions are determined as in the ε-expansion framework, but then ε is set to its physical value $\varepsilon = 1$ and the RG functions are then expansions in powers of the renormalized quartic couplings.

Since these expansions are divergent, summation methods must be used to obtain meaningful results.

Beta functions to 5-loop order in the MS-scheme

$$\beta_\lambda = -\varepsilon\lambda + \frac{\lambda\left(\lambda\left(n^2-15\right)+6nv\right)}{4n\pi^2} - \frac{\lambda\left(\lambda^2\left(1168-126n^2+3n^4\right)+2n^2\left(77+5n^2\right)v^2+\lambda\left(-800nv+68n^3v\right)\right)}{128n^2\pi^4} +$$

$$\frac{1}{4096n^3\pi^6}\lambda\left(2n^3v^3\left(2903-13n^4+2496\zeta[3]+2n^2(197+96\zeta[3])\right)+\right.$$

$$\lambda^3\left(13n^6-n^4(839+12\zeta[3])+12n^2(1730+649\zeta[3])-128(1139+957\zeta[3])\right)+$$

$$2\lambda^2 nv\left(68856+57600\zeta[3]+n^4(395+192\zeta[3])-2n^2(4969+3312\zeta[3])\right)$$

$$\left.+2\lambda n^2v^2\left(4n^4+n^2(1765+1728\zeta[3])-3(7739+6528\zeta[3])\right)\right)+$$

$$\frac{1}{1966080n^4\pi^8}\lambda\left(6n^4v^4\left(2368\pi^4+n^4\left(155+32\pi^4-6480\zeta[3]\right)+5n^6(-29+48\zeta[3])+\right.\right.$$

$$15n^2\left(-9269+32\pi^4-8144\zeta[3]-13440\zeta[5]\right)-15(43209+64016\zeta[3]+103040\zeta[5]))+$$

$$8\lambda n^3v^3\left(5n^6(7+72\zeta[3])+5n^2\left(-54875+64\pi^4-82104\zeta[3]\ -178560\zeta[5]\right)+\right.$$

$$\left.2n^4\left(88\pi^4-5(2663+2988\zeta[3]+1920\zeta[5])\right)+6\left(-3176\pi^4+15(58991+84388\zeta[3]+136960\zeta[5])\right)\right)+$$

$$8\lambda^3 nv\left(n^6\left(32\pi^4-15(1167+640\zeta[3]+800\zeta[5])\right)+n^4\left(-1369\pi^4+60(11239+7213\zeta[3]+9755\zeta[5])\right)-\right.$$

$$\left.96\left(1883\pi^4-60(8940+12253\zeta[3]+20155\zeta[5])\right)+3n^2\left(8387\pi^4-20(164828+151569\zeta[3]+234315\zeta[5])\right)\right)+$$

$$4\lambda^2n^2v^2\left(n^6\left(2895+16\pi^4-3600\zeta[3]\right)+n^4\left(472\pi^4-15(27977+16112\zeta[3]+37120\zeta[5])\right)+\right.$$

$$\left.720\left(236\pi^4-3(22245+31004\zeta[3]+50720\zeta[5])\right)+n^2\left(-19704\pi^4+30(261395+279456\zeta[3]+496800\zeta[5])\right)\right)+$$

$$\lambda^4\left(15n^8(-67+32\zeta[3]-80\zeta[5])+20n^4\left(-164827+222\pi^4+18072\zeta[3]+69600\zeta[5]\right)+\right.$$

$$24n^6\left(\pi^4-50(-59+67\zeta[3]+57\zeta[5])\right)+$$

$$\left.\left.768\left(1583\pi^4-15(29939+40648\zeta[3]+66940\zeta[5])\right)-8n^2\left(19283\pi^4-10(757837+513714\zeta[3]+566640\zeta[5])\right)\right)\right)-$$

$$\frac{1}{3963617280n^5\pi^{10}}\lambda\left(-6n^5v^5\left(63n^8\left(8\pi^4-5(61+80\zeta[3])\right)\right.\right.$$

Beta functions to 5-loop order in the MS-scheme

$$-105n^6\left(2825+96\pi^4+6288\zeta[3]-4608\zeta[5]\right)+$$

$$n^4\left(-327264\pi^4-60800\pi^6-105\left(-232007-555024\zeta[3]\quad+52992\zeta[3]^2-902016\zeta[5]-508032\zeta[7]\right)\right)-$$

$$280\left(50613\pi^4+11360\pi^6-18\left(396047+859814\zeta[3]\quad+159456\zeta[3]^2+1761816\zeta[5]+2603664\zeta[7]\right)\right)+$$

$$n^2\left(-3747072\pi^4-822400\pi^6+105\left(5662825+7977168\zeta[3]\quad+52992\zeta[3]^2+18031104\zeta[5]+21845376\zeta[7]\right)\right)\Big)+$$

$$2\lambda^5\Big(3n^{10}\left(-252\pi^4+100\pi^6+315\left(-109+488\zeta[3]+72\zeta[3]^2\quad-1240\zeta[5]\right)\right)+$$

$$12n^2\left(156072063\pi^4+27825800\pi^6+105\left(-320405039\quad-321971772\zeta[3]+62890272\zeta[3]^2-565354944\zeta[5]\quad-397542978\zeta[7]\right)\right.$$

$$n^8\left(150696\pi^4+16400\pi^6-315\left(-20135\quad+246942\zeta[3]+8688\zeta[3]^2+130440\zeta[5]+272538\zeta[7]\right)\right)-$$

$$5n^6\left(515088\pi^4+149780\pi^6-63\left(-2098453+7340118\zeta[3]\quad+1009464\zeta[3]^2+8776380\zeta[5]+14505372\zeta[7]\right)\right)-$$

$$3840\left(3201429\pi^4+733615\pi^6-63\left(8050214+15343559\zeta[3]\quad+2643726\zeta[3]^2+31822482\zeta[5]+46957239\zeta[7]\right)\right)-$$

$$n^4\left(62922384\pi^4+3049000\pi^6+315\left(-85910707+26500944\zeta[3]\quad+38740224\zeta[3]^2+47717148\zeta[5]+206290098\zeta[7]\right)\right)\Big)+$$

$$8\lambda^2n^3v^3\Big(315n^8(-1661+1428\zeta[3]-672\zeta[5])+$$

$$70n^6\left(4239\pi^4+800\pi^6+27\left(1223-26576\zeta[3]+3136\zeta[3]^2\quad-44808\zeta[5]-14112\zeta[7]\right)\right)-$$

$$2n^4\left(786114\pi^4+15800\pi^6+315\left(1734473+952590\zeta[3]\quad+558096\zeta[3]^2+2176752\zeta[5]+6435072\zeta[7]\right)\right)+$$

$$3n^2\left(-26741946\pi^4-6962000\pi^6+105\left(68941403+99050520\zeta[3]\quad+21284064\zeta[3]^2+212699952\zeta[5]+346604832\zeta[7]\right)\right)+$$

$$30\left(28341096\pi^4+6441920\pi^6-63\left(67774897+135943024\zeta[3]\quad+23837568\zeta[3]^2+282678720\zeta[5]+415570176\zeta[7]\right)\right)\Big)$$

Beta functions to 5-loop order in the MS-scheme

$$+4\lambda^4 nv \Big(n^4 \Big(90741294\pi^4 + 17659700\pi^6 + 945\,(-30955487 - 22179684\zeta[3] \quad +4169384\zeta[3]^2 - 43475712\zeta[5] - 34906032\zeta[7]$$

$$n^6 \Big(-3101868\pi^4 - 592100\pi^6 - 315\,(-3952403 - 1762320\zeta[3] \quad +561336\zeta[3]^2 - 4239408\zeta[5] - 2206764\zeta[7]\Big)\Big) +$$

$$12n^8 \Big(5649\pi^4 + 1100\pi^6 + 105\,\Big(-18283 - 10734\zeta[3] + 1008\zeta[3]^2 \quad -25128\zeta[5] - 19845\zeta[7])) +$$

$$768\,\Big(11653656\pi^4 + 2672375\pi^6 - 315\,(5811173 + 11180057\zeta[3] \quad +1930854\zeta[3]^2 + 23293326\zeta[5] + 34312005\zeta[7]\Big)\Big) -$$

$$12n^2 \Big(123583257\pi^4 + 25545650\pi^6 - 105\,(260175425 + 326240376\zeta[3] \quad +3571020\zeta[3]^2 + 620621748\zeta[5] + 757301076\zeta[7]\Big)\Big)\Big)$$

$$4\lambda^3 n^2 v^2 \Big(n^8 \Big(19908\pi^4 + 2000\pi^6 + 945\,(3553 - 1600\zeta[3] \quad +224\zeta[3]^2 - 4000\zeta[5]\Big)\Big) +$$

$$9n^6 \Big(96271\pi^4 + 29650\pi^6 - 105\,\Big(667717 + 422598\zeta[3] + 11484\zeta[3]^2 \quad +774168\zeta[5] + 1121904\zeta[7])) +$$

$$n^4 \Big(-47288934\pi^4 - 11068300\pi^6 + 315\,(53986819 + 50850816\zeta[3] \quad +2419512\zeta[3]^2 + 98341848\zeta[5] + 137184516\zeta[7]\Big)\Big) -$$

$$240\,\Big(22459038\pi^4 + 5134580\pi^6 - 63\,(55083763 + 107736592\zeta[3] \quad +18706920\zeta[3]^2 + 224643840\zeta[5] + 330506568\zeta[7]\Big)\Big) +$$

$$15n^2 \Big(54652101\pi^4 + 12461510\pi^6 - 21\,(602360767 + 824840334\zeta[3] \quad +92960964\zeta[3]^2 + 1673661168\zeta[5] + 2405102868\zeta[7]\Big)\Big)\Big)$$

$$3\lambda n^4 v^4 \Big(382059888\pi^4 + 86249600\pi^6 + 63n^8 \Big(-805 + 48\pi^4 - 3760\zeta[3]\Big) -$$

$$2n^6 \Big(73584\pi^4 + 6400\pi^6 + 105\,\Big(26815 - 6048\zeta[3] + 13824\zeta[3]^2 \quad -147456\zeta[5])) -$$

$$315\,(176904407 + 367505072\zeta[3] \quad +65743104\zeta[3]^2 + 759588096\zeta[5] + 1117670400\zeta[7]\Big) +$$

$$2n^2 \Big(6272784\pi^4 + 550400\pi^6 + 105\,(4881235 + 9364512\zeta[3] \quad +12524544\zeta[3]^2 + 22345344\zeta[5] + 81285120\zeta[7]\Big)\Big) -$$

$$672n^4 \Big(6196\pi^4 + 1300\pi^6 + 15\,\Big(-94137\zeta[3] + 4728\zeta[3]^2 \quad -8(7163 + 18336\zeta[5] + 19845\zeta[7])))))$$

Padé-Borel summation method

Borel summation

Borel transform:

$$A(\lambda x, vx) = \sum_{k=0}^{\infty} a_k(\lambda, v) x^k \qquad \longmapsto \qquad \mathcal{B}A(s) = \sum_{k=0}^{\infty} \frac{a_k}{k!} s^k$$

$A(x)$ is then obtained from the integral: $\quad A(x) = \int_0^\infty ds\ e^{-s}\ \mathcal{B}A(sx)$
for all the values of x for which it converges.

Padé approximants

If $\mathcal{B}A(s)$ is known only up to order l, it can be constructed a Padé approximant $\left[\frac{m}{n}\right]_{\mathcal{B}A}(s)$, with $m + n \leq l$, which is the best approximation of $\mathcal{B}A(s)$ by a rational function of order $\frac{m}{n}$.

$O(8)$ critical exponents

$O(8)$ fixed point

CP^2

$$u^* = v^* + \frac{\lambda^*}{6} = 6.85(0.69)$$

$O(8)$

$$u^* = 6.847(0.074)$$

Critical exponents

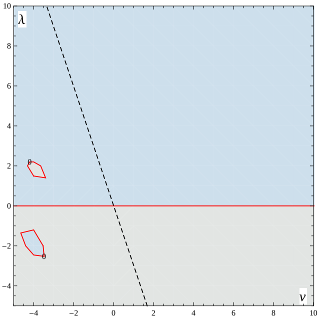
From the RG functions

$$\eta(u) = \frac{\partial \ln Z}{\partial \ln t^{\frac{1}{2}}}_{|u_o} \qquad \eta_2(u) = \frac{\partial \ln(Z_t/Z)}{\partial \ln t^{\frac{1}{2}}}_{|u_o}$$

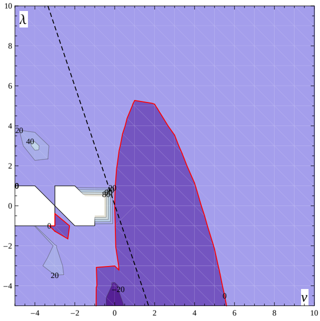
summed with different Padé approximants ($\left[\frac{4}{1}\right]$, $\left[\frac{3}{2}\right]$, $\left[\frac{2}{3}\right]$, $\left[\frac{3}{1}\right]$):

$$\eta \equiv \eta(u^*) = 0.0290(0.0026) \qquad \nu \equiv \frac{1}{\eta_2(u^*) + 2} = 0.828(0.015)$$

Zeros (red lines) of $\beta_\lambda^{5/1}$



Zeros (red lines) of $\beta_v^{5/1}$



Non trivial fixed point: $\{\lambda^* = 8.369 * 10^{-23}, v^* = 4.401(0.024)\}$

Eigenvalues of its stability matrix: $\omega_1 = -0.65(0.01)$ $\omega_2 = 0.82(0.02)$

No stable fixed points $\longrightarrow$ discontinuous transition