www.df.unipi.it/~anselmi

Damiano Anselmi

naturali.campusnet.unito.it/didattica/att/a08b.5472.file.pdf

S. Console, M. Roggero, D. Romagnoli

Lezioni di Matematica

C & F Sbordone, Matematica per le scienze della vita

. . .

V. Villani, G. Gentili, Matematica. Comprendere e interpretare fenomeni delle scienze della vita

Ricevimento : MARTEDì 8.30 - 9.30 portineria colla sbarra

damiano.anselmi@unipi.it FISICA ED.C UFF 158 1° piano a destra

15.00 - 17.00

Insiemi indicati con A B C ... X Y ...

Collezioni di oggetti

elementi indicati con a, b, c, ... x, y

"a" appartiene ad "A" : $a \in A$

$\emptyset$ insieme vuoto

$\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ l'insieme dei numeri naturali

$\mathbb{N}_+ = \{1, 2, 3, \ldots\} = \mathbb{N} / \{0\}$

← "privato di"

$\mathbb{Z}$ = numeri interi relativi = $\{\ldots, -3, -2, -1, 0, +1, +2, +3, \ldots\}$

$0 - 3 = -3$ $(-3) \cdot (-4) = 12$

$\mathbb{Q}$ = numeri razionali = $\left\{ \frac{m}{n} \right.$ dove $m \in \mathbb{Z}$ e $\left. n \in \mathbb{Z},\ n \neq 0 \right\}$

↑ "diverso da"

$$\frac{1}{2} = \frac{2}{4} = \frac{1 \cdot p}{2 \cdot p} \qquad p \in \mathbb{Z}$$

$$\frac{\cancel{3}^{1}}{\cancel{15}_{5}} = \frac{1}{5} \qquad q \in \mathbb{Q}$$

$q_1 + q_2$

$$q_1 = \frac{7}{11} \qquad q_2 = \frac{1}{3} \qquad q_1 \cdot q_2$$

$$q_1 \cdot q_2 = \frac{7}{11} \cdot \frac{1}{3} = \frac{7 \cdot 1}{11 \cdot 3} = \frac{7}{33}$$

$$q_1 = \frac{m_1}{n_1} \qquad q_2 = \frac{m_2}{n_2} \qquad q_1 \cdot q_2 = \frac{m_1 \cdot m_2}{n_1 \cdot n_2}$$

$$q_1 + q_2 = \frac{7}{11} + \frac{1}{3} = \frac{7 \cdot 3}{11 \cdot 3} + \frac{1 \cdot 11}{3 \cdot 11} =$$

$$= \frac{21}{33} + \frac{11}{33} = \frac{21 + 11}{33} = \frac{32}{33}$$

$$q_1 + q_2 = \frac{m_1}{n_1} + \frac{m_2}{n_2} = \frac{m_1 \cdot n_2}{n_1 \cdot n_2} + \frac{n_1 \cdot m_2}{n_1 \cdot n_2} =$$

$$= \frac{m_1 \cdot n_2 + n_1 \cdot m_2}{n_1 \cdot n_2}$$

$$\frac{1}{4} = 0{,}25 = \frac{\cancel{25}^{\cancel{5}\,1}}{\cancel{100}_{\cancel{20}\,4}}$$

$\mathbb{Q}$ = { numeri decimali che hanno un numero finito di cifre decimali o sono periodici }

numero decimale periodico: le sue cifre decimali contengono una sequenza di cifre che (da un certo punto in poi) si ripete indefinitamente

0,134565656656 ....

$\mathbb{R}$ = insieme dei numeri reali

= { tutti i numeri decimali }

36, 7845.....

i numeri reali che non sono razionali si dicono irrazionali

$\sqrt{2}$ irrazionale $(\sqrt{2}) \cdot (\sqrt{2}) = 2$

$\mathbb{C} = \{ (r_1, r_2) \; , \; r_1 \in \mathbb{R} \, , \; r_2 \in \mathbb{R} \}$

$\sqrt{-1} = ?$ $\sqrt{-1} \cdot \sqrt{-1} = -1$

Sottoinsiemi

si dice che $A \subseteq B$ se tutti gli elementi di A sono elementi di B

$\subseteq$ ← "è contenuto in"

"$\leq$"

$A \subseteq B$ se $\forall a \in A \quad a \in B$

$\forall$ ← "per ogni"

$\in$

Se B è più grande di A, si può scrivere $A \subset B$

$\subset$ ← "strettamente contenuto in"

$A \subset B$ se $\forall a \in A \quad a \in B$ e

$\exists\, b \in B \quad b \notin A$

$\exists$ ← "esiste"

$\notin$ ← "non appartiene a"

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$$

Altri esempi di insiemi : intervalli

$$[a, b] = \{ x \in \mathbb{R} \mid a \leq x \leq b \}$$

↖ "tale che"

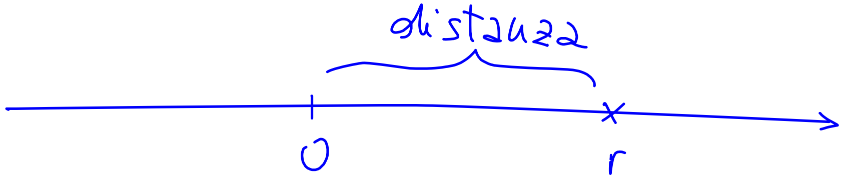
$$[2, 3]$$

$$(a, b) = \{ x \in \mathbb{R} \mid a < x < b \}$$

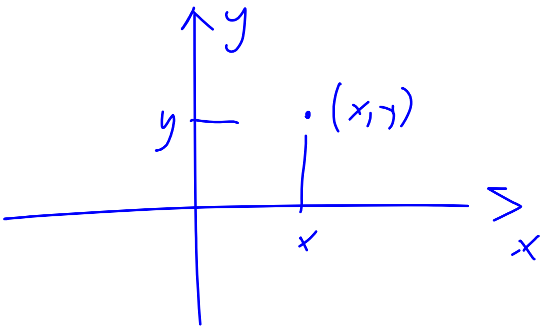
$$[a, b) = \{ x \in \mathbb{R} \mid a \leq x < b \}$$

$$(a, b] = \dots$$

Retta reale



$\mathbb{C} = \{ (x, y), \ x \in \mathbb{R}, \ y \in \mathbb{R} \}$



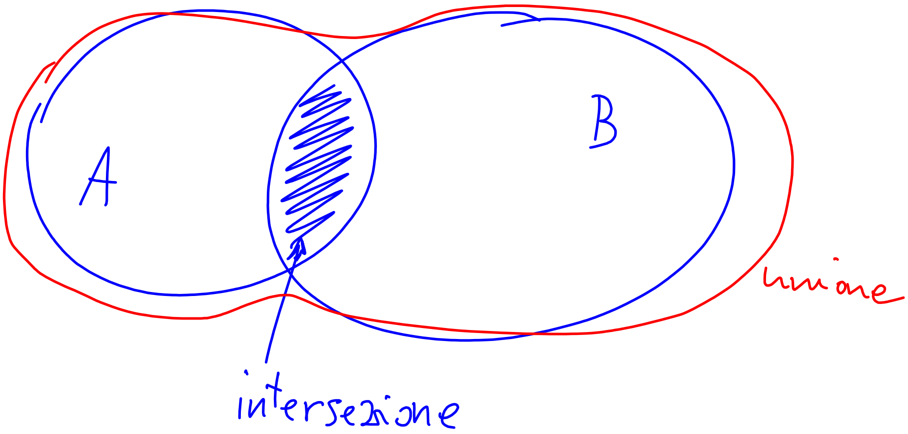
"piano complesso"

Unione di insiemi

$A \cup B = \{ x \mid x \in A \text{ o } x \in B \}$

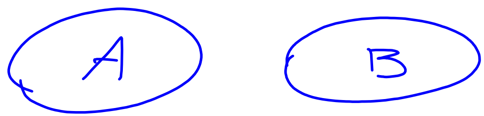
Intersezione

$A \cap B = \{ x \mid x \in A \text{ e } x \in B \}$

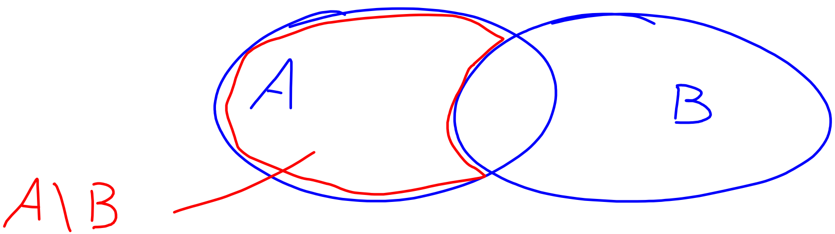


Due insiemi A e B sono disgiunti se

$A \cap B = \emptyset$



Differenza $A \setminus B = \{ x \mid x \in A, x \notin B \}$



Prodotto cartesiano di insiemi

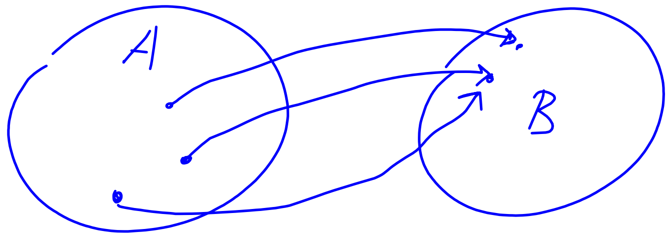
$$A \times B = \{ (a, b) \text{ dove } a \in A \text{ e } b \in B \}$$

Una funzione f tra due insiemi A e B è una corrispondenza (relazione, associazione) che associa a ciascun elemento di A un elemento di B

$$f : A \longrightarrow B$$

$$f : a \longmapsto b$$

$$b = f(a)$$



A = dominio della funzione

Esempi: $f : [0,1] \longrightarrow \mathbb{R}$

$A = [0,1]$

$f(x) = 2 \quad \forall x \in [0,1]$

oppure $f(x) = x$

$f(x) = x+2$

$f(x) = x^2 \qquad f(x) = -x$

....

$f : A \longrightarrow B \qquad b = f(a)$ a si dice controimmagine di b

A = dominio

b si chiama immagine di a

Immagine di f (o codominio) $= \{ b \in B$ tale che $\exists\, a \in A$ tale che $b = f(a) \}$

Una funzione si dice <u>iniettiva</u> se elementi distinti di A hanno immagini distinte

Si dice <u>suriettiva</u> se ogni elemento di B ha <u>almeno</u> una controimmagine

Si dice biunivoca se è iniettiva e suriettiva

Se è biunivoca è invertibile

$$f : A \to B \qquad\qquad f^{-1} : B \to A$$

$$a \mapsto b = f(a) \qquad\qquad a = f^{-1}(b)$$

Permutazioni

Dato un insieme di n oggetti distinti si dice permutazione un qualunque loro ordinamento o allineamento

$A = \{B, R, V\}$ $n = 3$ $3! = 3 \cdot 2 \cdot 1 = 6$

BRV BVR RBV RVB VRB VBR

il numero di permutazioni di n elementi

è $n! = n(n-1)(n-2) \dots 3 \cdot 2 \cdot 1$

"n fattoriale" $10! = 3628800$

$1! = 1$ $2! = 2$ $3! = 6$ $4! = 24$ $5! = 120$ $6! = 720$

Disposizioni con ripetizione :

di n elementi scelti in un insieme

B di m elenti

ogni ordinamento (detto anche disposizione) con la possibilità di usare più volte lo stesso elemento

$$B = \{ x_1, \dots \quad x_m \}$$

disposizione di $\underline{\underline{n}}$ elementi : $\underbrace{m \cdot m \cdot m \cdot \quad \cdot m}_{n \text{ volte}} = m^n$

Disposizioni semplici (cioè <u>senza</u> ripetizione):

$$(m-0)(m-1)(m-2)\dots\underset{m-(n-1)}{(m-n+1)} = D_{m,n}$$

$$D_{m,n} = m(m-1)(m-2)\cdots(m-n+1) =$$

$$= \frac{m(m-1)(m-2)\cdots(m-n+1)\overbrace{(m-n)(m-n-1)\cdots 1}}{\underline{(m-n)}(m-n-1)\cdots 3\cdot 2\cdot 1} =$$

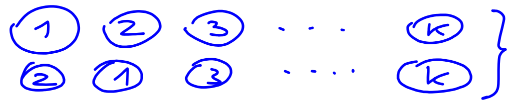
$$= \frac{m!}{(m-n)!}$$

Sia A un insieme di n oggetti diversi.

Combinazioni: si dice combinazione di n oggetti di ordine k ogni sottoinsieme di A fatto di k oggetti.

Quante sono le combinazioni degli n oggetti di ordine k ?

$$C_{n,k} = \frac{D_{n,k}}{k!}$$



$D_{n,k}$ = disposizioni semplici di k oggetti scelti in un insieme di n oggetti diversi

$C_{n,k}$ = combinazioni di k oggetti scelti in un insieme di n oggetti diversi

$C_{n,k}$ = # di sottoinsiemi di k elementi scelti in un insieme di n elementi

Esempio $A = \{ R, V, G \}$ $n = 3$

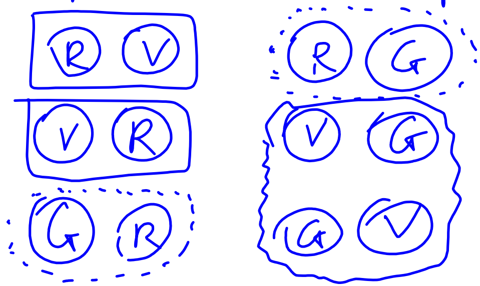
$$D_{3,2} = \frac{3!}{(3-2)!} = 6$$

$k = 2$

$$D_{n,k} = \frac{n!}{(n-k)!}$$

$$C_{n,k} = \frac{D_{n,k}}{k!} = \frac{6}{2!} = 3$$

disposizioni semplici:



combinazioni

RV RG GR

$$C_{n,k} = \frac{D_{n,k}}{k!} = \frac{n!}{(n-k)!\,k!} = \binom{n}{k}$$

coefficiente binomiale

$$(x+y)^n = \binom{n}{0} x^n y^0 + \binom{n}{1} x^{n-1} y^1 + \dots \binom{n}{n} x^0 y^n$$

$\binom{n}{0} = 1$, $\binom{n}{1} = n$, $\binom{n}{n} = 1$

$$(x+y)^2 = (x+y)\cdot(x+y) =$$

$$= (x+y)\,x + (x+y)\,y =$$

$$= x^2 + yx + xy + y^2 =$$

$$= x^2 + 2xy + y^2$$

1   2   1

$$\binom{n}{0} = \frac{n!}{n!\,0!} = 1 \qquad \binom{n}{n} = \frac{n!}{0!\,n!} = 1$$

$$\binom{n}{1} = \frac{n!}{(n-1)!\,1!} = \frac{n\,(n-1)(n-2)(n-3)\cdots 3\cdot 2\cdot 1}{(n-1)(n-2)(n-3)\cdots 3\cdot 2\cdot 1} =$$

$$= n$$

Operazioni

A = insieme di numeri $A = \mathbb{R}$

Somma $+ : A \times A \longrightarrow A$

↖ prodotto cartesiano di insiemi

proprietà

associativa : $a + (b+c) = (a+b)+c$

commutativa : $a + b = b + a$

esistenza dell'elem. neutro (0) $a + 0 = a$ $\forall a$

esistenza dell'opposto : $\forall a \ \exists b \ (= -a)$ tale che $a + b = 0$

Prodotto : $A \times A \longrightarrow A$

associativa : $a(bc) = (ab)c$

commutativa : $ab = ba$!!!!

esiste elem. neutro (1) : $a \cdot 1 = a \quad \forall a$

esistenza del reciproco : $a \neq 0 \Rightarrow$
"implica"

$\exists\, b \left(= \frac{1}{a}\right) \,/\, ab = 1$

Somma e prodotto :

distributiva $a \cdot (b + c) = ab + bc$

$21 = 3 \cdot (5+2) = 3 \cdot 5 + 3 \cdot 2 = 15 + 6$

Successioni

è una funzione $f: \mathbb{N} \to \mathbb{R}$

$f(n)$ $\quad a(n)$ $\quad a_n$

$n \in \mathbb{N}$

successione aritmetica

Esempio $a_n = 3n + 2$ $\quad n = 0, 1, 2, \dots$

$a_0 = 2 \quad a_1 = 5 \quad a_2 = 8 \quad a_3 = 11 \dots$

$a_n = 2^n$ $\quad a_0 = 1, \ a_1 = 2, \ a_2 = 4, \ a_3 = 8$

successione geometrica

$a_4 = 16 \dots$

Successione aritmetica:

$a_{n+1} - a_n$ non dipende da n

$\downarrow$

$$3(n+1)+2-(3n+2) = \cancel{3n}+3+\cancel{2}-\cancel{3n}-\cancel{2} = 3$$

Successione geometrica:

$\dfrac{a_{n+1}}{a_n}$ non dipende da n

$$\frac{2^{n+1}}{2^n} = \frac{\overbrace{2 \cdot 2 \cdot 2 \cdots\cdots 2}^{n+1}}{\underbrace{2 \cdot 2 \cdot 2 \cdots 2}_{n}} = 2$$

Successione di Fibonacci

$a_0 = 1 \qquad a_1 = 1 \qquad a_n = a_{n-2} + a_{n-1} \quad \forall n \geq 2$

1 1 2 3 5 8 13

$a_0 \; a_1 \; a_2 \; a_3 \; a_4 \; a_5 \; a_6$

ESERCIZI

Superenalotto $\quad A = \{1, 2, \dots, 90\}$

ne vengono estratti 6 numeri diversi

non importa l'ordine

1€ 1 sestina

$$C_{90,6} = \binom{90}{6} = \frac{90!}{6!\,84!} =$$

$$C_{n,k} = \frac{n!}{k!\,(n-k)!}$$

$$= \frac{90 \cdot 89 \cdot 88 \cdot 87 \cdot 86 \cdot 85 \cdot 84 \cdot \ldots}{6! \;\; 84 \cdot 83 \cdot 82 \cdot 81 \cdot \ldots}$$

$$= \frac{90 \cdot 89 \cdot 88 \cdot 87 \cdot 86 \cdot 85}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2} = 11 \cdot 89 \cdot 87 \cdot 86 \cdot 85$$

$$= 622\,614\,630$$

Ambo: $\sim \frac{1}{22}$

Probabilità di fare ambo con una sestina

$$A = \{1, \dots, 90\} = A_1 \cup A_2$$

$$A_1 = \{n_1, n_2, n_3, n_4, n_5, n_6\} = \text{numeri estratti}$$

$$A_2 = A \setminus A_1 = \text{numeri non estratti}$$

Vostra sestina $B = \{m_1, m_2, m_3, m_4, m_5, m_6\}$

$$= B_1 \cup B_2$$

$B_1 = \{a_1, a_2\}$ ambo $\qquad B_2 = B \setminus B_1$

$B_1 = \{a_1, a_2\} \subset A_1 \qquad C_{6,2} = \binom{6}{2}$

$B_2 \subset A_2 \qquad C_{84,4} = \binom{84}{4}$

Probabilità dell'ambo:

$$\frac{\binom{6}{2} \cdot \binom{84}{4}}{\binom{90}{6}} \sim \frac{1}{21.51}$$

2 giusti — 4 sbagliati

terno

$$\frac{\binom{6}{3}\binom{84}{3}}{\binom{90}{6}}$$

Roulette

$A = \{0, 1, 2, \dots 36\}$

1 € sul 29 Se esce 29

posta guadagnamo 36 €

37 €

Blackjack

Interesse semplice, interesse composto

deposito 1000 € in banca coll'interesse del 5% all'anno

Quanto guadagno in 10 anni nei 2 casi seguenti:

1) prelevo il guadagno ogni anno
2) non lo prelevo

1)

$5\% = \frac{\cancel{5}^{1}}{\cancel{100}_{20}}$

5% di 1000 €

vuol dire

$\frac{5}{\cancel{100}} \cdot 10\cancel{00} = 50$ €

g_n = guadagno realizzato dopo n anni

$g_1 = 50$ € $\quad g_2 = 100$ € $\quad g_3 = 150$ €

$g_n = \underline{\underline{50 \cdot n}}$ € $\qquad g_n - g_{n-1} = 50$ €

Successione aritmetica

$g_{10} = 50 \cdot 10$ € $= 500$ €

2) S_n = somma depositata in banca dopo n anni

$S_0 = 1000$ € $S_1 = 1050$ € $S_2 = ?$

Se in banca abbiamo X dopo un anno abbiamo

$$X + 5\% \cdot X = X\left(1 + \frac{5}{100}\right) =$$

$$= X\left(1 + \frac{1}{20}\right) = X\,\frac{20+1}{20} =$$

$$= X\,\frac{21}{20}$$

$$S_{n+1} = \frac{21}{20} S_n$$

$$\begin{array}{r} 21 \\ 5 \\ \hline 105 \end{array}$$

$S_0 = 1000\,€$

$S_1 = \frac{21}{\cancel{20}} \overset{50}{100\cancel{0}}\,€ = 1050\,€$

$S_2 = \frac{21}{20} 1050\,€ = \frac{21}{20}\frac{21}{20} 1000$

$S_3 = \frac{21}{20}\frac{21}{20}\frac{21}{20} 1000 = \left(\frac{21}{20}\right)^3 1000\,€$

$1 + 5\%$

$S_n = \left(\frac{21}{20}\right)^n 1000\,€$

successione geometrica

$S_{10} = \left(\frac{21}{20}\right)^{10} 1000 \sim 1628\,€$

guadagno 628 €

Successione aritmetica

$a_{n+1} - a_n$ non dipende da n

$a_{n+1} - a_n \equiv d$ $\qquad$ $\underline{a_{n+1} = a_n + d}$

$a_1 \quad a_2 \quad a_3 \quad a_4 \quad \dots \quad a_n$

$a_2 = a_1 + d \qquad a_3 = a_2 + d = a_1 + d + d =$

$= a_1 + 2d$

$a_4 = a_3 + d = a_1 + 2d + d = a_1 + 3d$

$$\boxed{a_n = a_1 + (n-1)d}$$

lineare in n

$1 \; 2 \; 3 \dots \; n$

Successione geometrica

$$\frac{a_{n+1}}{a_n} = q \quad \text{non dipende da } n$$

$$a_{n+1} = q\, a_n \qquad a_0 \quad a_1 \quad a_2 \quad a_3 \quad \dots$$

$$a_2 = q\, a_1 \qquad a_3 = q\, a_2 = q \cdot q \cdot a_1 = q^2 a_1$$

$$a_4 = q\, a_3 = q \cdot q^2 a_1 = q^3 a_1$$

$$a_n = q^{n-1} a_1 = \frac{q^n}{q} a_1 = q^n \frac{a_1}{q}$$

$$\frac{a_1}{q} = a_0 \rightarrow 1000 \text{ €}$$

$$q = \frac{21}{20}$$

$$a_n = q^n a_0$$

$$a_0 + a_1 + a_2 + \cdots + a_n \equiv S_n$$

$$\underbrace{a_0 + a_1 + a_2 + \cdots + a_n}_{n+1}$$

$$S_0 = a_0$$

$$S_1 = a_0 + a_1 = a_0 + q a_0 = (1+q) a_0$$

$$S_2 = \underbrace{a_0 + a_1}_{S_1} + a_2 = S_1 + a_2 = (1+q)a_0 + q^2 a_0 =$$

$$= (1+q+q^2) a_0$$

$$S_3 = \underbrace{a_0 + a_1 + a_2}_{S_2} + a_3 = S_2 + a_3 = (1+q+q^2) a_0 +$$

$$+ q^3 a_0 = (1+q+q^2+q^3) a_0$$

$$S_n = \underbrace{(1 + q + q^2 + \dots + q^n)}a_0$$

$$t_n \equiv \frac{S_n}{a_0} = 1 + q + q^2 + \dots + q^n$$

$$q\,t_n = q(1 + q + q^2 + \dots + q^{n-1} + q^n) =$$

$$= \underbrace{q + q^2 + q^3 + \dots + q^n}_{t_n - 1} + q^{n+1} =$$

$$= t_n - 1 + q^{n+1}$$

$$q\,t_n = t_n - 1 + q^{n+1}$$ equazione con incognita t_n

Sommo $-q\,t_n$ ad entrambi i membri

$$0 = t_n - 1 + q^{n+1} - q t_n =$$

$$= t_n(1-q) - 1 + q^{n+1}$$

Aggiungo $1 - q^{n+1}$ ad entrambi i membri

$$1 - q^{n+1} = t_n(1-q)$$

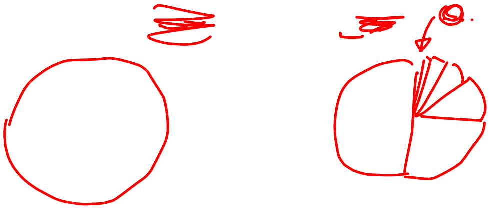
divido per $1-q$ (se $q \neq 1$) entrambi i membri

$$t_n = \frac{1-q^{n+1}}{1-q} \quad (q \neq 1)$$

$$1 + q + q^2 + q^3 + \dots + q^{n-1} + q^n = \frac{1-q^{n+1}}{1-q}$$

$$q = \frac{1}{2}$$

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \dots = 2$$



1

$= 2$ pizze

$$S_n = \frac{1 - \frac{1}{2^{n+1}}}{1 - \frac{1}{2}} = \frac{1 - \frac{1}{2^{n+1}}}{\frac{1}{2}} \cdot \frac{2}{2} = 2 - \frac{1}{2^n}$$

$n = \infty$

$$\lim_{n \to \infty} S_n = 2 \qquad \lim_{n \to \infty} \frac{1}{2^n} = 0$$

$$0,99999\ldots = 0,\bar{9} = 1$$

$$\| $$

$$0,9 + 0,09 + 0,009 + 0,0009 + \ldots =$$

$$= \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \frac{9}{10000} + \ldots =$$

$$= \frac{9}{10}\left(1 + \frac{1}{10} + \frac{1}{100} + \ldots\right) =$$

$$= \frac{9}{10}\left(1 + \frac{1}{10} + \left(\frac{1}{10}\right)^2 + \left(\frac{1}{10}\right)^3 + \ldots\right) =$$

$$q = \frac{1}{10} \qquad S_n = \frac{1 - q^{n+1}}{1 - q} = \frac{1 - \frac{1}{10^{n+1}}}{1 - \frac{1}{10}}$$

$$S_n = \frac{1-\frac{1}{10^{n+1}}}{1-\frac{1}{10}} = \frac{1-\frac{1}{10^{n+1}}}{\frac{9}{10}} \cdot \frac{\frac{10}{9}}{\frac{10}{9}} =$$

$$= \frac{10}{9}\left(1-\frac{1}{10^{n+1}}\right) = \frac{10}{9} - \frac{1}{9 \cdot 10^n}$$

$$1 + \frac{1}{10} + \left(\frac{1}{10}\right)^2 + \left(\frac{1}{10}\right)^3 + \cdots = \frac{10}{9}$$

$$0.\overline{9} = \frac{9}{10}\left(1 + \frac{1}{10} + \left(\frac{1}{10}\right)^2 + \cdots\right) = \frac{9}{10} \cdot \frac{10}{9} = 1$$

Il tempo di dimezzamento del ^{14}C è di 5730 anni circa

(a) età di un fossile in cui la concentrazione di ^{14}C è $\sim$ 12% di quella dell'analogo organismo vivente

$$12\% = \frac{12}{100} = \left(\frac{1}{2}\right)^n \qquad n = ?$$

$$\frac{1}{2} = 0{,}5 \qquad \frac{1}{4} = 0{,}25 \qquad \frac{1}{8} = 0{,}125 \sim 12{,}5\%$$

$$12\% \sim \frac{1}{8} \sim \left(\frac{1}{2}\right)^3 \qquad 5730 \cdot 3 = 17190 \text{ anni}$$

(b) determinare la concentrazione di ^{14}C in un fossile di circa 23000 anni

$\frac{1}{2}$ ogni 5730

Si è dimezzata $\frac{23000}{5730}$ volte $\sim$ 4 volte

$$\frac{23000}{5730} \sim 4$$

$$\begin{array}{r} 5730 \\ 4 \\ \hline 22920 \end{array}$$

$$\frac{12,5}{2}$$

$$\frac{1}{2}\,\frac{1}{2}\,\frac{1}{2}\,\frac{1}{2} = \left(\frac{1}{2}\right)^4 = \frac{1}{16} = \underline{\underline{6,25\ \%}}$$

In una coltura batterica ci sono inizialmente N batteri che raddoppiano ogni 160'

(a) quanti batteri ci sono dopo 22 h ?

$22\,h = 22 \cdot 60$ min. $= 1320$ minuti

Sono raddoppiati $\frac{1320}{160}$ volte ~ 8 volte

$160 \cdot 8 = 1280$

$$N \cdot \underbrace{2 \cdot 2 \cdot 2 \cdot \ldots 2}_{8 \text{ volte}} = 2^8 N = 256\,N$$

1ª prova in itinere (Matematica): 2 NOV 9.30 - 11.30
Aula Magna

oggi ricevimento a fisica tra le 14,00 e le 15,30
(invece che 15 - 17)

MATRICI

Sono tabelle di numeri reali

$$\begin{pmatrix} 1 & 3 & -2 \\ 0 & \pi & 5 \end{pmatrix}$$

righe (rows of the matrix), colonne (columns of the matrix)

$\pi = 3{,}14\ldots$ numero irrazionale

$M_{m,n}$ — M = "matrice"; m = # righe; n = # colonne

$A_{2,3}$: matrice 2×3

m, n si dicono dimensioni della matrice

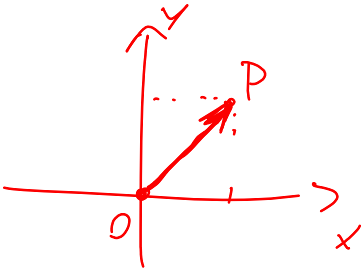
$$A_{m,n} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix}$$

elementi di matrice a_{ij}

$$A_{m,n} = (a_{ij}) \qquad 1 \le i \le m \qquad 1 \le j \le n$$

Matrici con una riga sola: si dicono <u>vettori</u>

$$V_{1,n} = (v_1, v_2, v_3, \dots v_n)$$



$$P = (x_P, y_P)$$

Matrici quadrate A_{mm} stesso numero di righe e colonne

$$A_{22} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \qquad A_{33} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Somma di Matrici delle stesse dimensioni:

Si somma termine a termine

$$A_{mn} = (a_{ij}) \qquad B_{mn} = (b_{ij})$$

$$A_{mn} + B_{mn} \equiv C_{mn} = (c_{ij}) \qquad c_{ij} = a_{ij} + b_{ij}$$

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix}$$

$$A + B = \begin{pmatrix} 1 & 3 \\ 3 & 3 \end{pmatrix}$$

Proprietà:

Associativa $A+(B+C)=(A+B)+C$

Commutativa $A+B=B+A$

$\exists$ l'elemento neutro $\mathbf{0}=\begin{pmatrix} 0 & 0 & 0 & \\ 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & \\ \dots & & & \end{pmatrix}$

$$A+\mathbf{0}=A$$

$\exists$ l'opposto $-A=(-a_{ij})$

$$A+(-A)=\mathbf{0}$$

Moltiplicazione di una matrice per un numero $\lambda \in \mathbb{R}$

$$\lambda A = \begin{pmatrix} \lambda a_{11} & \lambda a_{12} & \lambda a_{13} & \dots & \lambda a_{1n} \\ \vdots & & & & \\ \lambda a_{m1} & \lambda a_{m2} & \dots & & \lambda a_{mn} \end{pmatrix}$$

$$A = (a_{ij}) \qquad \lambda A = (\lambda a_{ij})$$

Prodotto di matrice

$$A = (a_1, a_2 \dots a_n) \qquad B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

$$A \times B = a_1 b_1 + a_2 b_2 + a_3 b_3 \dots + a_n b_n$$

$$= \text{numero (matrice } 1 \times 1)$$

$$A = (1, 2) \qquad B = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$$

$$A \times B = 1 \cdot 4 + 2 \cdot 6 = 4 + 12 = 16$$

Posso moltiplicare le matrici

$A_{m,n} \times B_{p,q}$ se $n = p$

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & \dots & & a_{2n} \\ a_{31} & \dots & & a_{3n} \\ \vdots & & & \\ a_{m1} & \dots & & a_{mn} \end{pmatrix} \times \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1q} \\ \vdots & & & \\ b_{p1} & b_{p2} & \dots & b_{pq} \end{pmatrix} =$$

$$= C = \begin{pmatrix} & & \\ & c_{ij} & \\ & & \end{pmatrix}$$

c_{ij} = prodotto della i esima riga di A per la j esima colonna di B

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix}$$

$$A \times B = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \times \begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1\cdot 0 + 1\cdot 1 & 1\cdot 2 + 1\cdot 2 \\ 2\cdot 0 + 1\cdot 1 & 2\cdot 2 + 1\cdot 2 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 1 & 6 \end{pmatrix}$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

$$A \cdot B = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix}$$

$$B \cdot A = \begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 4 & 2 \\ 5 & 3 \end{pmatrix} \neq A \cdot B$$

↖ "diverso da"

Il prodotto di matrici non soddisfa la proprietà <u>commutativa</u>

Proprietà associativa : $(AB)C = A(BC)$ OK

" distributiva ; $A\cdot(B+C) = A\cdot B + A\cdot C$

$(B+C)\cdot A = B\cdot A + C\cdot A$

$\exists$ elemento neutro $\mathbb{1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ matrice quadrata

diagonale

$A\cdot\mathbb{1} = \mathbb{1}\cdot A \quad \forall A$

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$

$B = \mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$A\cdot B = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$

$e = h = 1 \quad g = f = 0$

$A\cdot B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = A$

Esercizio. $A = \mathbb{1}$
B qualsiasi
$A\cdot B = B$

$a \neq 0 \quad a \in \mathbb{R} \quad \exists \; b \in \mathbb{R} \quad \left(b = \frac{1}{a} \right)$

tale che $ab = 1$

e per le matrici? $\exists$ l'inversa?
Quando

Data $A \quad \exists \; B \;/\; AB = \mathbb{1} = BA$?

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$$

$$A \cdot B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$$

MATRICI QUADRATE

L'inversa esiste ed è unica se la matrice ha <u>determinante</u> diverso da zero

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \underline{\det A = ad - bc} \neq 0$$

$$A^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \frac{1}{ad-bc} = \begin{pmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix}$$

$$A \cdot A^{-1} = A^{-1} \cdot A = \mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A \cdot A^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \frac{1}{ad-bc} =$$

$$= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \frac{1}{ad-bc} =$$

$$= \begin{pmatrix} ad-bc & 0 \\ 0 & -bc+da \end{pmatrix} \frac{1}{ad-bc} =$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Riduzione di una matrice
(metodo di Gauss)

Data una matrice A (anche non quadrata) facciamo le seguenti operazioni:

i) moltiplichiamo tutti gli elementi di una riga per una stessa costante $\lambda \neq 0$

ii) scambiamo due righe

iii) sommiamo (termine a termine) a una riga un'altra riga moltiplicata per $\lambda \neq 0$

Riduzione: aumentare il numero di elementi uguali a 0, fino a ridurre la matrice a una matrice a scalini (cioè una matrice con tutti zeri sotto la diagonale)

$$\begin{pmatrix} a & e & f & g \\ 0 & b & h & i \\ 0 & 0 & c & l \\ 0 & 0 & 0 & d \end{pmatrix}$$

- Il determinante di una matrice a scalini è il prodotto dei suoi elementi diagonali

- L'operazione iii) non cambia il determinante

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Se $c = 0$ $\quad \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$

$\det = ad$

OK

Supponiamo $c \neq 0$, $a \neq 0$

Uso iii) per ottenere una matrice della forma $\begin{pmatrix} a' & b' \\ 0 & d' \end{pmatrix}$

2^a riga $\longrightarrow$ 2^a riga $+ \lambda\, 1^a$ riga

$$\begin{pmatrix} a & b \\ c+\lambda a & d+\lambda b \end{pmatrix}$$

$c + \lambda a = 0$

$\lambda = -\dfrac{c}{a}$

$$\begin{pmatrix} a & b \\ 0 & d - \frac{cb}{a} \end{pmatrix} \qquad \underline{\det = ab - bc}$$

ok!

$\underline{a = 0}$

$$\begin{pmatrix} 0 & b \\ c & d \end{pmatrix}$$

L'operazione ii) cambia il segno al determinante

uso ii)

$$\begin{pmatrix} c & d \\ 0 & b \end{pmatrix} \qquad \det = cb$$

$$\det \begin{pmatrix} 0 & b \\ c & d \end{pmatrix} = -cb$$

L'operazione i) moltiplica il determinante per λ

Per calcolare l'inversa: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Affianco alla matrice $A_{n,n}$ l'identità $n \times n$ e costruisco una matrice $n \times 2n$

$$\begin{pmatrix} a & b & 1 & 0 \\ c & d & 0 & 1 \end{pmatrix}$$

Poi uso i) ii) iii) per arrivare a

$$\begin{pmatrix} 1 & 0 & a' & b' \\ 0 & 1 & c' & d' \end{pmatrix}$$

$$\begin{pmatrix} a & b & 1 & 0 \\ c & d & 0 & 1 \end{pmatrix}$$

$a \neq 0$ o $c \neq 0$ (altrimenti l'inversa non esiste)

Eventualmente scambiando le righe, possiamo supporre $a \neq 0$

Moltiplico la 1^a riga per $\frac{1}{a}$: (i)

$$\begin{pmatrix} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ c & d & 0 & 1 \end{pmatrix}$$

Sottraggo alla 2^a, la $1^a \times c$ (iii)

$$\begin{pmatrix} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ c & d & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & d-\frac{bc}{a} & -\frac{c}{a} & 1 \end{pmatrix}$$

Moltiplico la 2^a per $\frac{1}{d-\frac{bc}{a}}$

$$\begin{pmatrix} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & 1 & \frac{-\frac{c}{a}}{d-\frac{bc}{a}} & \frac{1}{d-\frac{bc}{a}} \end{pmatrix}$$

$$\begin{pmatrix} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & 1 & \dfrac{-\frac{c}{a}}{d-\frac{bc}{a}} & \dfrac{1}{d-\frac{bc}{a}} \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & 1 & \dfrac{-c}{ad-bc} & \dfrac{a}{ad-bc} \end{pmatrix}$$

$$1^A \longrightarrow 1^A - \frac{b}{a} \times 2^A$$

$$\begin{pmatrix} 1 & 0 & \frac{1}{a} + \dfrac{bc}{a(ad-bc)} & \dfrac{-b}{ad-bc} \\ 0 & 1 & \dfrac{-c}{ad-bc} & \dfrac{a}{ad-bc} \end{pmatrix}$$

$$\frac{1}{a} + \frac{bc}{a(ad-bc)} = \frac{ad - bc + bc}{a(ad-bc)} =$$

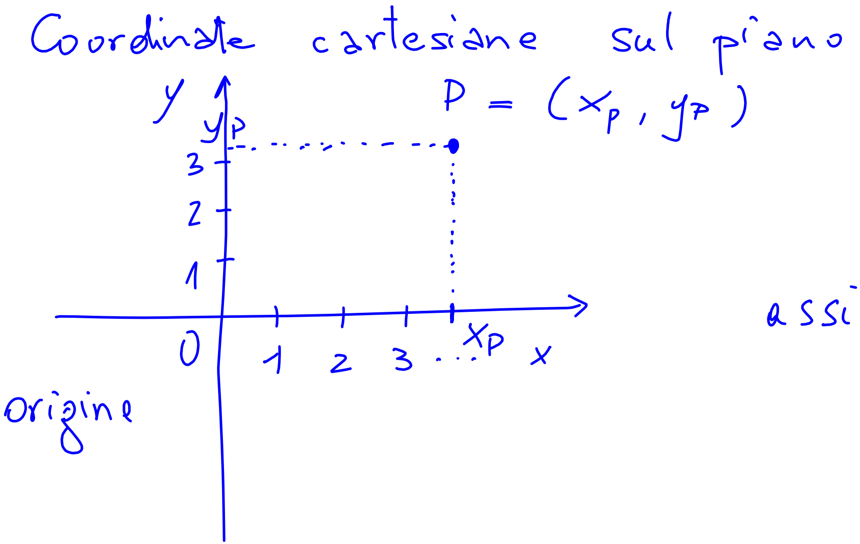
$$= \frac{ad}{a(ad-bc)} = \frac{d}{ad-bc}$$

$$\left(\begin{array}{cc|cc} 1 & 0 & \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array} \right)$$

inversa

GEOMETRIA ANALITICA

Coordinate cartesiane sul piano



Retta: i punti (x,y) che soddisfano un'equazione lineare (cioè non possono apparire potenze di x e y più alte di 1, né prodotti xy):

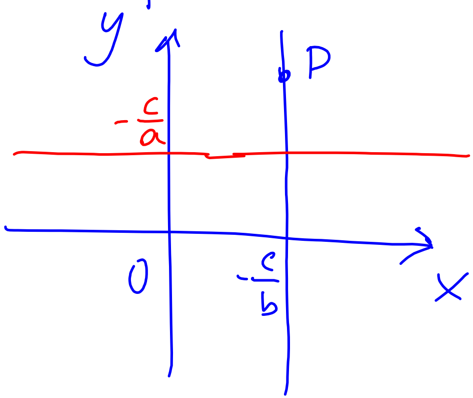
$$ay + bx + c = 0$$

$ay + bx + c = 0 \qquad a, b, c \in \mathbb{R}$ dati

Le soluzioni (x, y) stanno su una retta

Esempi: 1) $a = 0 \quad b \neq 0 \qquad bx + c = 0$

$$x = -\frac{c}{b}$$



2) $b = 0 \quad a \neq 0$

$ay + c = 0 \qquad y = -\dfrac{c}{a}$

[$a = b = 0 \Rightarrow c = 0$ non ha senso]

3) $a \neq 0 \quad b \neq 0$

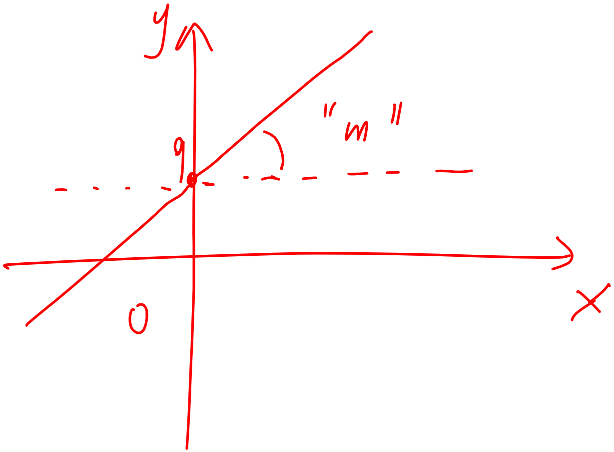
$ay + bx + c = 0$

$ay = -bx - c$

$$y = -\frac{b}{a}x - \frac{c}{a} \equiv mx + q$$

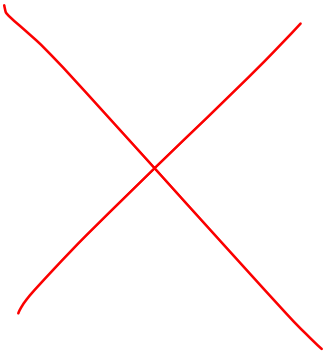
$$m = -\frac{b}{a}$$ si chiama coefficiente angolare

$$q = -\frac{c}{a}$$



$$x = 0 \Rightarrow y = q$$

Due rette



$$\begin{cases} ax + by + c = 0 \\ a'x + b'y + c' = 0 \end{cases}$$

l'intersezione è la soluzione del sistema fatto da quelle due equazioni

$$\begin{cases} ax + by = -c \\ a'x + b'y = -c' \end{cases}$$

$$AX = C$$

$$A = \begin{pmatrix} a & b \\ a' & b' \end{pmatrix} \quad X = \begin{pmatrix} x \\ y \end{pmatrix} \quad C = \begin{pmatrix} -c \\ -c' \end{pmatrix}$$

$$AX = \begin{pmatrix} a & b \\ a' & b' \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ a'x + b'y \end{pmatrix}$$

$$AX = C = \begin{pmatrix} -c \\ -c' \end{pmatrix}$$

$$ax + by = -c$$
$$a'x + b'y = -c'$$

Supponiamo che $\exists\ A^{-1}$ $\quad (A^{-1}A = AA^{-1} = \mathbb{1})$

$AX = C$ $\quad$ moltiplico per A^{-1} a des e a sin

$$\underbrace{A^{-1}A}_{\mathbb{1}} X = A^{-1}C = X$$

A^{-1} esiste se e solo se $\det A \neq 0$

$$A = \begin{pmatrix} a & b \\ a' & b' \end{pmatrix} \qquad \det A = ab' - ba'$$

Le due rette sono parallele se l'intersezione non esiste, cioè se $\det A = 0$

$$ab' = ba'$$

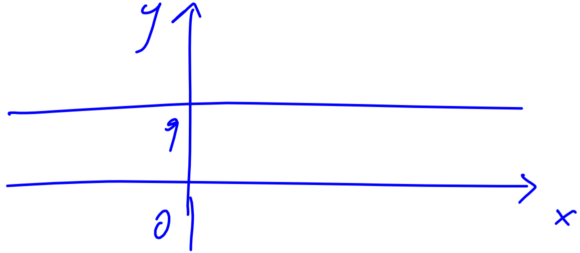
Se $a \neq 0$ $a' \neq 0$

$$\frac{b}{a} = \frac{b'}{a'} \qquad -\frac{b}{a} = -\frac{b'}{a'}$$

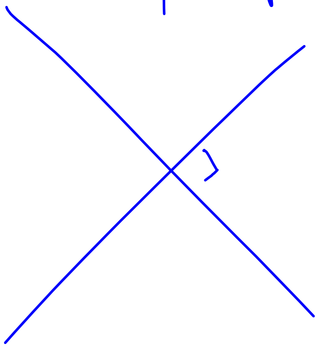
Due rette parallele hanno lo stesso coefficiente angolare $\underline{\underline{m = m'}}$

$$y = mx + q$$

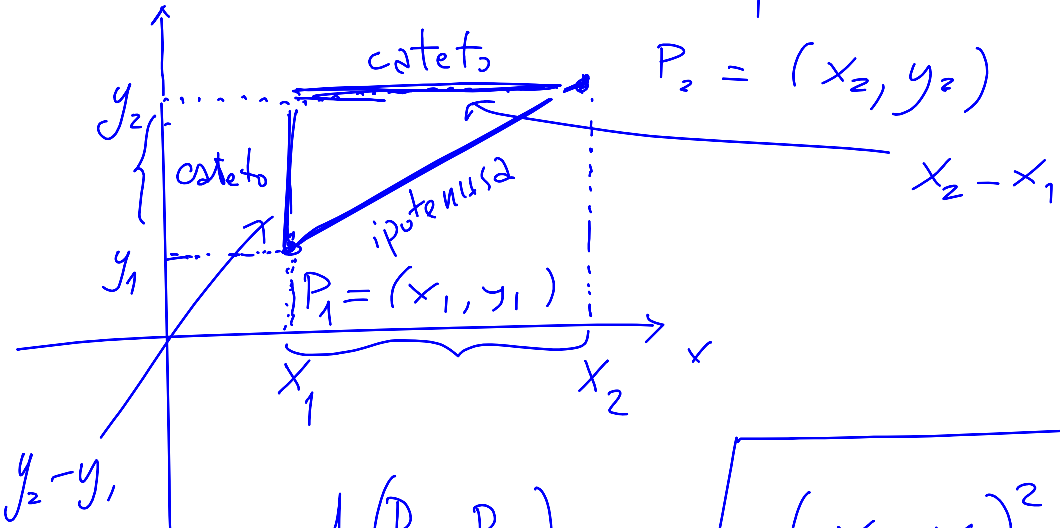
$m = 0 \quad y = q$



Rette perpendicolari: $m = -\frac{1}{m'}$



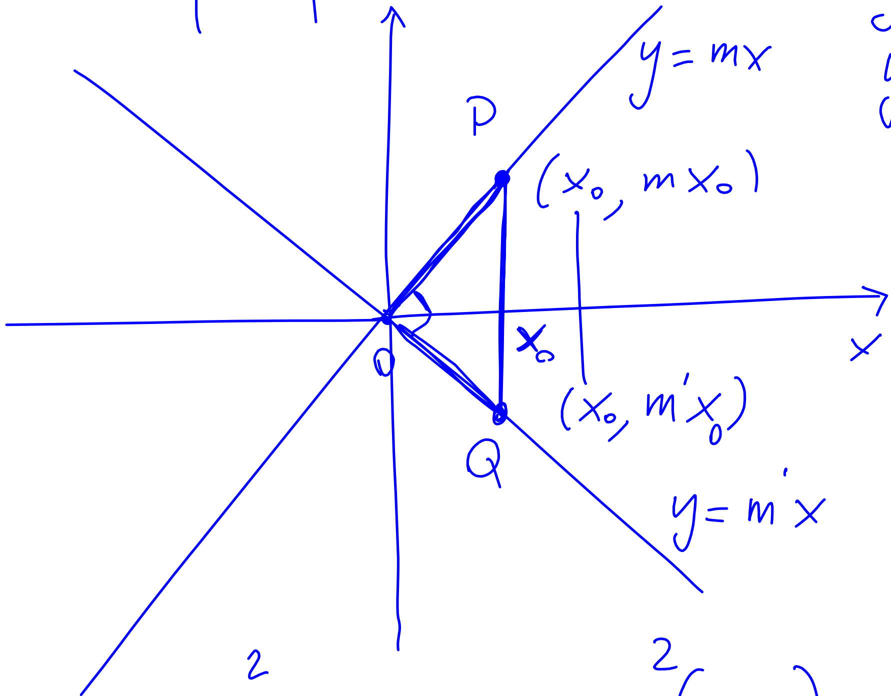
Distanza tra due punti



$$d(P_2, P_1) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Rette perpendicolari

$y = mx$
$y = m'x$



Teorema di Pitagora

$$d^2(P,Q) = d^2(P,O) + d^2(Q,O)$$

$$\cancel{(x_0 - x_0)^2} + (mx_0 - m'x_0)^2 = (x_0 - \cancel{0})^2 + (mx_0 - \cancel{0})^2 +$$
$$+ (x_0 - \cancel{0})^2 + (m'x_0 - \cancel{0})^2$$

$$(x_0 - x_0)^2 + (m x_0 - m' x_0)^2 = (x_0 - 0)^2 + (m x_0 - 0)^2 +$$
$$+ (x_0 - 0)^2 + (m' x_0 - 0)^2$$

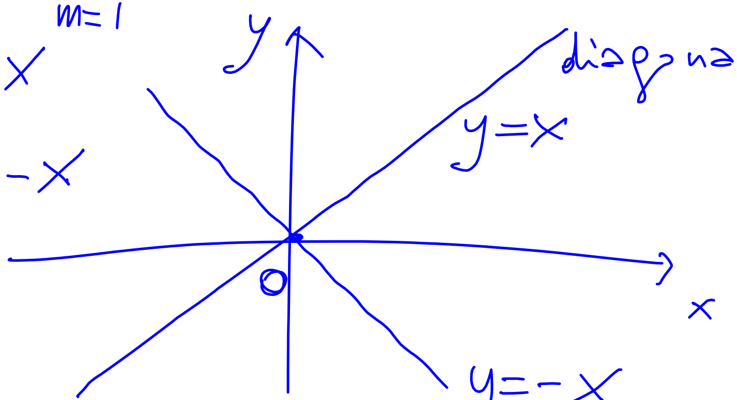
$$m^2 x_0^2 + m'^2 x_0^2 - 2 m m' x_0^2 = 2 x_0^2 + m^2 x_0^2 + m'^2 x_0^2$$

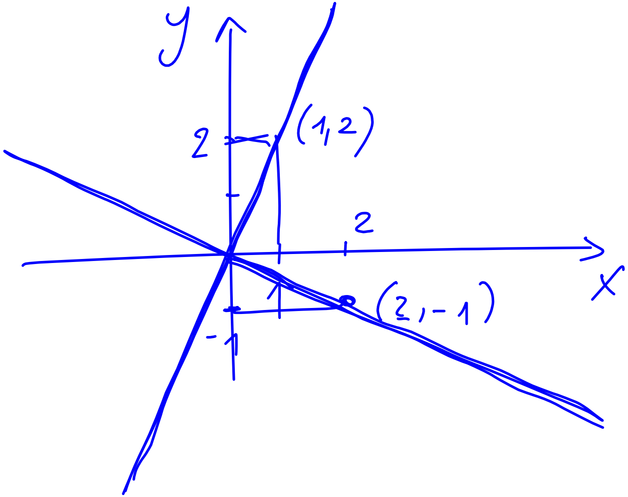
$$m^2 + m'^2 - 2 m m' = 2 + m^2 + m'^2$$

$$-2 m m' = 2 \qquad -m m' = 1 \qquad m = -\frac{1}{m'}$$

Esempio

$$\begin{cases} y = x & m = 1 \\ y = -x \end{cases}$$





$y = 2x \quad (1,2)$

$y = -\frac{1}{2}x \quad (2,-1)$

Equazioni lineari $\quad ay + bx + c = 0$

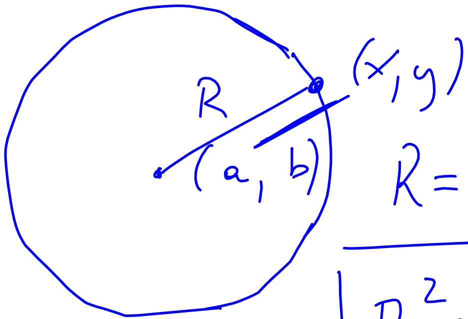
Equazioni quadratiche (potenza max 2 in x e y)

$ax^2 + by^2 + cxy + dx + ey + f = 0$

Le soluzioni sono delle curve notevoli:

circonferenza, ellisse, parabola, iperbole

Circonferenza: insieme di punti che hanno la stessa distanza R da un punto (a, b) dato



$$R = \text{distanza} = \sqrt{(x-a)^2+(y-b)^2}$$

$$R^2 = (x-a)^2 + (y-b)^2$$

R = raggio (a, b) = centro

Sistemi di equazioni lineari

incognite $x_1 \ldots x_n$

$$\underbrace{a_1 x_1 + a_2 x_2 + \cdots + a_n x_n}_{\text{combinazione lineare}} = b$$

una equazione

Sistema di m equazioni

$$\begin{cases} a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n = b_1 \\ a_{21} x_1 + a_{22} x_1 + \cdots + a_{2n} x_n = b_2 \\ \vdots \\ a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n = b_n \end{cases}$$

$$A_{mn} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \qquad X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$B = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

$$\boxed{AX = B}$$

Applichiamo i) ii) e iii) ad A e B (X non la tocchiamo)

Regole di riduzione di una matrice

i) moltiplichiamo tutti gli elementi di una riga per una stessa costante $\lambda \neq 0$

ii) scambiamo due righe

iii) sommiamo (termine a termine) a una riga un'altra riga moltiplicata per $\lambda \neq 0$

iii)

$$\begin{cases} \text{riga } 1 = b_1 \\ \text{riga } 2 = b_2 \end{cases}$$

$$\begin{cases} \text{riga } 1 = b_1 \\ \text{riga } 2 + \lambda(\text{riga } 1) = b_2 + \lambda b_1 \end{cases}$$

$$\lambda \neq 0$$

$$\begin{cases} 2x - 3y + z = -1 \\ 2x - 3y - z = -2 \\ 2x + y + 3z = -2 \end{cases}$$

$$A = \begin{pmatrix} 2 & -3 & 1 \\ 2 & -3 & -1 \\ 2 & 1 & 3 \end{pmatrix}$$

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad B = \begin{pmatrix} -1 \\ -2 \\ -2 \end{pmatrix}$$

$$AX = B$$

Cerchiamo A^{-1}

$$\underbrace{A^{-1}A}_{= \mathbb{1}} X = A^{-1}B$$

$$\boxed{X = A^{-1}B}$$

$$\left(\begin{array}{ccc|ccc} 2 & -3 & 1 & 1 & 0 & 0 \\ 2 & -3 & -1 & 0 & 1 & 0 \\ 2 & 1 & 3 & 0 & 0 & 1 \end{array}\right)$$

$$2^{a} \to 2^{a} - 1^{a}$$

$$3^{a} \to 3^{a} - 1^{a}$$

$$\begin{pmatrix} 2 & -3 & 1 & 1 & 0 & 0 \\ 0 & 0 & -2 & -1 & 1 & 0 \\ 0 & 4 & 2 & -1 & 0 & 1 \end{pmatrix} \leftrightarrow \begin{pmatrix} 2 & -3 & 1 & 1 & 0 & 0 \\ 0 & 4 & 2 & -1 & 0 & 1 \\ 0 & 0 & -2 & -1 & 1 & 0 \end{pmatrix}$$

$2' \rightarrow 2' + 3'$

$$\begin{pmatrix} 2 & -3 & 1 & 1 & 0 & 0 \\ 0 & 4 & 0 & -2 & 1 & 1 \\ 0 & 0 & -2 & -1 & 1 & 0 \end{pmatrix}$$

$1' \rightarrow 1' \frac{1}{2} \qquad 2' \rightarrow 2' \frac{1}{4} \qquad 3' \rightarrow 3' \frac{-1}{2}$

$$\begin{pmatrix} 1 & -\frac{3}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix}$$

$1' \rightarrow 1' + 2' \frac{3}{2}$

$$\begin{pmatrix} 1 & 0 & \frac{1}{2} & & & \\ & & & & & \\ & & & & & \end{pmatrix}$$

$$\left(\begin{array}{cccc|cc} 1 & -\frac{3}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & 0 \end{array}\right)$$

$$1^A \to 1^A + 2^A \frac{3}{2}$$

$$\begin{pmatrix} 1 & 0 & \frac{1}{2} & -\frac{1}{4} & \frac{3}{8} & \frac{3}{8} \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix}$$

$$1^A - \frac{1}{2} 3^A$$

$$\begin{pmatrix} 1 & 0 & 0 & -\frac{1}{2} & \frac{5}{8} & \frac{3}{8} \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix}$$

$$\frac{3}{8} - \frac{1}{2}\left(-\frac{1}{2}\right) = \frac{3}{8} + \frac{1}{4} = \frac{3}{8} + \frac{2}{8} = \frac{5}{8}$$

$$A^{-1} = \begin{pmatrix} -\frac{1}{2} & \frac{5}{8} & \frac{3}{8} \\ -\frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix}$$

Soluzione del sistema

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1} B = \begin{pmatrix} -\frac{1}{2} & \frac{5}{8} & \frac{3}{8} \\ -\frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} -1 \\ -2 \\ -2 \end{pmatrix} =$$

$$= \begin{pmatrix} -\frac{1}{2}(-1) + \frac{5}{8}(-2) + \frac{3}{8}(-2) \\ -\frac{1}{2}(-1) + \frac{1}{4}(-2) + \frac{1}{4}(-2) \\ \frac{1}{2}(-1) + \left(-\frac{1}{2}\right)(-2) + 0(-2) \end{pmatrix} = \begin{pmatrix} -\frac{3}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

Verifica

$$\begin{cases} 2x - 3y + z = -1 \\ 2x - 3y - z = -2 \\ 2x + y + 3z = -2 \end{cases}$$

$x = -\frac{3}{2}$

$y = -\frac{1}{2}$

$z = \frac{1}{2}$

$$2x - 3y + z = -3 + \frac{3}{2} + \frac{1}{2} = -3 + 2 = -1$$ ok

$$-3 + \frac{3}{2} - \frac{1}{2} = -3 + 1 = -2$$ ok

$$-3 - \frac{1}{2} + \frac{3}{2} = -3 + 1 = -2$$ ok

Eseraizio

$$x_1 + 0x_2 + 2x_3 + x_4 = 0$$
$$x_1 - x_2 + 3x_3 + 3x_4 = 1$$
$$x_1 + x_2 + 3x_3 = -1$$
$$2x_1 - x_2 + 7x_3 + 8x_4 = 2$$

$$AX = B$$

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 1 & -1 & 3 & 3 \\ 1 & 1 & 3 & 0 \\ 2 & -1 & 7 & 8 \end{pmatrix} \qquad B = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 2 \end{pmatrix}$$

$$2^A \to 2^A - 1^A \qquad 3^A \to 3^A - 1^A \qquad 4^A \to 4^A - 2 \cdot 1^A$$

$$A = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 1 & -1 & 3 & 3 \\ 1 & 1 & 3 & 0 \\ 2 & -1 & 7 & 8 \end{pmatrix} \qquad B = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 2 \end{pmatrix}$$

$$2^A \to 2^A - 1^A \qquad 3^A \to 3^A - 1^A \qquad 4^A \to 4^A - 2 \cdot 1^A$$

$$A \to \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & -1 & 1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & -1 & 3 & 6 \end{pmatrix} \qquad B = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 2 \end{pmatrix}$$

$$3^A \to 3^A + 2^A \qquad 4^A \to 4^A - 2^A$$

$$A \to \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & -1 & 1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & -1 & 3 & 6 \end{pmatrix} \qquad B = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 2 \end{pmatrix}$$

$$3^A \to 3^A + 2^A \qquad 4^A \to 4^A - 2^A$$

$$A \to \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & -1 & 1 & 2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 2 & 4 \end{pmatrix} \qquad B \to \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$4^A \to 4^A - 3^A$$

$$A \rightarrow \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & -1 & 1 & 2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 2 & 4 \end{pmatrix} \qquad B \rightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$4^{A} \rightarrow 4^{A} - 3^{A}$$

$$A \rightarrow \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & -1 & 1 & 2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix} = A' \qquad B \rightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} = B'$$

Abbiamo fatto <u>solo</u> operazioni iii)
che non cambiano il determinante

$$\det A = \det A' = 1 \cdot (-1) \cdot 2 \cdot 3 = -6$$

$$A'X = B'$$

$$\begin{cases} x_1 + 2x_3 + x_4 = 0 \\ -x_2 + x_3 + 2x_4 = 1 \\ 2x_3 + x_4 = 0 \\ 3x_4 = 1 \end{cases}$$

$$x_4 = \frac{1}{3}$$

$$x_3 = -\frac{1}{2}x_4 = -\frac{1}{6}$$

$$-x_2 + \frac{-1}{6} + \frac{2}{3} = 1$$

$$-x_2 - \frac{1}{6} = \frac{1}{3}$$

$$-x_2 = \frac{1}{3} + \frac{1}{6} = \frac{2}{6} + \frac{1}{6} = \frac{3}{6} = \frac{1}{2}$$

$$x_2 = -\frac{1}{2} \qquad x_1 = 0$$

$$x = \begin{pmatrix} 0 \\ -\frac{1}{2} \\ -\frac{1}{6} \\ \frac{1}{3} \end{pmatrix}$$

Verificare se $x^2+y^2-2x-4=0$ è una circonferenza. Se sì trovare il centro e il raggio

$$R^2=(x-a)^2+(y-b)^2$$

R = raggio (a,b) = centro

$$x^2-2x=\underbrace{x^2-2x+1}-1=(x-1)^2-1$$

$$(x-1)^2-1+y^2-4=0$$

$$(x-1)^2+y^2=5$$

$R=\sqrt{5}$ $(a,b)=(1,0)$

Idem per $x^2+y^2+6x+3y+1=0$

$$x^2+6x=(x+3)^2-9$$

$$y^2+3y=\left(y+\frac{3}{2}\right)^2-\frac{9}{4}$$

$$\frac{3^2}{2^2}$$

$$(x+3)^2-9+\left(y+\frac{3}{2}\right)^2-\frac{9}{4}+1=0$$

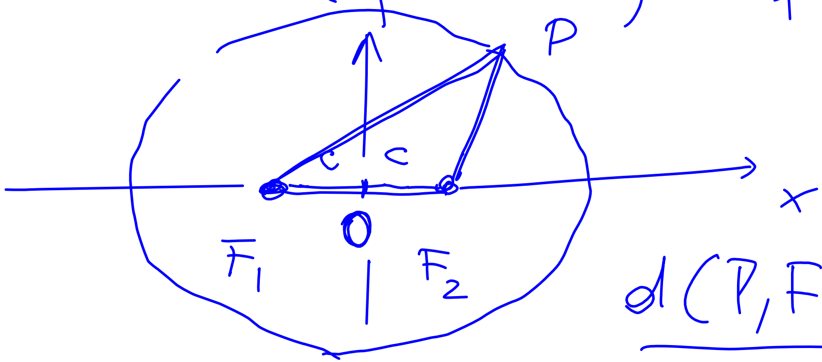
$$(x+3)^2+\left(y+\frac{3}{2}\right)^2=9-1+\frac{9}{4}=8+\frac{9}{4}=$$

$$=\frac{41}{4} \qquad R=\frac{\sqrt{41}}{2} \qquad (a,b)=\left(-3,-\frac{3}{2}\right)$$

$$x^2 + y^2 = -1 \quad \text{impossibile}$$

$$(x-a)^2 + (y-b)^2 = -1$$

<u>Ellisse</u> insieme dei punti $(x, y) = P$ tali che è costante $(\equiv 2a)$ la somma delle distanze tra P e due punti dati (fuochi) F_1 e F_2



$$d(F_1, F_2) = 2c$$

$$\underline{d(P, F_1) + d(P, F_2) = 2a}$$

$P : (x, y) \quad F_1 : (-c, 0) \quad F_2 : (c, 0)$

$$d(P, F_2) = \sqrt{(x-c)^2 + y^2}$$

$+$

$$d(P, F_1) = \sqrt{(x+c)^2 + y^2}$$

$\|$

$$2a = \sqrt{(x-c)^2 + y^2} + \sqrt{(x+c)^2 + y^2}$$

$$2a - \sqrt{(x-c)^2 + y^2} = \sqrt{(x+c)^2 + y^2}$$

Elevo al quadrato

$$4a^2 + (x-c)^2 + y^2 - 4a\sqrt{(x-c)^2 + y^2} = (x+c)^2 + y^2$$

$$4a^2 + x^2 + c^2 - 2xc - 4a\sqrt{(x-c)^2+y^2} =$$

$$= x^2 + c^2 + 2xc$$

$$4a^2 - 4xc = 4a\sqrt{(x-c)^2+y^2}$$

$$a^2 - xc = a\sqrt{(x-c)^2+y^2}$$

$$a^4 + x^2c^2 - 2xca^2 = a^2\left(x^2 + c^2 - 2cx + y^2\right)$$

$$a^4 + x^2c^2 - 2xca^2 = a^2x^2 + a^2c^2 - 2cxa^2 + a^2y^2$$

$$a^4 - a^2c^2 = (a^2 - c^2)x^2 + a^2y^2$$

$$a^4 - a^2 c^2 = (a^2 - c^2)x^2 + a^2 y^2$$

$$\parallel$$

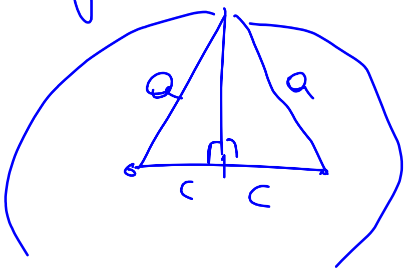
$$a^2(a^2 - c^2) \qquad \boxed{b^2 = a^2 - c^2} \qquad a > c$$

$$a^2 b^2 = \underline{\underline{b^2 x^2}} + \underline{a^2 y^2}$$

divido per $a^2 b^2$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Circonferenza $a = b = R$ $(c = 0)$

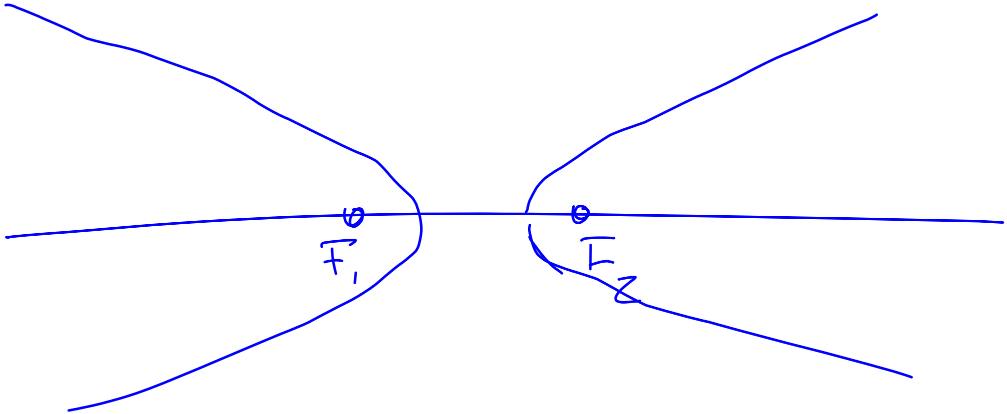


Iperbole insieme dei punti $(x, y) = P$ tali che è costante ($\equiv 2a$) la differenza delle distanze tra P e due punti dati (fuochi) F_1 e F_2

[Tutto come prima, ma $a < c$]

$b^2 = c^2 - a^2$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$



FUNZIONI

$f: D \rightarrow \mathbb{R}$

$y = f(x)$

$x \in \mathbb{R}$ variabile indipendente

$y \in \mathbb{R}$ variabile dipendente

$f(x)$ si dice immagine del punto x

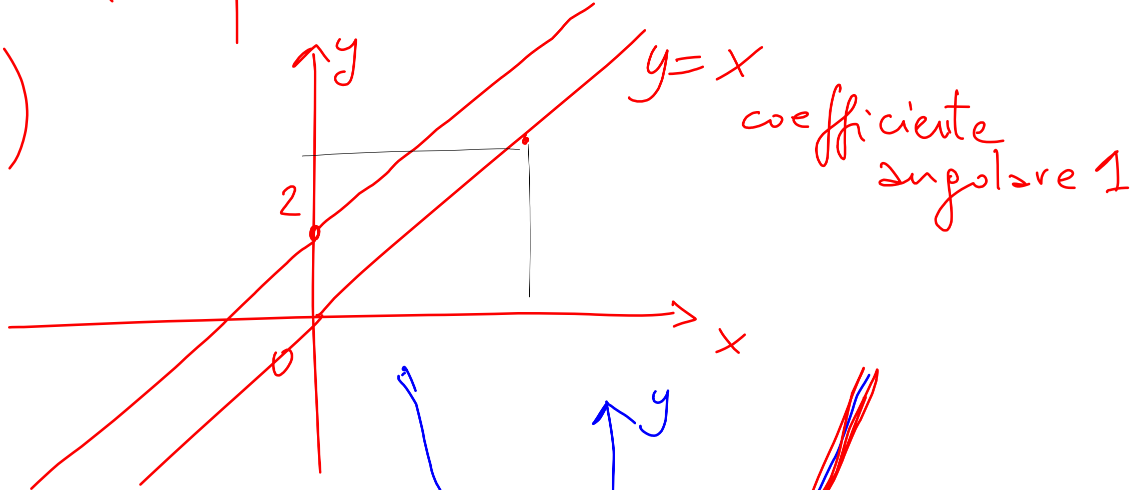
Esempi:

$y = x + 2$ retta — traslazione di $y = x$

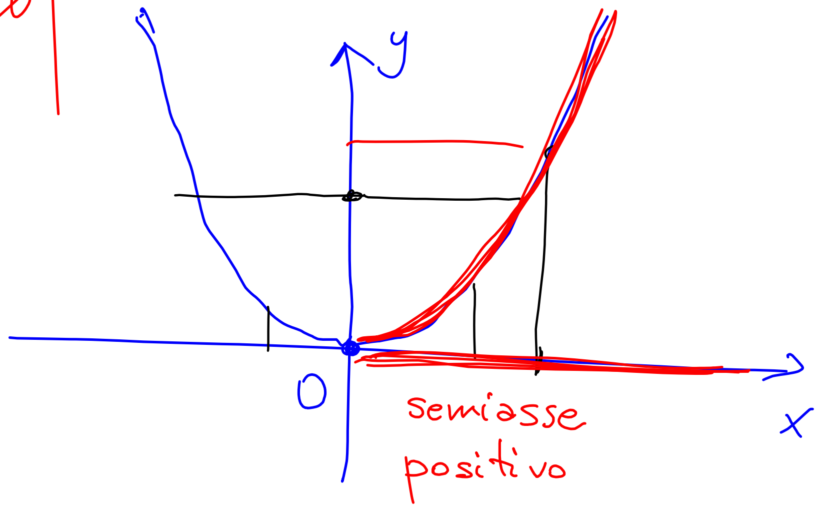
$y = x^2$ (parabola)

Data $y = f(x)$ il suo grafico è l'insieme dei punti che hanno coordinate

$(x, f(x))$



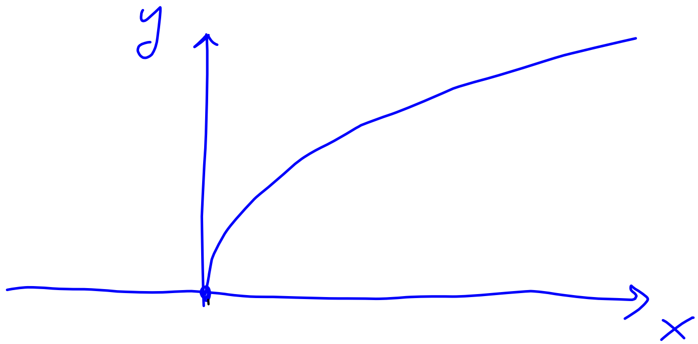
$y = x + 2$



$y = x^2$

$y = \sqrt{x}$ $\qquad \sqrt{a} = b \qquad b^2 = a \geq 0$

Il dominio è $\mathbb{R}_+ = \{ x \in \mathbb{R} : x \geq 0 \}$



$y^2 = x$

$x = y^2$

(parabola)

$y = \sqrt{x-3}$ $\qquad D = \{ x \in \mathbb{R} : x \geq 3 \}$

$= [3, \infty)$ $\qquad \infty$ "infinito"

$$y = \frac{1}{x(x-1)} = \frac{1}{x} \quad \frac{1}{x-1} \qquad \frac{1}{0}$$

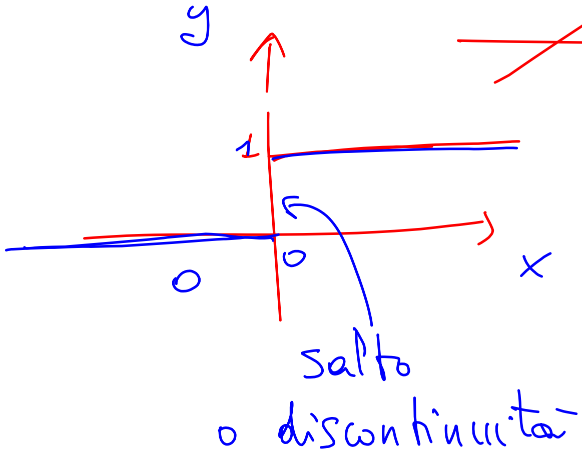
$$D = \mathbb{R} \setminus \{0, 1\}$$

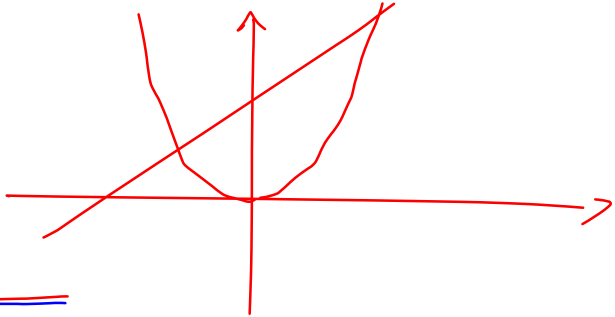
- Funzioni polinomiali
- Funzioni esponenziali e logaritmiche
- Funzioni trigonometriche

Operazioni tra funzioni

- algebriche
- composizione
- inversa

Continuità





$$y = f(x) = \begin{cases} 1 & \text{per } x > 0 \\ ? & \text{per } x = 0 \\ 0 & \text{per } x < 0 \end{cases}$$

Una funzione $f: D \to \mathbb{R}$ è continua in $x_0 \in D$

se $\forall \varepsilon > 0 \ \exists \delta > 0$:

$\forall x \in (x_0 - \delta, x_0 + \delta)$ vale $f(x_0) - \varepsilon < f(x) < f(x_0) + \varepsilon$

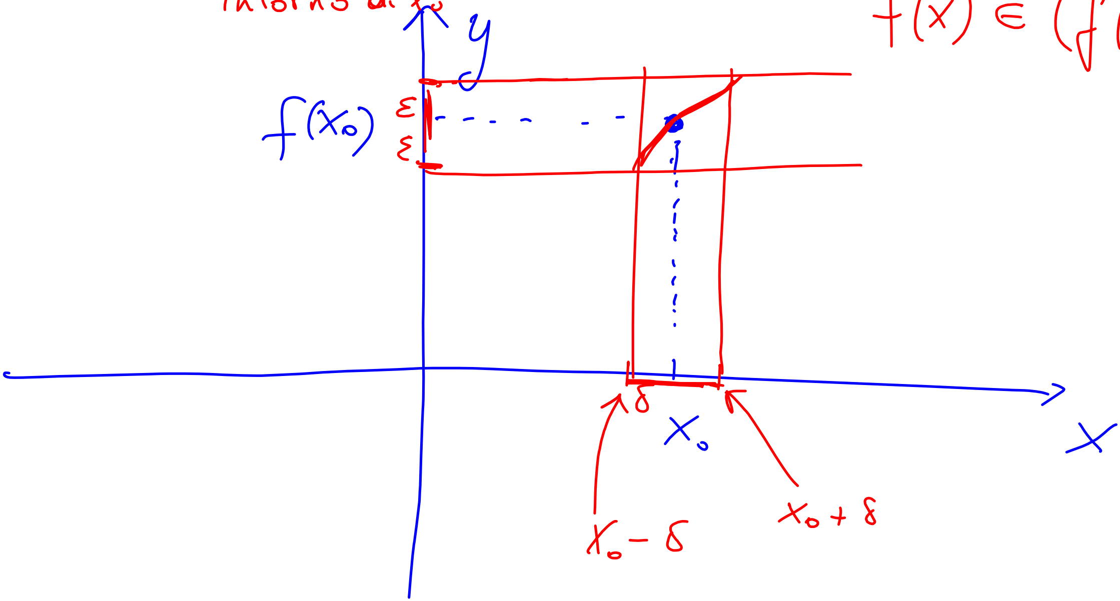
Una funzione $f : D \to \mathbb{R}$ è continua in $x_0 \in D$

se $\forall \varepsilon > 0 \quad \exists \delta > 0$:

$\forall x \in (x_0 - \delta, x_0 + \delta)$ vale $f(x_0) - \varepsilon < f(x) < f(x_0) + \varepsilon$

intorno di x_0

$f(x) \in (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$



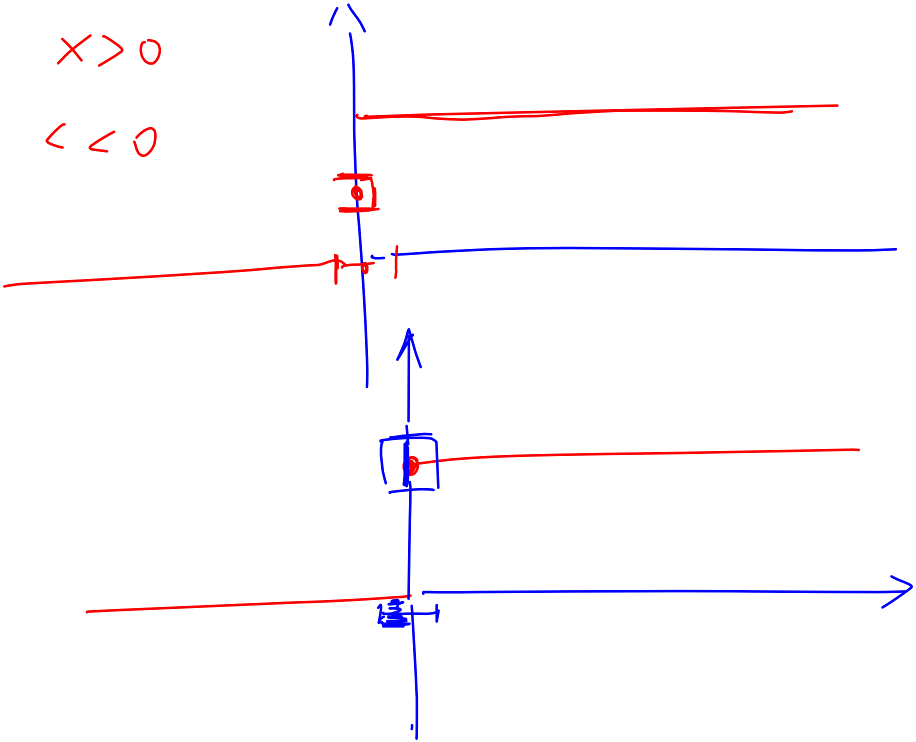
$$f(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$$

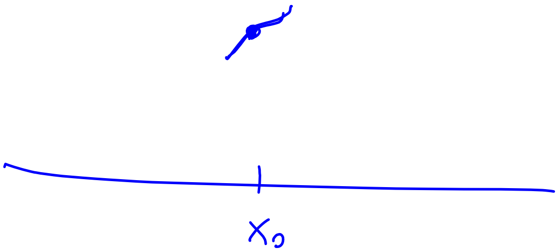
$$x_0 = 0$$

$$f(x_0) = \frac{1}{2}$$

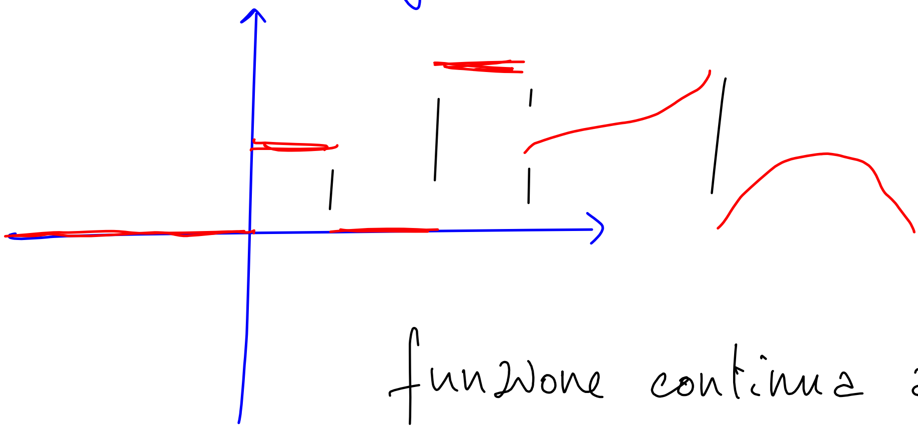
$$f(x_0) = 1$$

$$f(x_0) = 0$$



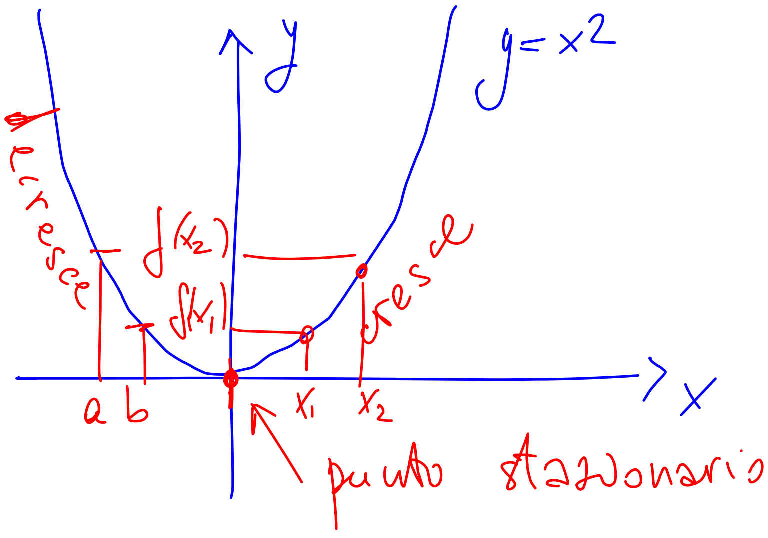


f è continua nell'intervallo (a,b) se è continua in ogni $x \in (a,b)$.



funzione continua a tratti

Crescenza e decrescenza



$f(x)$ è crescente in $[a, b]$ se

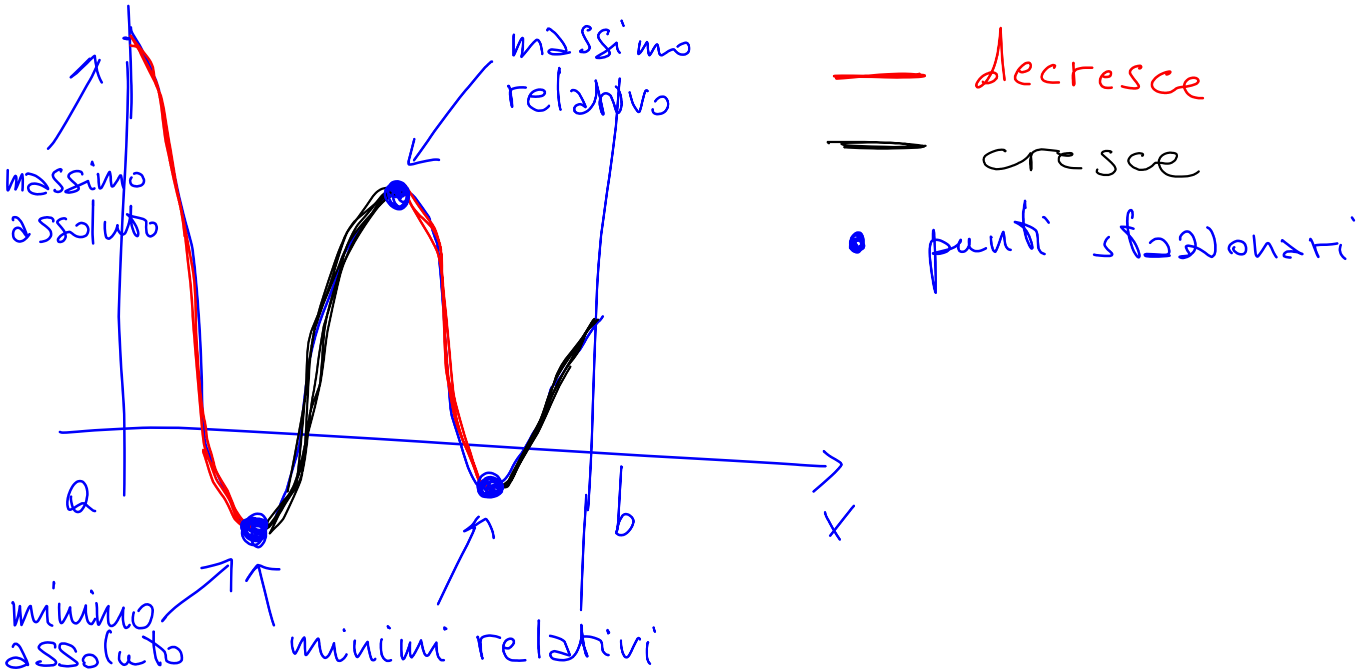
$\forall\, x_1, x_2 \in [a, b]$

$$x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$$

f è decrescente se

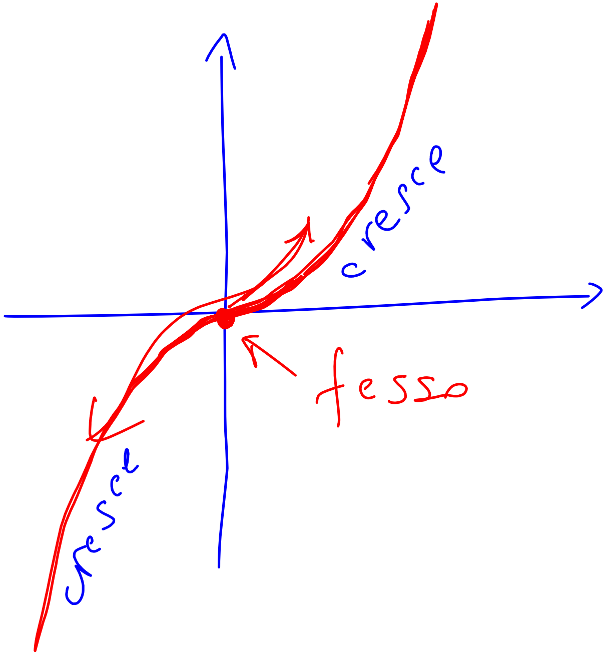
$$x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$$

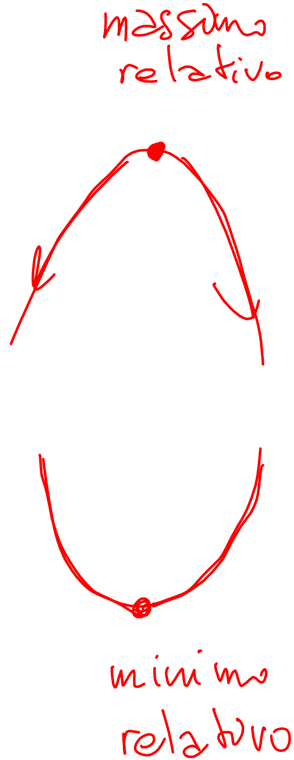
Massimi e minimi



Flesso

$$y = x^3$$





Operazioni tra funzioni

$y = f(x) \qquad y = g(x)$

$y = f(x) + g(x) \qquad y = f(x)\,g(x) \qquad y = \dfrac{f(x)}{g(x)}$

dominio $D_f \cap D_g$ tranne per il quoziente

$y = \dfrac{1}{x} \qquad D = \mathbb{R} \setminus \{0\}$

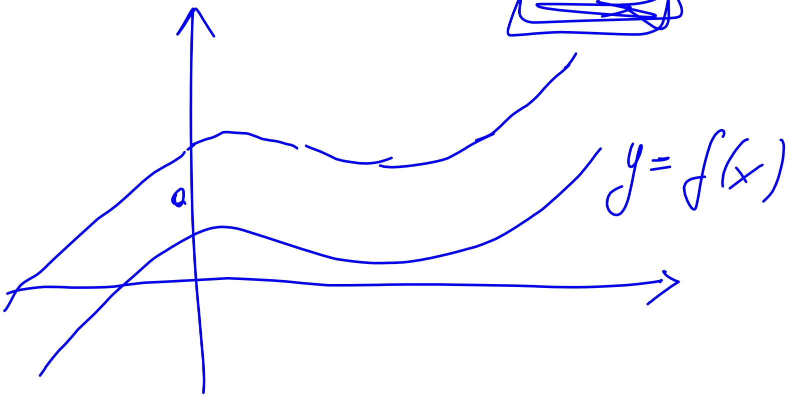
$y = x \qquad D = \mathbb{R}$

$\dfrac{0}{0}$ indeterminata

$y = x \, \dfrac{1}{x} = 1 \qquad D = \mathbb{R} \setminus \{0\}$

Esempi: $y = g(x) = a$ costante

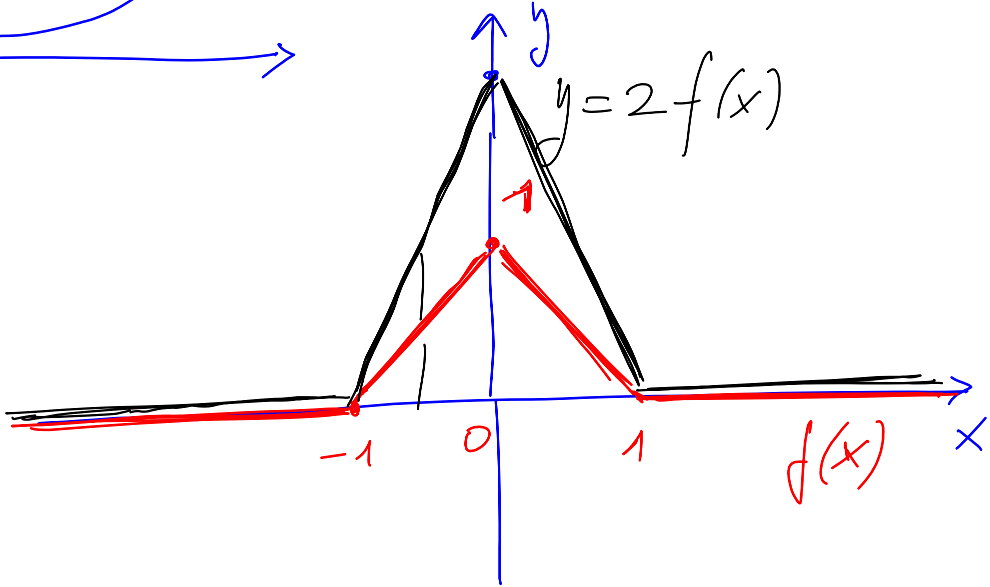
$y = f(x)$ $\qquad$ $y = f(x) + a$ $\qquad$ traslazione lungo la y



$y = a\, f(x)$

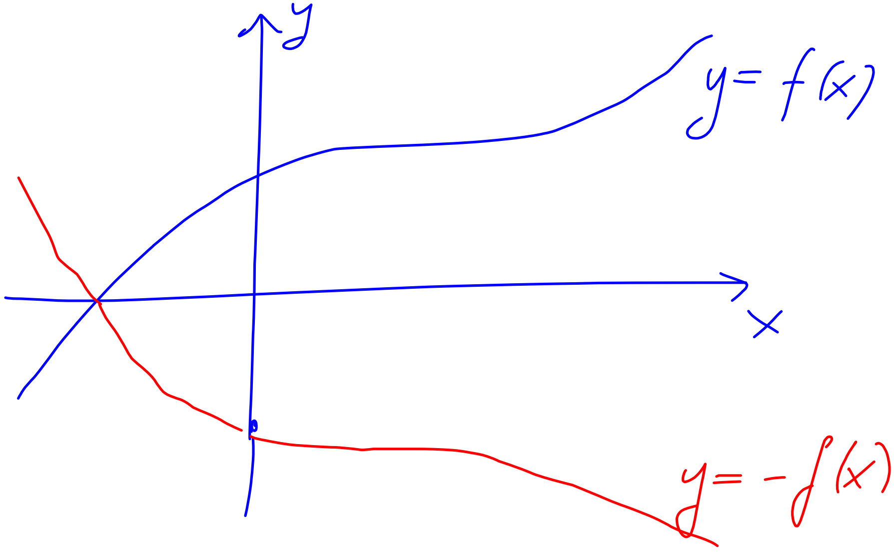
$a = 2$

$y = 2 f(x)$



$$y = -f(x)$$

riflessione rispetto all'asse x



Composizione di funzioni

$$y = f(x) \qquad y = g(x) \qquad y = f(g(t))$$

$$x = g(t)$$

← indipendente

Esempio

$y = f(x) = \sqrt{x}$ $\quad y = g(x) = x + 2$

$y = f(g(t))$ $\quad x = t + 2$

$y = \sqrt{x} = \sqrt{t+2}$

$y = \sqrt{t+2}$

$y = f(g(x)) = \sqrt{x+2}$ $\quad f \circ g\,(x)$

$y = g(f(x))$ $\quad g \circ f$

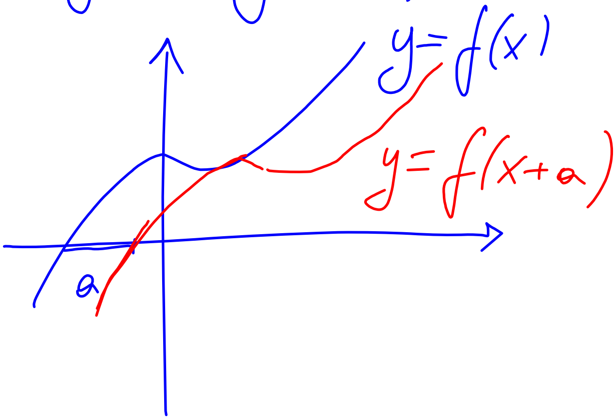
$y = \sqrt{x}$ $\quad y = x + 2$ $\quad y = x + 2 = \sqrt{t} + 2$

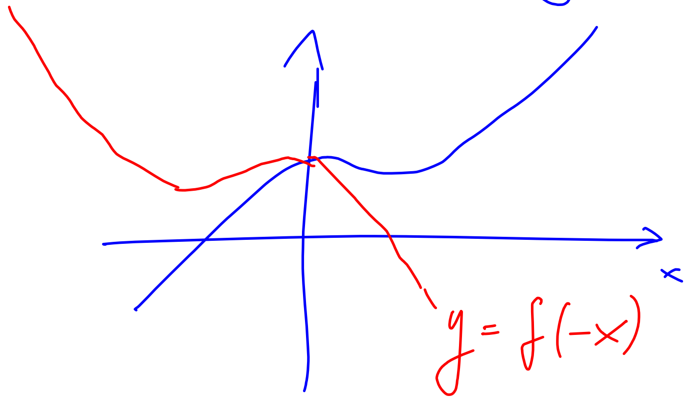
$x = \sqrt{t}$

Esempi notevoli

$y = f(x)$ $\quad$ $y = g(x) = x + a$ $\quad$ a costante

$f \circ g$ $\quad$ $y = f(\underline{x+a})$ $\quad$ traslazione lungo x





$g(x) = -x$ $\quad$ $f \circ g$: $y = f(-x)$

riflessione rispetto all'asse y

Funzione inversa

$y = f(x) \qquad x = g(y) = f^{-1}(y) \quad / \quad y = f(g(x)) = x$

$f \circ f^{-1} = \text{identità}$

Esempi

$y = f(x) = \frac{1}{x} \qquad y = \frac{1}{x} \qquad x = \frac{1}{y}$

$$\boxed{y = g(x) = \frac{1}{x}} \qquad x = \frac{1}{t}$$

$$y = \frac{1}{x} = \frac{1}{\frac{1}{t}} = t = y(t)$$

$y = x^2 \quad (x > 0)$

$x = \sqrt{y} \longrightarrow$ (scambio $x \leftrightarrow y$) $\quad y = \sqrt{x}$

$y = x^2 \qquad y = \sqrt{x}$

$x = \sqrt{t}$

$y = x^2 = (\sqrt{t})^2 = t \qquad$ identità

Funzioni elementari

Potenze a^b

ha senso se

$a > 0 \quad b \in \mathbb{R}$

$a = 0 \quad b > 0$

$a < 0 \quad b = \frac{m}{n} \quad m, n \in \mathbb{Z}$, n dispari

$a^{-1} = \frac{1}{a}$

$a^{-n} = \frac{1}{a^n} \quad n \in \mathbb{N}$

$a^{\frac{1}{2}} = \sqrt{a}$ non definita per $a < 0$

$a^{\frac{1}{3}} = \sqrt[3]{a} \equiv b$

è definita anche per $a < 0$

$a = b^3 \quad a = -8 \quad b = -2$

$a > 0$

$$a^0 = 1 \qquad a^{-b} = \frac{1}{a^b} \qquad a^{\frac{1}{b}} = \sqrt[b]{a}$$

$$a^{b+c} = a^b a^c \qquad a^{bc} = (a^b)^c = (a^c)^b$$

$$a^3 = a^{3+0} = a^3 a^0$$

$a^0 = 1$ elemento neutro

$c = \frac{1}{b}$

$$a^{b\frac{1}{b}} = a^1 = a = (a^b)^{1/b} = \sqrt[b]{a^b} =$$

$$= \left(\sqrt[b]{a}\right)^b = a$$

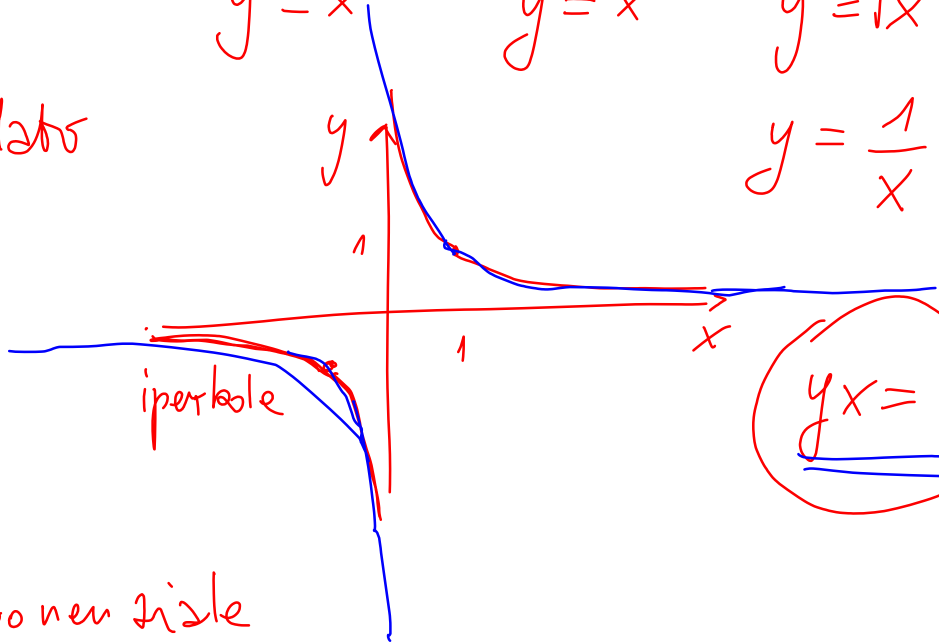
Funzione potenza

$$y = x^b$$

b è dato

$y = x^2$ $\quad$ $y = x^3$ $\quad$ $y = \sqrt{x}$

$$y = \frac{1}{x} = x^{-1}$$



$$yx = 1$$

Funzione esponenziale

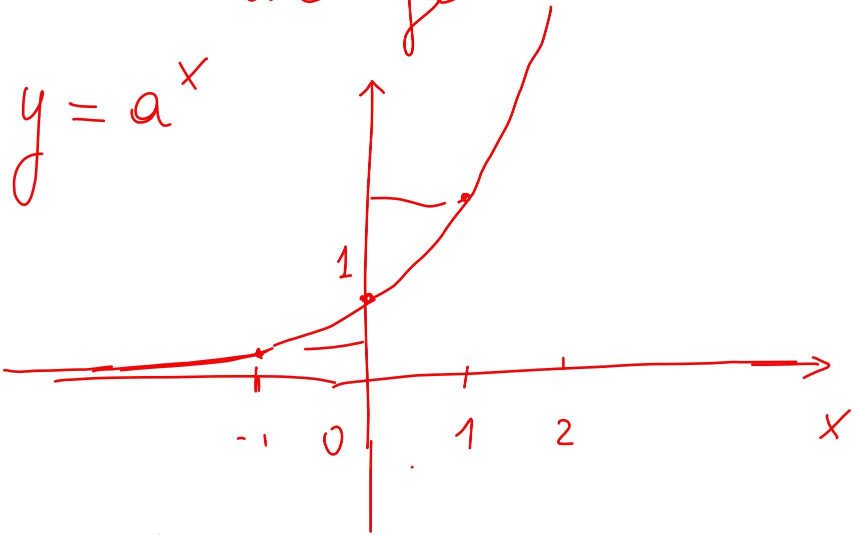
$$y = a^x$$

a dato (si chiama base)

Successione geometrica $a_n = \frac{1}{2^n} = \left(\frac{1}{2}\right)^n$

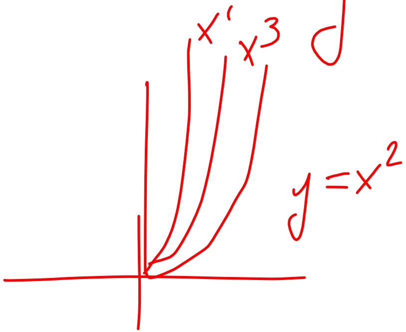
↑ base

$y = a^x$

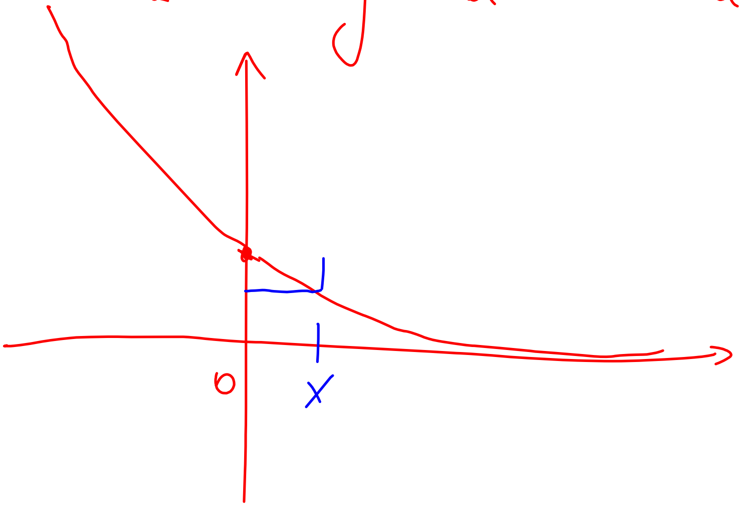


$a^0 = 1$

$a > 1$ $y = 2^x$ $x = -1$ $2^{-1} = \frac{1}{2}$



$a < 1$ $\qquad y = a^x \qquad a = \frac{1}{2}$



$\left(\frac{1}{2}\right)^2 \qquad \left(\frac{1}{2}\right)^{10}$

$\left(\frac{1}{2}\right)^{-1} = 2$

$\left(\frac{1}{2}\right)^{-2} = 4$

Una funzione sempre crescente o sempre decrescente si dice monotona

10^6 milione — mega $\qquad 10^9$ miliardo — Giga $\qquad 10^3$ kilo

e numero di Nepero (o di Eulero)

$$e = 2.718 \ldots = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots$$

$$\boxed{y = e^x} \qquad e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \frac{x^n}{n!} + \ldots$$

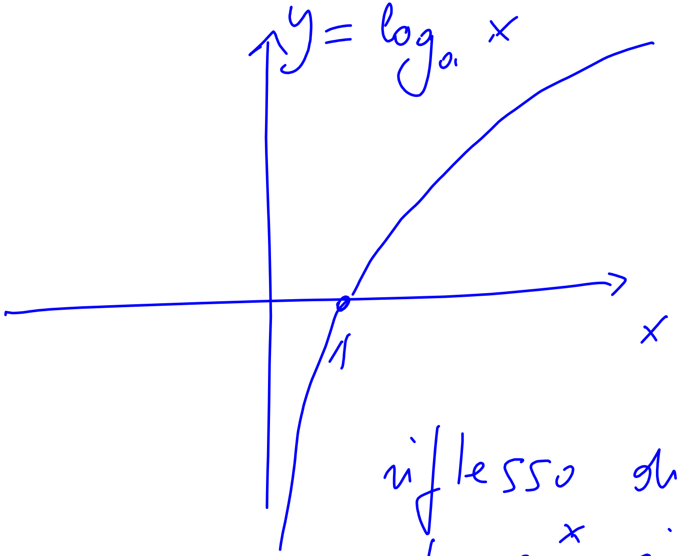
Logaritmo è l'inversa dell'esponenziale

$y = a^x \quad a > 1 \qquad \mathbb{R} \to \mathbb{R}_+$

dominio codominio

$x = \log_a y$ scambio di nome x e y

$y = \log_a x \quad : \quad \mathbb{R}_+ \to \mathbb{R}$



$a > 1$

$x = 1 \qquad \log_a 1 = 0$

$a^0 = 1$

riflesso di $y = a^x$ rispetto alla diagonale $y = x$

$$a^{b+c} = a^b a^c \qquad a^{bc} = (a^b)^c = (a^c)^b$$

$$a^s = bc$$
$$a^t = b$$
$$a^u = c$$

$$\underbrace{\log_a (bc)}_{=s} = \underbrace{\log_a b}_{=t} + \underbrace{\log_a c}_{=u}$$

$$a^s = a^t a^u = a^{t+u}$$

$\log_{10} x$

$\log_{10} 10^9 = 9$ ← ordine di grandezza

$\log_{10}$ 1 milione di miliardi $= 15$

$\log_e x = \ln x$

↑ base naturale

↑ logaritmo naturale

Cambio di base

$$y = \log_a x = \log_a b \cdot \log_b x$$

$\log_a x = s$, $\log_a b = t$, $\log_b x = u$

$a^s = x$

$a^t = b$

$b^u = x$

$$a^s = x = b^u = (a^t)^u = a^{tu} \qquad s = tu$$

$\log_a 1 = 0 \qquad (a^0 = 1)$

$\log_a a = 1 \qquad a^1 = a$

$a^{\log_a b} = b$ definizione di $\log_a b$

$\log_a b \cdot \log_b a = 1$ Infatti

$\log_a a = \log_a b \cdot \log_b a = 1$

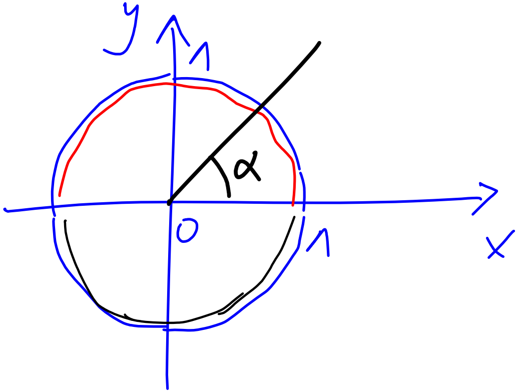
$\log_a \frac{b}{c} = \log_a b - \log_a c$

$$\log_a \frac{b}{c} = \log_a b - \log_a c$$

$$\log_a \left(\frac{b}{c}\right) + \log_a c = \log_a \frac{b}{\not c} \not c$$

Funzioni trigonometriche

Considero una circonferenza di raggio 1



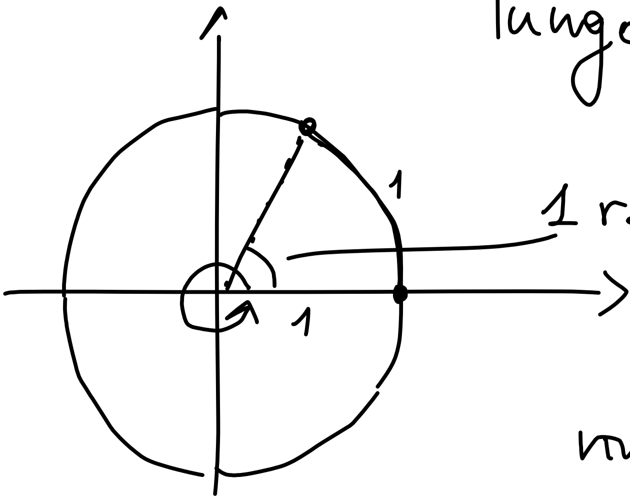
$$x^2 + y^2 = 1$$

$$y = \sqrt{1 - x^2}$$

$$y = -\sqrt{1 - x^2}$$

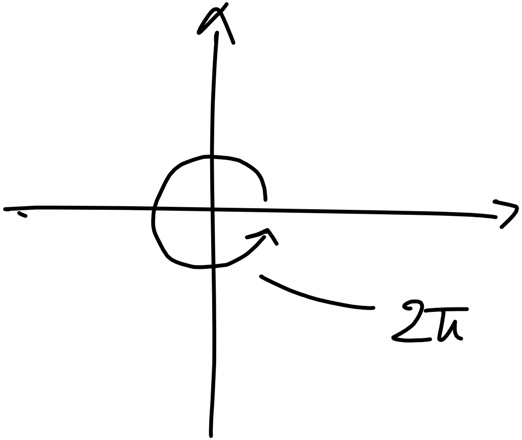
Misuro gli angoli in radianti

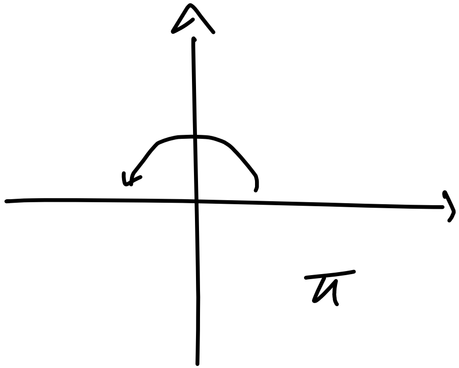
Si dice radiante l'angolo individuato da un arco lungo quanto il raggio



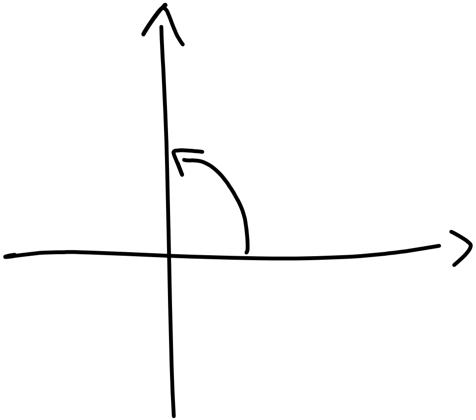
l'intera circonferenza misura 2π (raggio = 1)

l'angolo sotteso da tutta la circonferenza (360°) è 2π radianti





(180°)

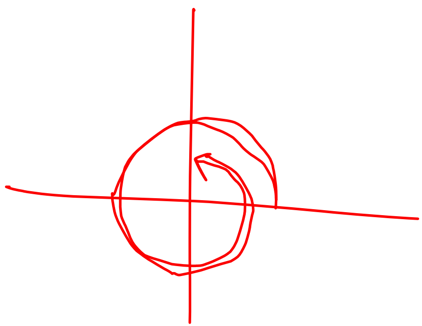


angolo retto $\frac{\pi}{2}$ (90°)

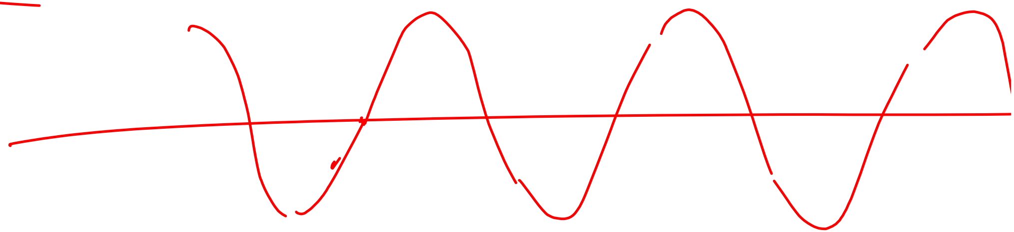
Sono funzioni periodiche di periodo 2π.

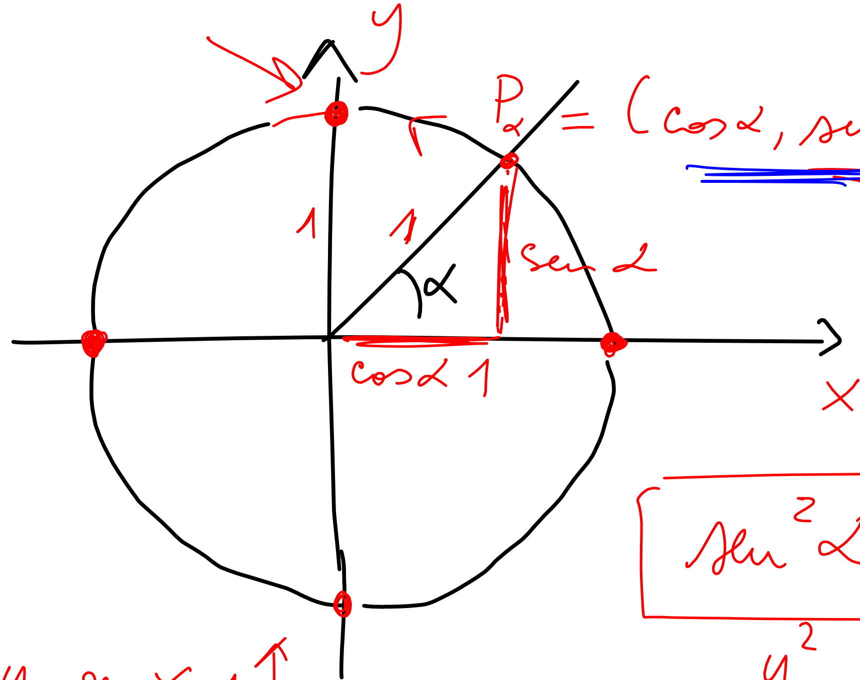
$f(x)$ tale che $f(x) = f(x+a)$ $\forall x$
(a fissato)

si dice periodica di periodo a



$450^\circ = \frac{5}{2}\pi$





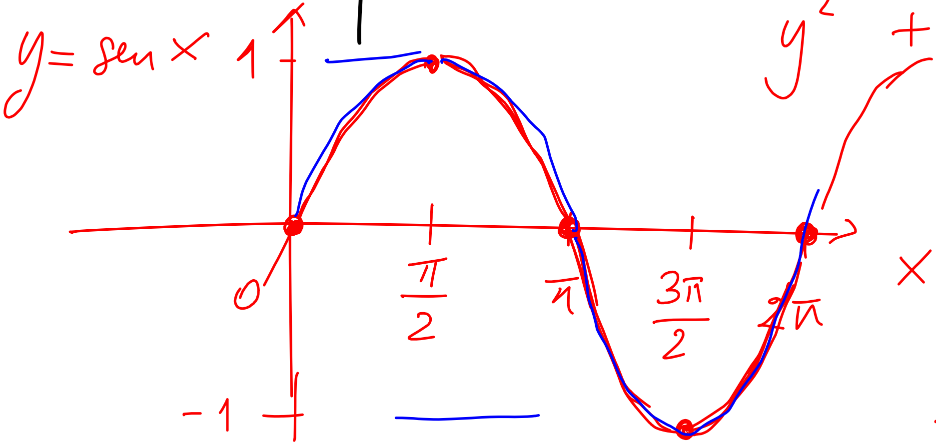
$P_\alpha = (\cos\alpha, \operatorname{sen}\alpha)$

$1 : \operatorname{sen}\alpha = l : l\operatorname{sen}\alpha$

Teorema di Pitagora

$$\operatorname{sen}^2\alpha + \cos^2\alpha = 1$$

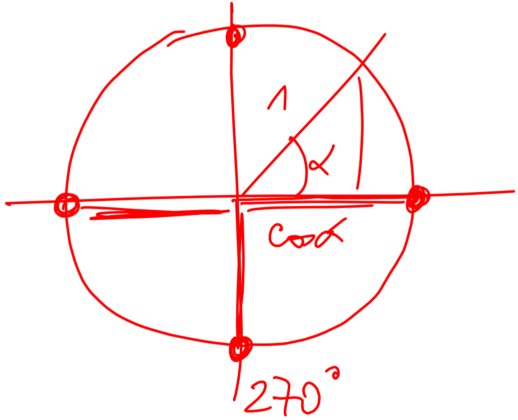
$$y^2 + x^2 = 1$$

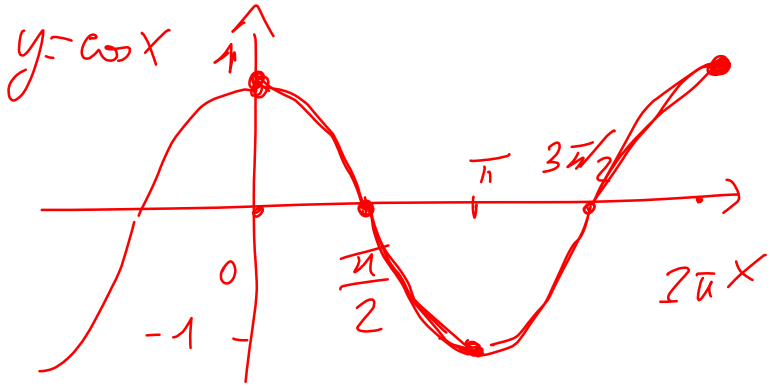


$\operatorname{sen} 0 = 0$

$\operatorname{sen} \frac{\pi}{2} = 1$

le onde

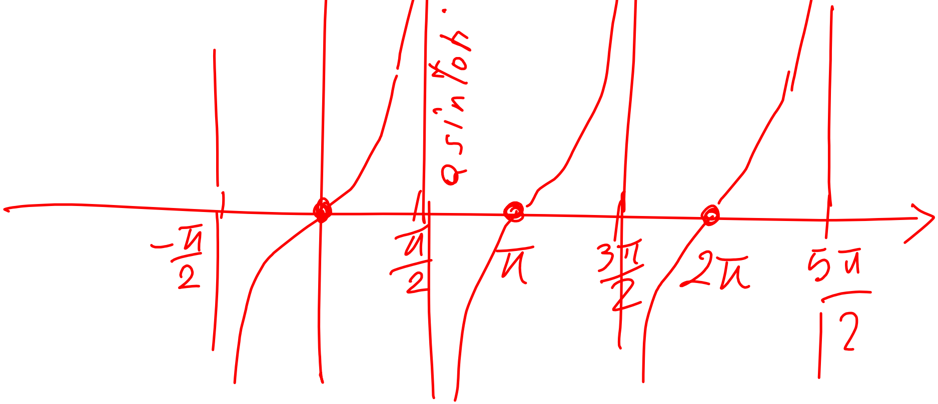


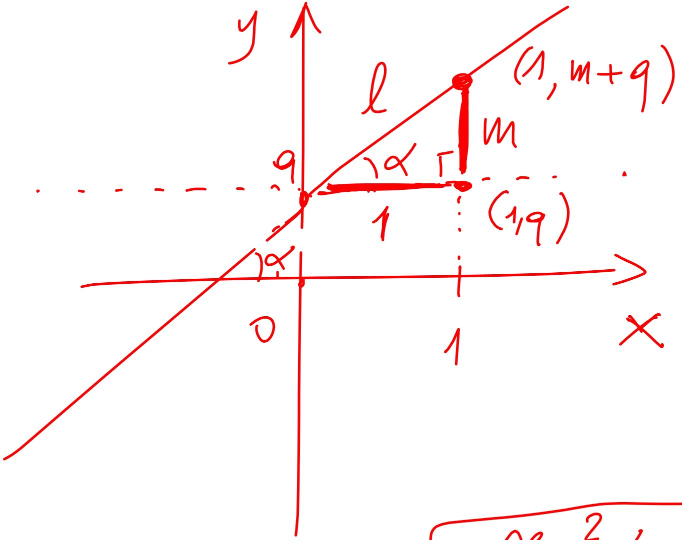


$$tg\,x = \frac{sen\,x}{cos\,x}$$

tangente

$$cotg = \frac{cos\,x}{sen\,x}$$





$$y = mx + q$$

$$\begin{cases} \operatorname{sen}\alpha = \dfrac{m}{l} \\ \cos\alpha = \dfrac{1}{l} \end{cases}$$

$$\boxed{\operatorname{sen}^2\alpha + \cos^2\alpha = 1}$$

$$\frac{m^2}{l^2} + \frac{1}{l^2} = 1$$

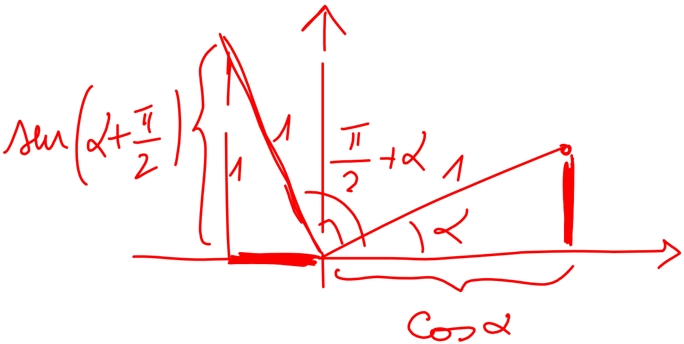
$$m^2 + 1^2 = l^2$$

$$\operatorname{tg}\alpha = \frac{\operatorname{sen}\alpha}{\cos\alpha} = \frac{\frac{m}{l}}{\frac{1}{l}} = m$$

Identità

$$\begin{cases} \operatorname{sen}(\alpha+\beta) = \operatorname{sen}\alpha \cos\beta + \operatorname{sen}\beta \cos\alpha \\ \cos(\alpha+\beta) = \cos\alpha \cos\beta - \operatorname{sen}\alpha \operatorname{sen}\beta \end{cases}$$

$\beta = \frac{\pi}{2}$ $\qquad \operatorname{sen}\left(\alpha + \frac{\pi}{2}\right) = \cos\alpha$

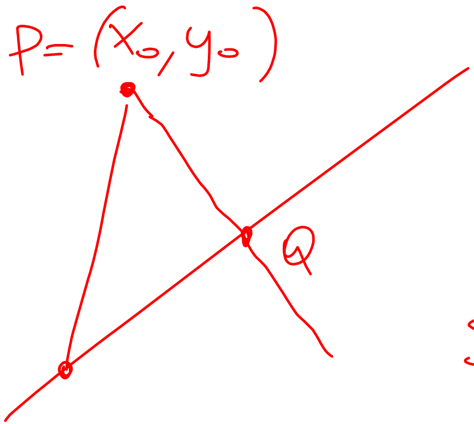


$\beta = \alpha$:

$\operatorname{sen} 2\alpha = 2 \operatorname{sen}\alpha \cos\alpha$

$\cos 2\alpha = \cos^2\alpha - \operatorname{sen}^2\alpha$

Dato un punto $P = (x_0, y_0)$ e una retta $ax + by + c = 0$ trovare la distanza tra il punto e la retta



$ax + by + c = 0$

distanza $= d(P, Q)$

Sia $b \neq 0$ $\quad y = -\frac{a}{b}x - \frac{c}{b}$

$= mx + q$

$m = -\frac{a}{b} \qquad q = -\frac{c}{b}$

$r: y = mx + q$ Retta perpendicolare:

$$r_{\perp}: y = -\frac{1}{m}x + q'$$

deve passare per P:

$$y_0 = -\frac{1}{m}x_0 + q' \qquad q' = y_0 + \frac{1}{m}x_0$$

$$r_{\perp}: y = -\frac{1}{m}x + y_0 + \frac{1}{m}x_0$$

$$Q: \begin{cases} y = mx + q \\ y = -\frac{1}{m}x + y_0 + \frac{1}{m}x_0 \end{cases}$$

$$mx + q = -\frac{1}{m}x + y_0 + \frac{1}{m}x_0$$

$$\left(m + \frac{1}{m}\right)x = y_0 + \frac{1}{m}x_0 - q$$

$$x_Q = \frac{1}{m + \frac{1}{m}}\left[y_0 + \frac{1}{m}x_0 - q\right]$$

$$y_Q = m x_Q + q \qquad P : (x_0, y_0)$$

$$d(P, Q) = \sqrt{(x_P - x_Q)^2 + (y_P - y_Q)^2}$$

$$d^2(P, Q) = \left(x_0 - \frac{1}{m + \frac{1}{m}}\left(y_0 + \frac{1}{m}x_0 - q\right)\right)^2 + (y_0 - y_Q)^2$$

$$x_0 - x_2 = x_0 - \frac{1}{m+\frac{1}{m}}\left(y_0 + \frac{1}{m}x_0 - q\right) = \frac{m x_0 + \frac{1}{m}x_0 - y_0 - \frac{x_0}{m} + q}{m+\frac{1}{m}} =$$

$$= -\frac{y_0 - m x_0 - q}{m+\frac{1}{m}}$$

$$r: \; y - mx - q = 0$$

$$= +\frac{y_0 + \frac{a}{b}x_0 + \frac{c}{b}}{+\frac{a}{b}+\frac{b}{a}} \cdot \frac{ab}{ab} =$$

$$= \frac{a x_0 + b y_0 + c}{a^2 + b^2} \, a$$

$$y_0 - y_Q = y_0 - m x_Q - q =$$

$$= \underbrace{y_0} + \underbrace{m(x_0 - x_Q)} - \underline{m x_0 - q}$$

$$= \underbrace{y_0 - m x_0 - q} + m\left(- \frac{y_0 - m x_0 - q}{m + \frac{1}{m}}\right) =$$

$$= (y_0 - m x_0 - q)\left(1 - \frac{m}{m + \frac{1}{m}}\right) =$$

$$= \frac{y_0 - m x_0 - q}{m + \frac{1}{m}} \cdot \frac{1}{m} = \frac{a x_0 + b y_0 + c}{a^2 + b^2} b$$

$$d^2(P,Q) = \frac{(a x_0 + b y_0 + c)^2}{(a^2+b^2)^{\not{2}}} \quad \cancel{(a^2+b^2)}$$

$$d(P,Q) = \frac{|a x_0 + b y_0 + c|}{\sqrt{a^2+b^2}}$$

$|x| = x$ se $x > 0$

$|x| = -x$ se $x < 0$

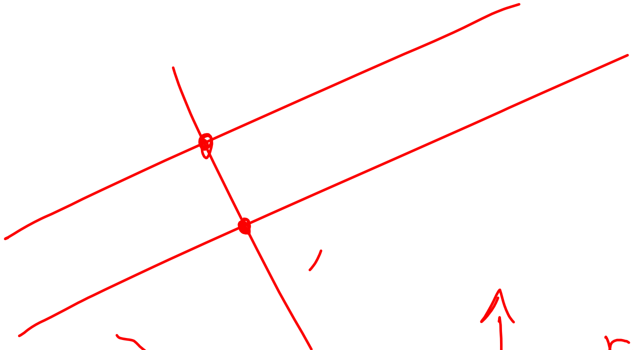
Determinare la distanza tra

$B = (-1, 3)$ e la retta $2x + 3y = 1$

$x_0 \quad y_0 \qquad a \quad b \quad c = -1$

$$d = \frac{|-2 + 3 \cdot 3 - 1|}{\sqrt{2^2 + 3^2}} = \frac{6}{\sqrt{13}}$$

Trovare la distanza tra le rette parallele

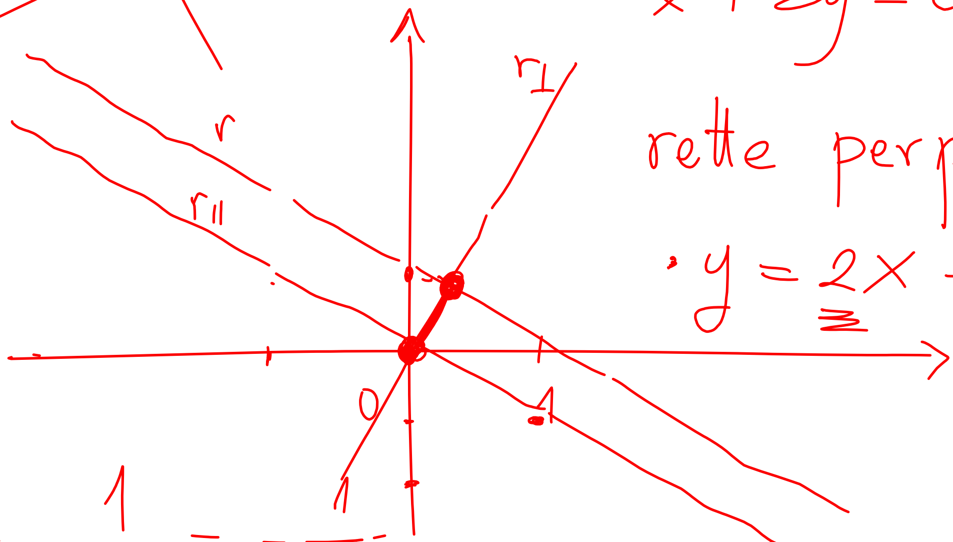


- $x + 2y = 1 \qquad y = -\frac{x}{2} + \frac{1}{2}$
- $x + 2y = 0 \qquad y = -\frac{x}{2}$

rette perpendicolari:

- $y = 2x + q'$

Scelgo $q' = 0$



$m' = -\frac{1}{m} = -\frac{1}{-\frac{1}{2}} = 2$

$r: \quad y = -\frac{x}{2} + \frac{1}{2}$

$r_{\parallel}: \quad y = -\frac{x}{2}$ } intersezione $(0,0)$

$r_{\perp}: \quad y = 2x$ }

intersezione tra r e $r_{\perp}$ $\left(\frac{1}{5}, \frac{2}{5}\right)$

$$\begin{cases} y = -\frac{x}{2} + \frac{1}{2} \\ y = 2x \end{cases}$$

$2x = -\frac{x}{2} + \frac{1}{2}$

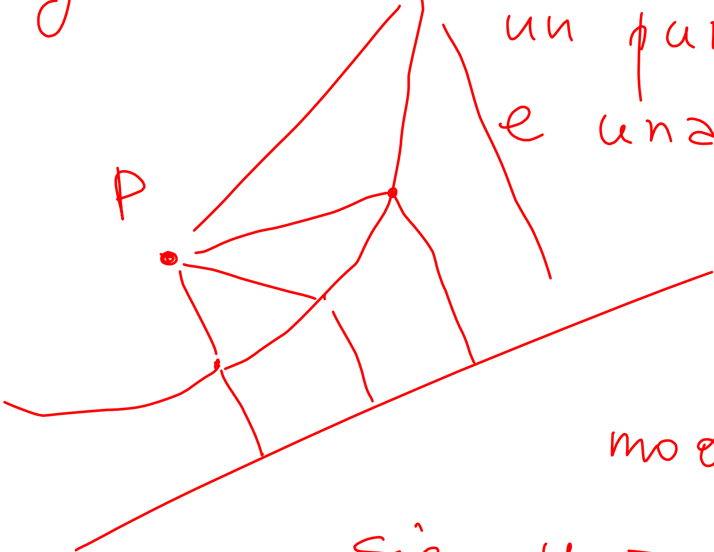
$\frac{5}{2}x = \frac{1}{2}$ $\quad x = \frac{1}{5}$

$$\text{distanza} = \sqrt{\frac{1}{5^2} + \frac{2^2}{5^2}} = \frac{\sqrt{5}}{5} = \frac{1}{\sqrt{5}}$$

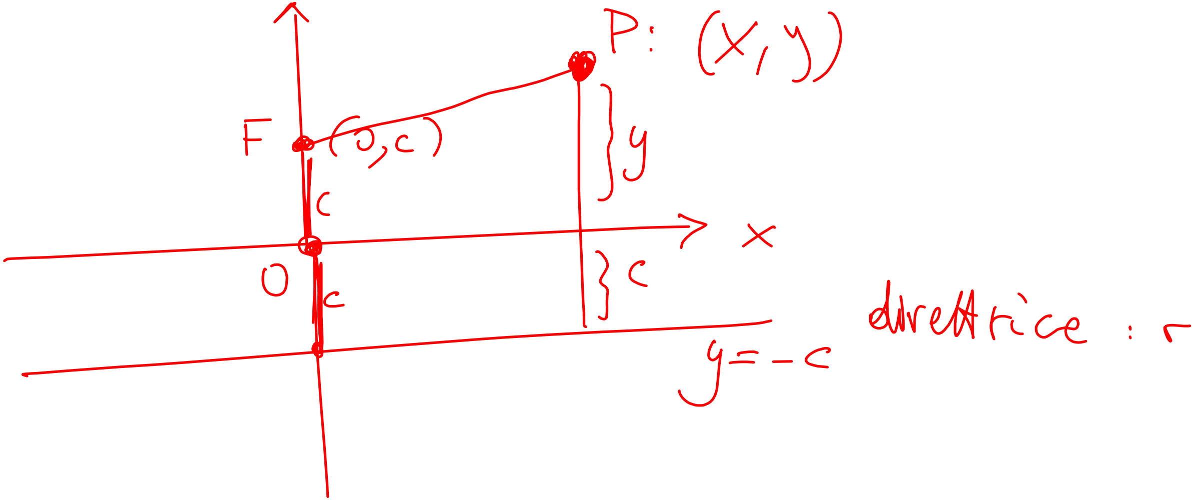
Parabola

$y = ax^2$

è l'insieme dei punti (x,y) equidistanti da un punto dato (fuoco) e una retta data (direttrice)



Scegliamo le coordinate in modo tale che la direttrice sia $y = -c$ e il fuoco sia $(0, c)$



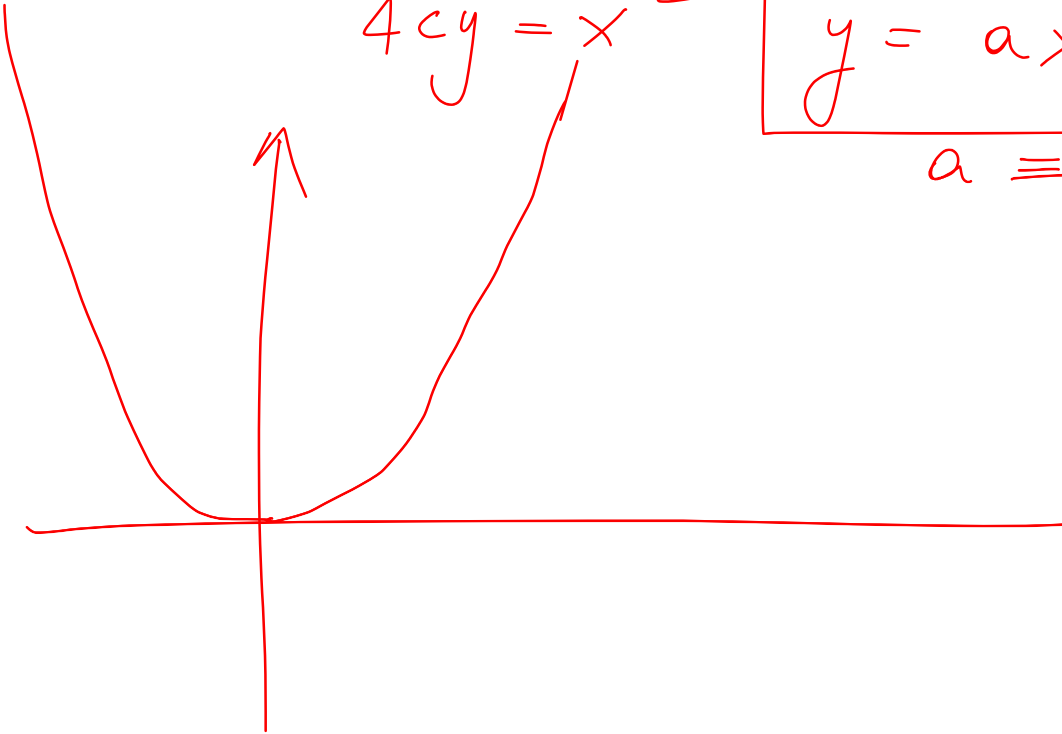
distanza tra P e r = $|y+c|$

$$d(P,F) = \sqrt{x^2 + (y-c)^2}$$

$$|y+c| = \sqrt{x^2 + (y-c)^2}$$

$$y^2 + c^2 + 2cy = x^2 + y^2 + c^2 - 2cy$$

$$4cy = x^2$$

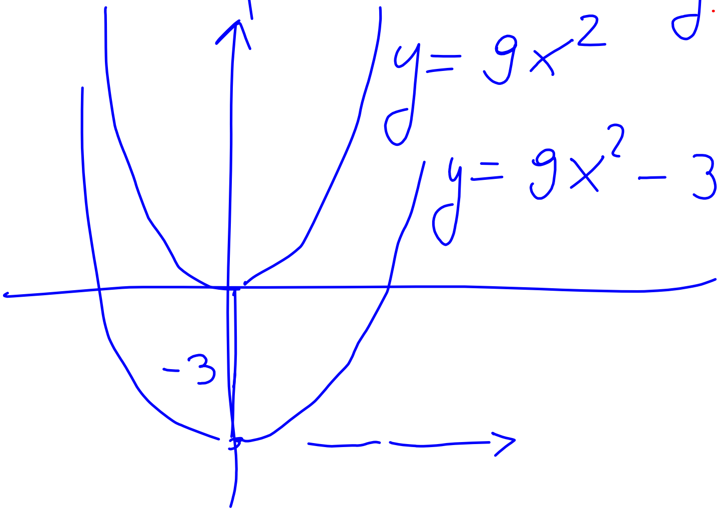


$$\boxed{y = ax^2}$$

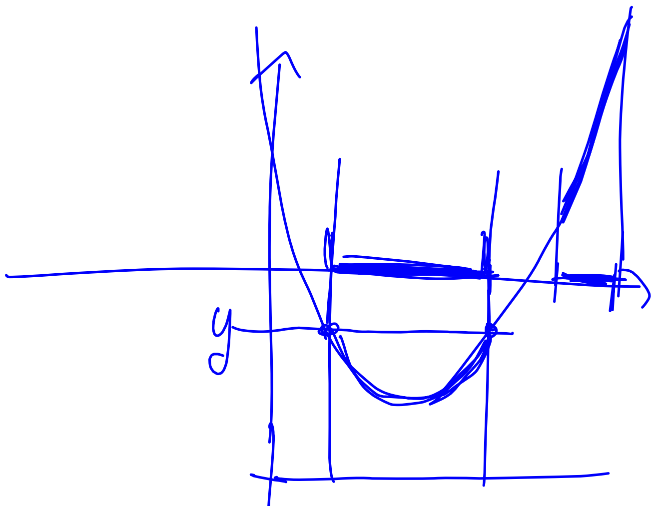
$$a = \frac{1}{4c}$$

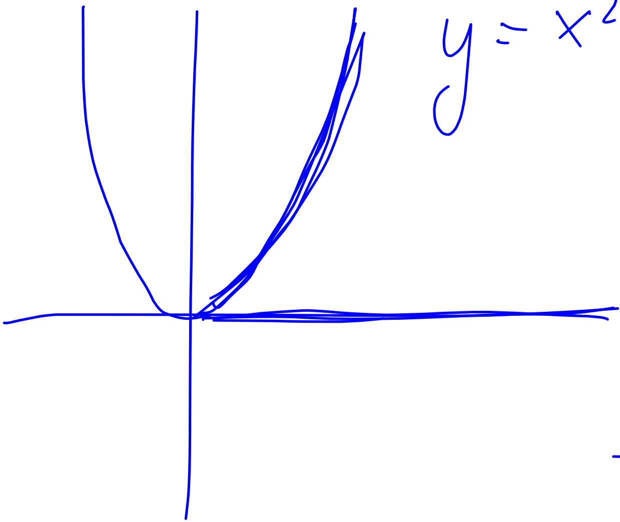
Determinare un intervallo in cui la funzione $y = f(x) = (3x+2)^2 - 3$ è invertibile e invertirla

È una parabola

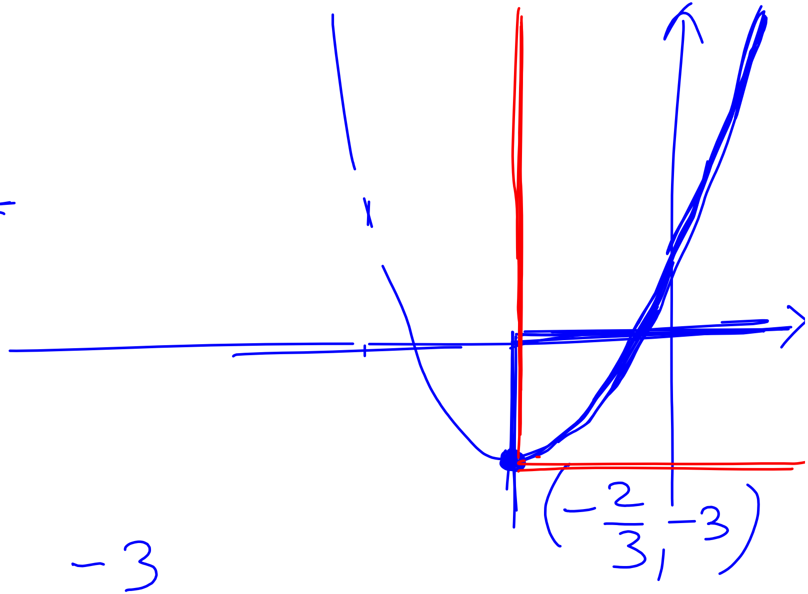


$$y = 9\left(x + \frac{2}{3}\right)^2 - 3$$





$y = x^2$ $\quad x > 0$



$$y = \underbrace{(3x+2)^2}_{\geq 0} - 3 \geq -3$$

$x = -\frac{2}{3}$ $\quad y = -3$ minimo

La funzione è invertibile $\quad x \geq -\frac{2}{3}$

$y \geq -3$

$$y+3=(3x+2)^2 \qquad \sqrt{y+3}=3x+2$$

$$y+3 \geq 0 \qquad 3x+2 \geq 0$$

$$3x=\sqrt{y+3}-2$$

$$x=\frac{1}{3}\sqrt{y+3}-\frac{2}{3}$$

scambio x con y

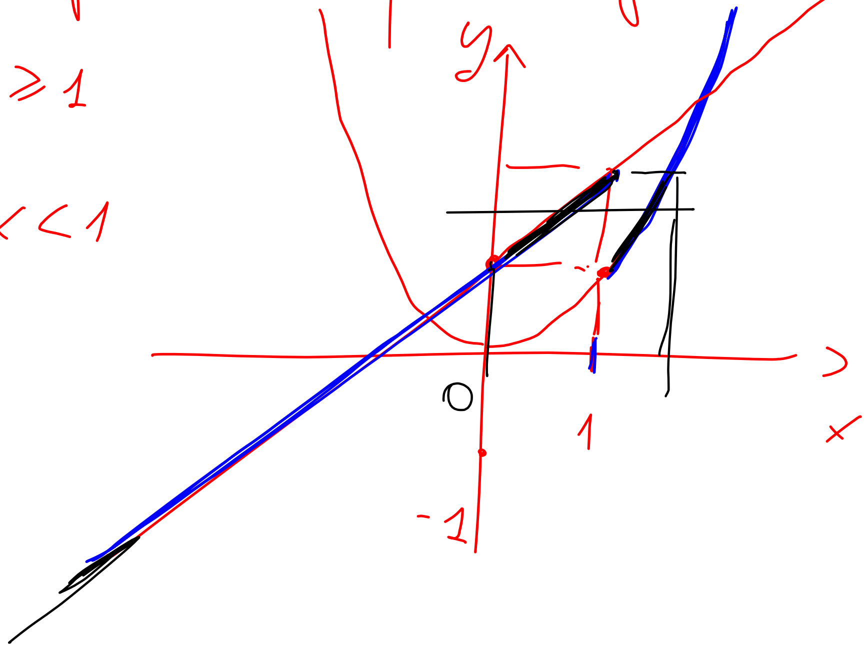
$$y=\frac{1}{3}\sqrt{x+3}-\frac{2}{3}$$

$$\begin{cases} x \geq -3 \\ y \geq -\frac{2}{3} \end{cases}$$

$y = ax^2 + bx + c$ è sempre una parabola

Studiare i grafici di queste funzioni

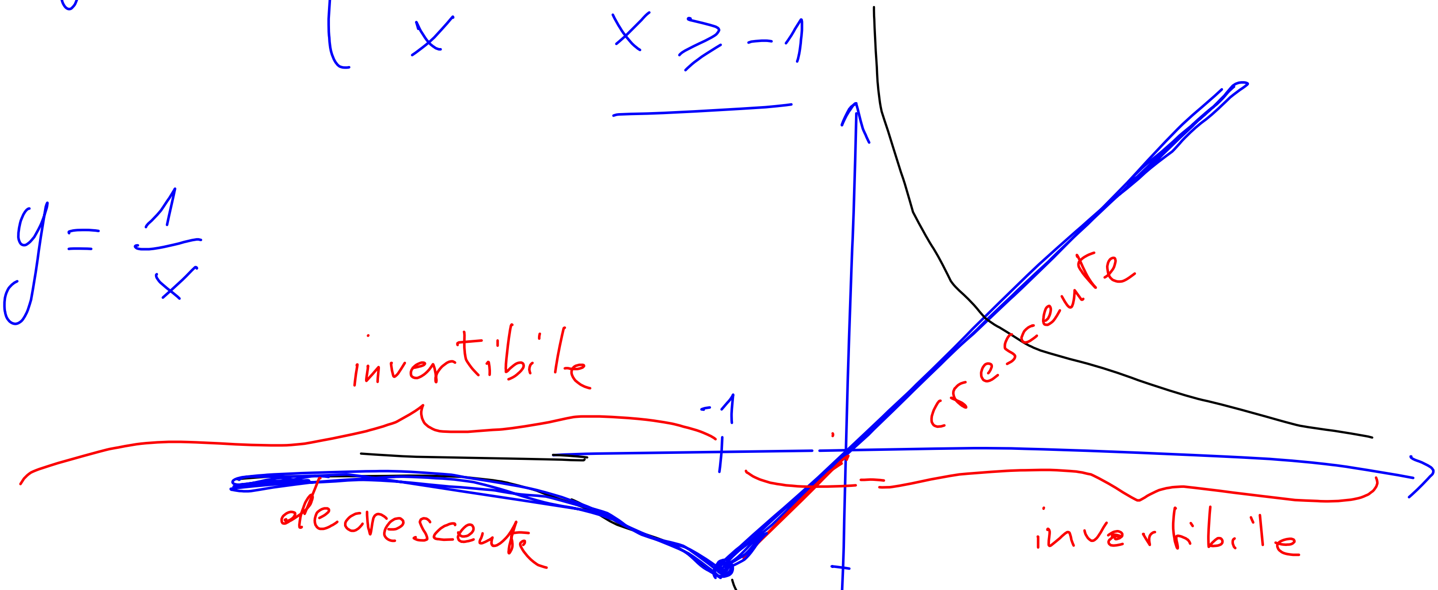
$$f(x) = \begin{cases} x^2 & x \geq 1 \\ x+1 & x < 1 \end{cases}$$



la funzione ha un salto in $x = 1$

$$g(x) = \begin{cases} \dfrac{1}{x} & x < -1 \\ x & x \geq -1 \end{cases}$$

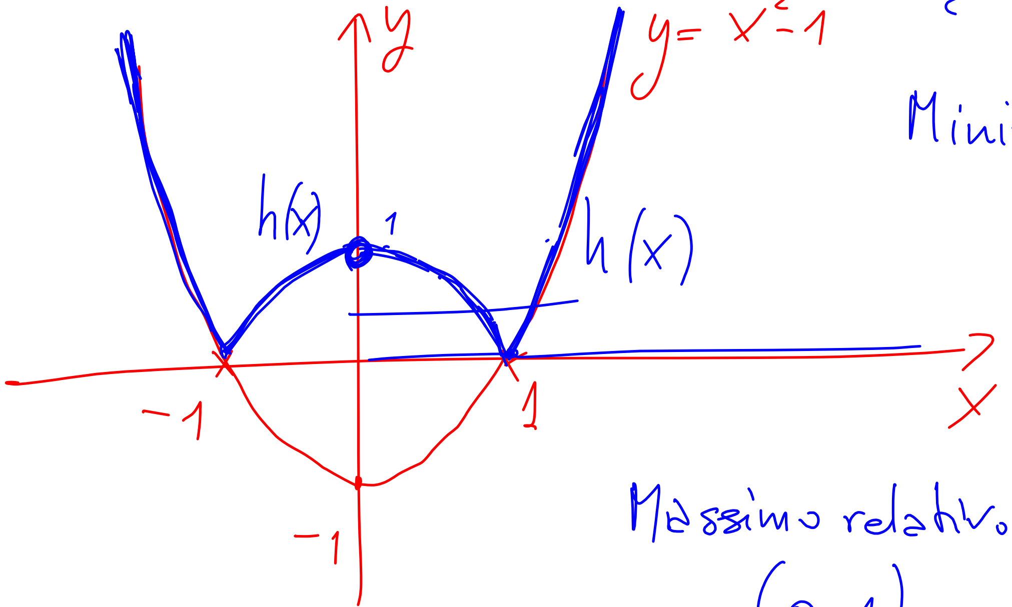
$$y = \frac{1}{x}$$



È continua? Sì

Minimo? Sì -1.

$h(x) = |x^2 - 1|$



È continua? Sì

Minimi: $(-1, 0)$

$(1, 0)$

Massimi

(∞, ∞)

$(-\infty, \infty)$

Massimo relativo

$(0, 1)$

È invertibile per $x \geq 1$ (per esempio)

Invertire

$y = 2x^3 + 3$

$y = -3x + 4$

$y = \frac{2}{3x-4}$

invertibile

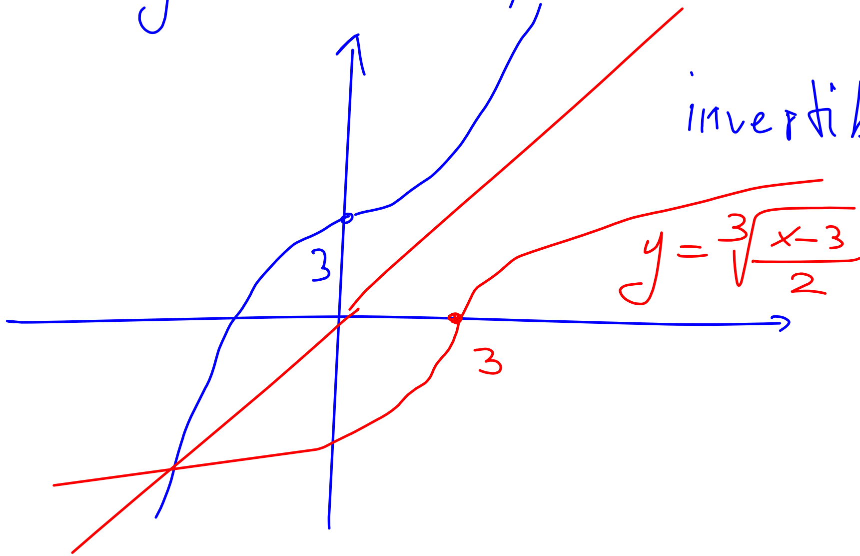
$y = 2x^3 + 3$

$y - 3 = 2x^3$

$x^3 = \frac{1}{2}y - \frac{3}{2}$

$x = \sqrt[3]{\frac{y-3}{2}}$

$x \leftrightarrow y \quad y = \sqrt[3]{\frac{x-3}{2}}$



$$y = -3x + 4$$

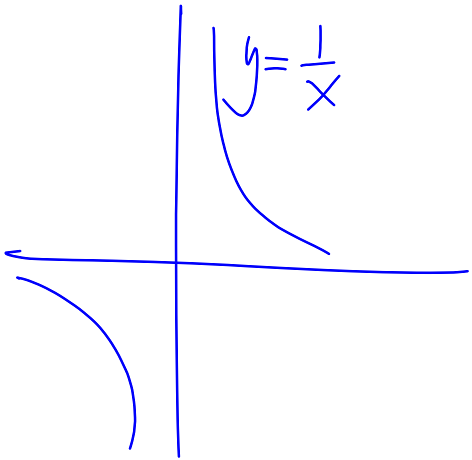
$$y - 4 = -3x$$

$$x = -\frac{y}{3} + \frac{4}{3}$$

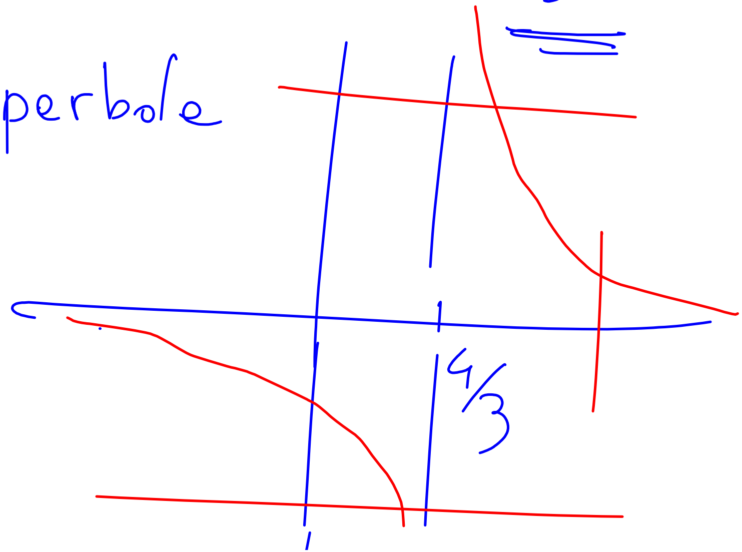
$$y = -\frac{x}{3} + \frac{4}{3}$$

$$y = \frac{2}{3x - 4}$$

dominio : $\mathbb{R} \setminus \left\{ \frac{4}{3} \right\}$



iperbole



$$y = \frac{2}{3x-4}$$

$$y(3x-4) = 2$$

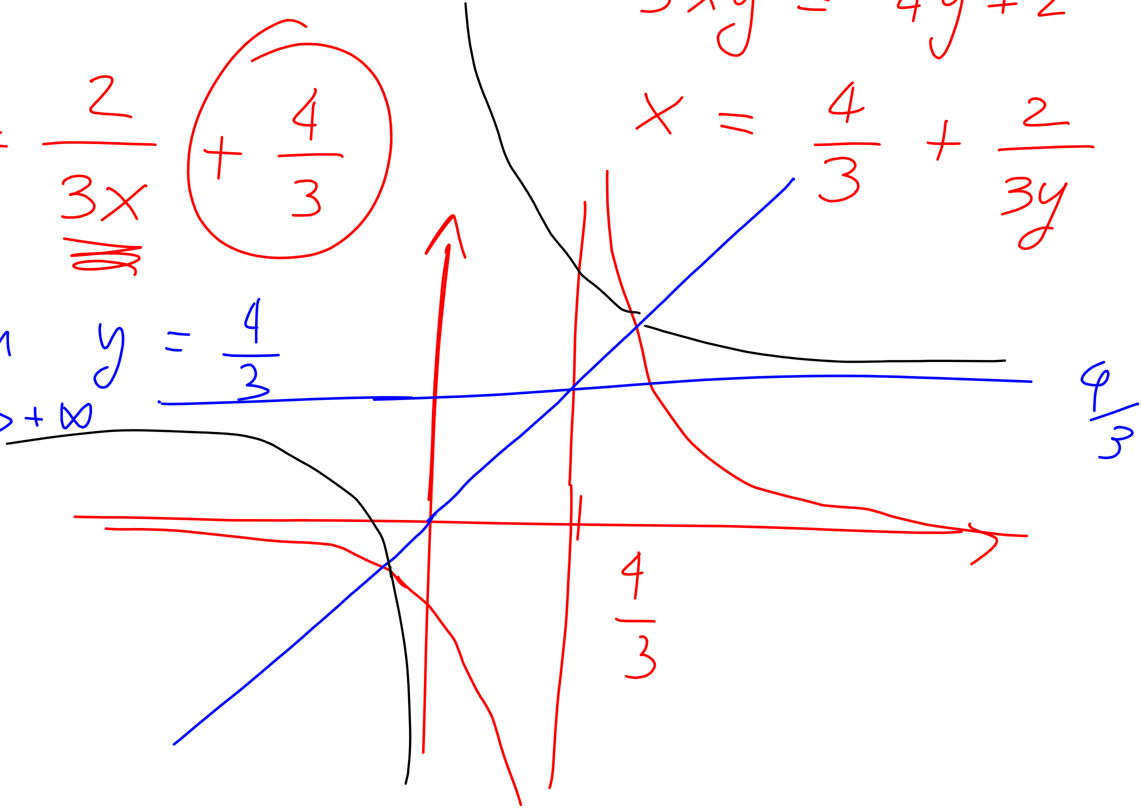
$$3xy - 4y = 2$$

$$3xy = 4y + 2$$

$$x = \frac{4}{3} + \frac{2}{3y}$$

$$y = \frac{2}{3x} + \frac{4}{3}$$

$$\lim_{x \to +\infty} y = \frac{4}{3}$$

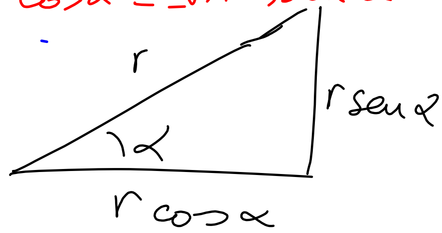


A.M. BIGATTI, L. ROBBIANO

MATEMATICA DI BASE

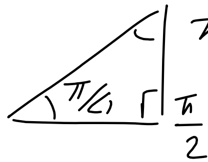
$\text{sen}^2\alpha + \cos^2\alpha = 1$

$\cos\alpha = \pm\sqrt{1-\text{sen}^2\alpha}$



	α	$\cos\alpha$		$\text{sen}\,\alpha$
	0	1		0
$30°$	$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$		$\frac{1}{2}$
$45°$	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$=$	$\frac{\sqrt{2}}{2}$
$60°$	$\frac{\pi}{3}$	$\frac{1}{2}$		$\frac{\sqrt{3}}{2}$

→



$\pi - \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$

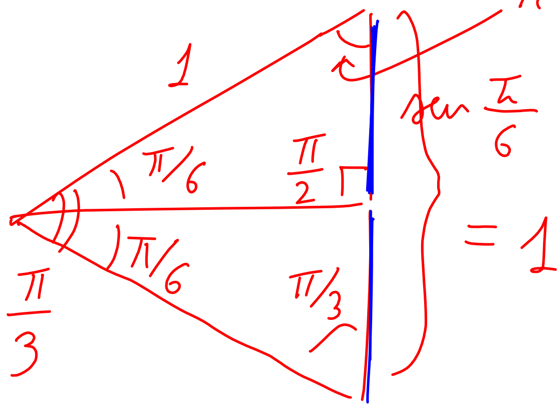
isoscele

$\text{sen}\frac{\pi}{4} = \cos\frac{\pi}{4}$

$\text{sen}^2\frac{\pi}{4} + \cos^2\frac{\pi}{4} = 1$

$2\,\text{sen}^2\frac{\pi}{4} = 1 \qquad \text{sen}\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

La somma degli angoli interni di un triangolo è $180°$ cioè π rad



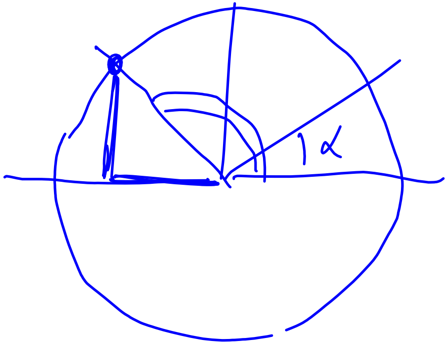
$$\pi - \frac{\pi}{2} - \frac{\pi}{6} = \pi\left(\frac{1}{2} - \frac{1}{6}\right)$$

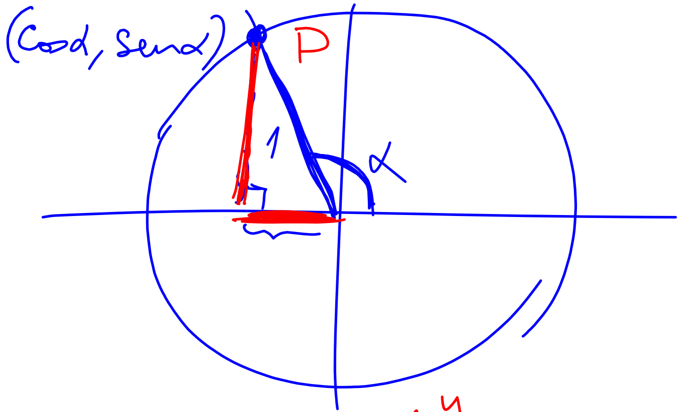
$$= \pi \frac{3-1}{6} = \frac{\pi}{3}$$

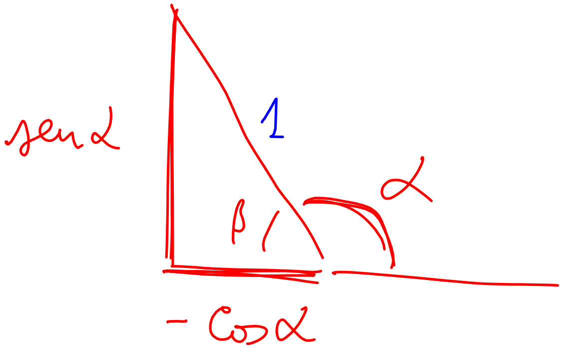
$$1 = 2 \operatorname{sen} \frac{\pi}{6}$$

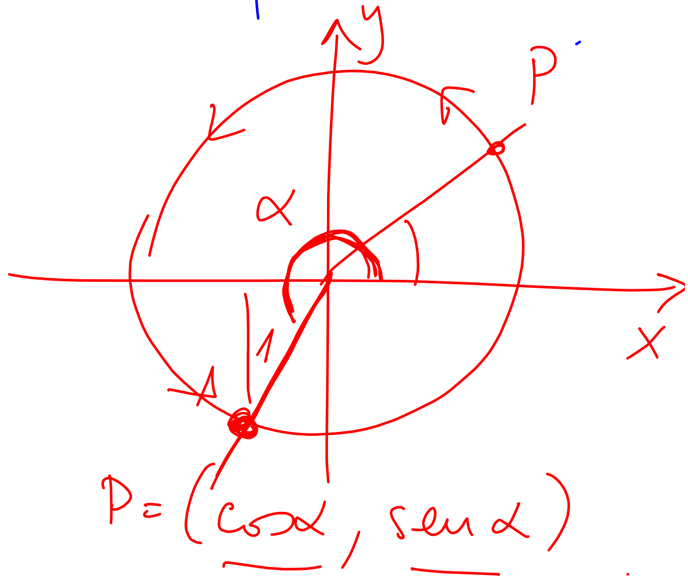
$$\operatorname{sen} \frac{\pi}{6} = \frac{1}{2} = \operatorname{sen} 30^\circ$$

$$\cos \frac{\pi}{6} = \sqrt{1 - \operatorname{sen}^2 \frac{\pi}{6}} = \sqrt{1 - \frac{1}{4}} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$$



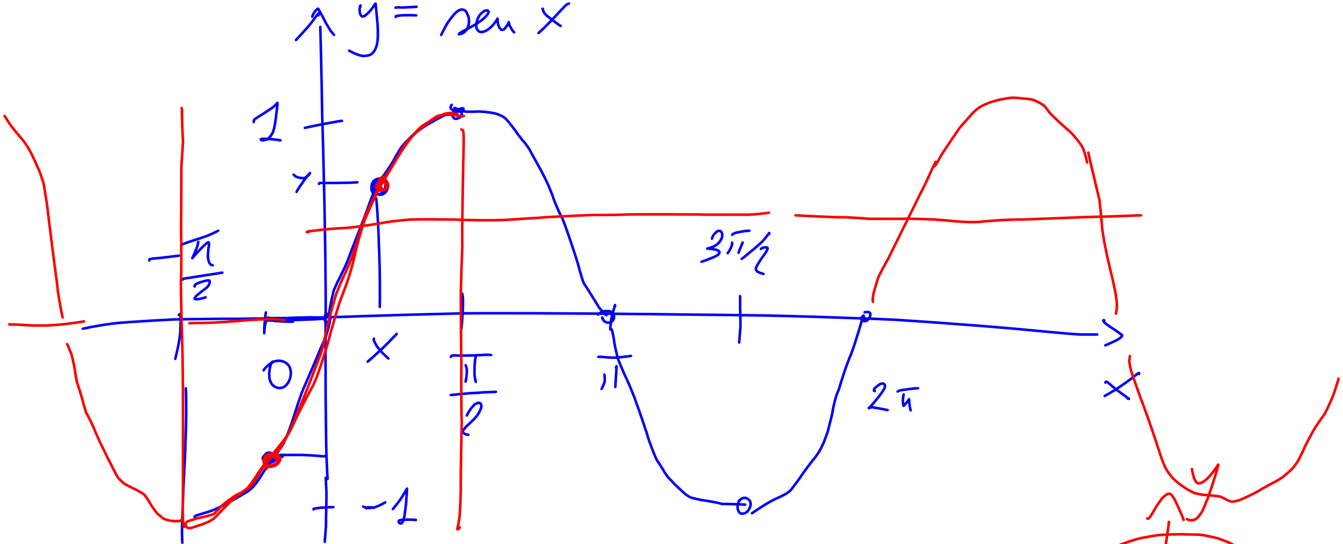


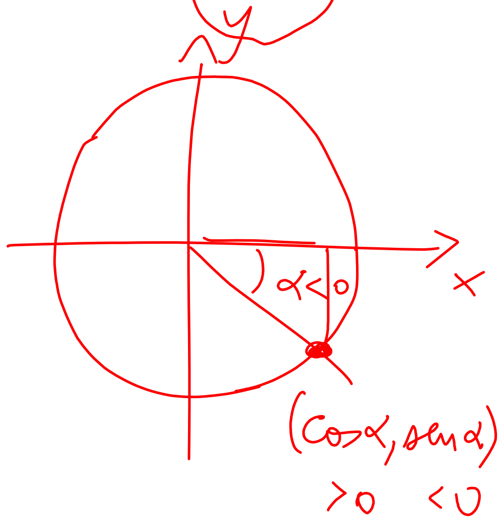




$$-1 \le \operatorname{sen} \alpha \le 1$$

$$-1 \le \cos \alpha \le 1$$





È una funzione dispari

$$f(x) = -f(-x)$$

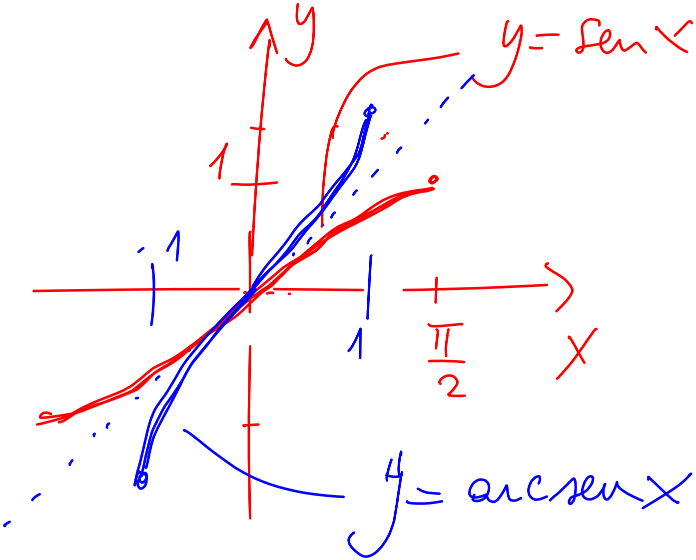
È invertibile tra $-\frac{\pi}{2}$ e $\frac{\pi}{2}$

$y = \text{sen}\, x \qquad x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \qquad y \in [-1, 1]$

dominio codominio

Qui la funzione è invertibile

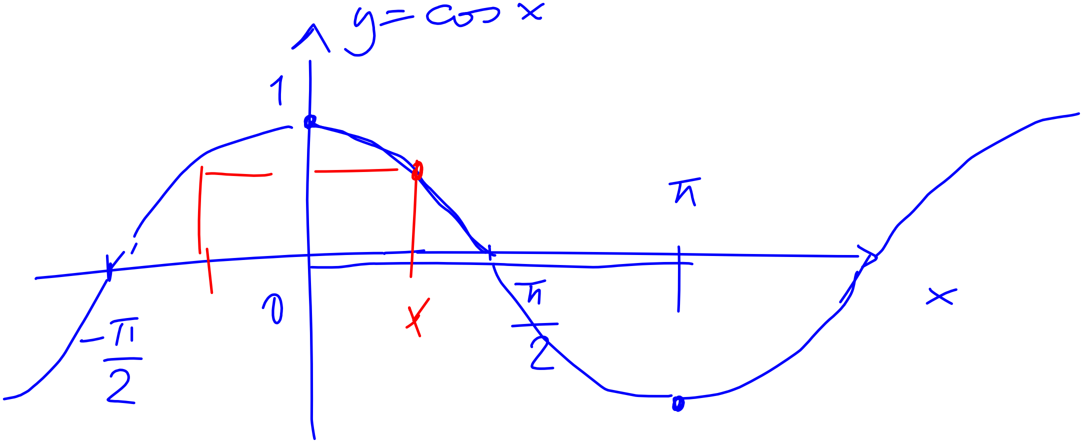
l'inversa si chiama $\text{arcsen} = \text{sen}^{-1}$



$\text{sen}\,\frac{\pi}{2} = 1$

$\frac{\pi}{2} = \frac{3{,}14\ldots}{2} > 1$

$-1 \leq x \leq 1 \qquad -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$



Io posso invertire per $x \in [0, \pi]$

L'inversa si chiama arccos $= \cos^{-1}$

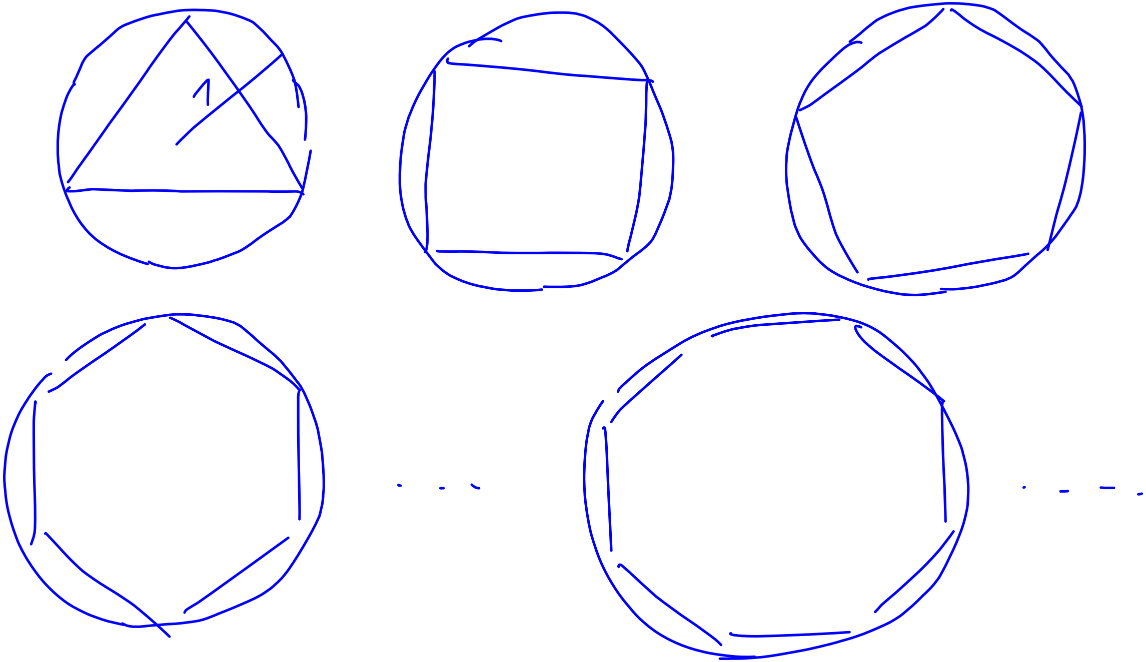
è una funzione pari :

$$f(x) = + f(-x)$$

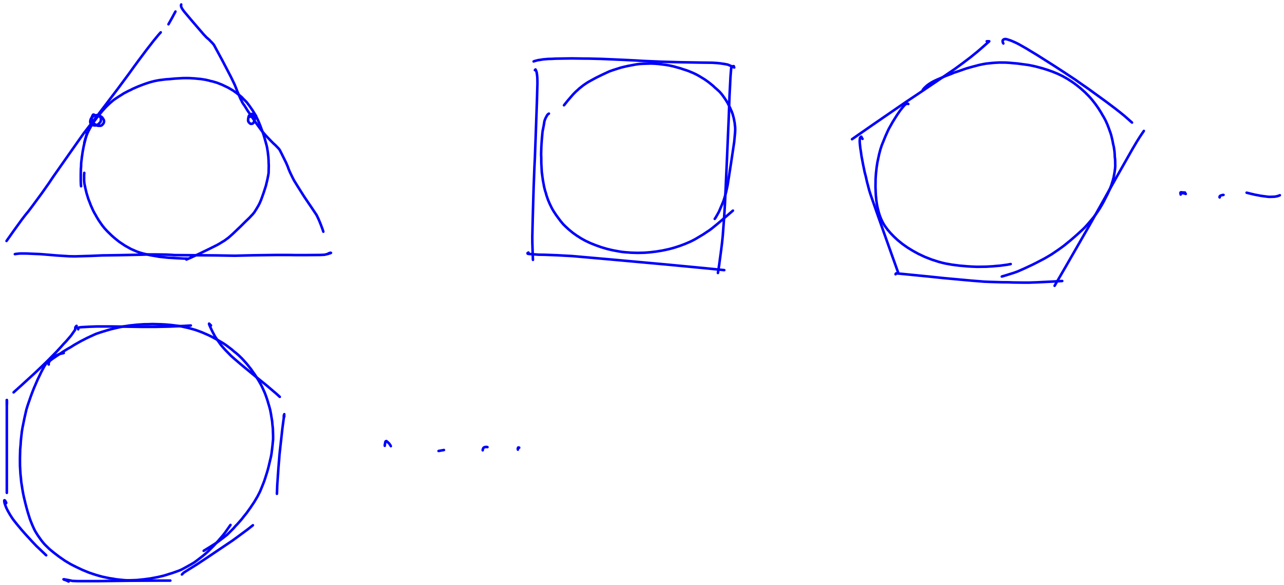
I LIMITI

CALCOLO INFINITESIMALE

ARCHIMEDE



2π



$$\lim_{n\to\infty} \frac{1}{n} = 0 \qquad 1 \quad \frac{1}{2} \quad \frac{1}{3} \quad \frac{1}{4} \quad \dots$$

$$\dots \quad \frac{1}{10} \quad \frac{1}{100} \quad \frac{1}{1000} \quad \dots$$

$$\lim_{n\to\infty} a_n = \ell$$

$$\forall \varepsilon > 0 \quad \exists N \ / \ \forall n > N \quad \underbrace{|\ell - a_n| < \varepsilon}$$

(ε "piccolo a piacere")

$\to$ $a_n = \frac{1}{n}$ $\quad \ell = 0$:

$$\forall \varepsilon > 0 \quad \exists N \ / \ \forall n > N \quad \left|0 - \frac{1}{n}\right| < \varepsilon$$

$$\forall \varepsilon > 0 \quad \exists N \ / \ \forall n > N \quad \left| 0 - \frac{1}{n} \right| < \varepsilon$$

$$\frac{1}{n} < \varepsilon$$

$$a < b \iff \lambda a < \lambda b$$

se $\lambda > 0$

$$a < b \iff \lambda a > \lambda b$$

se $\lambda < 0$

$$2 < 3 \qquad 4 < 6 \qquad -4 > -6$$

$$\frac{1}{n} < \varepsilon \qquad 1 < \varepsilon n \qquad \frac{1}{\varepsilon} < n$$

Se scelgo $N = \frac{1}{\varepsilon}$

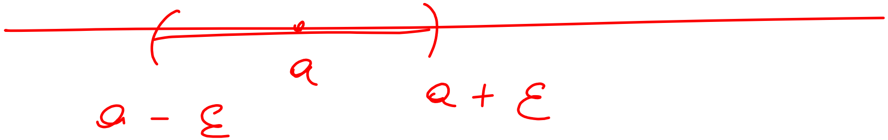
$$\forall n > \frac{1}{\varepsilon} \Rightarrow \frac{1}{n} < \varepsilon \qquad (?)$$

Intorni

Si dice intorno di $a \in \mathbb{R}$ un qualunque intervallo aperto che contiene a

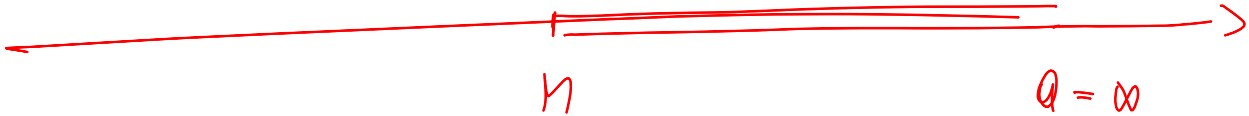
il punto a deve essere dentro (gli estremi sono esclusi)

a è finito ($a \neq \pm\infty$) un intorno è $\overline{}$ $(a-\varepsilon, a+\varepsilon)$ $\varepsilon > 0$



Se $a = +\infty$ un intorno è

$$(M, \infty)$$



Se $a = -\infty$ un intorno è

$$(-\infty, M)$$

Diciamo che una funzione $f(x)$ definita in un intorno di x_0 (ma non necessariamente in x_0) tende a un limite l (che potrebbe essere ∞) per x che tende a x_0

[e si scrive $\lim_{x \to x_0} f(x) = l$]

se $\forall I_l$ intorno di l $\exists I_{x_0}$ intorno di x_0 tale che

$$x \in I_{x_0} \quad x \neq x_0 \implies f(x) \in I_l$$

Consideriamo il caso in cui sia ℓ che x_0 sono finiti

Allora $\lim_{x \to x_0} f(x) = \ell$ se

$\forall \varepsilon > 0 \ \exists \delta > 0 \ /$

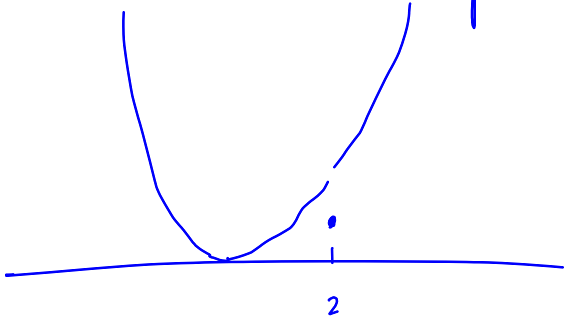
$$x \in (x_0 - \delta, x_0 + \delta), \ x \neq x_0 \implies f(x) \in (\ell - \varepsilon, \ell + \varepsilon)$$

$$\lim_{x \to 1} x^2 = 1 \qquad \lim_{x \to 2} x^2 = 4$$

Se f è continua in x_0 allora $l = f(x_0)$ è il suo limite

$$\lim_{x \to x_0} f(x) = f(x_0)$$

$$y = f(x) = \begin{cases} x^2 & \text{per } x \neq 2 \\ 1 & \text{per } x = 2 \end{cases}$$



$$\lim_{x \to 2} f(x) = 4 \neq f(2)$$

f non è continua in $x = 2$

$$\lim_{x \to \infty} \frac{x+1}{x-1} = 1 \qquad f(x) = \frac{x+1}{x-1}$$

$$\frac{x+1}{x-1} = \frac{x-1+2}{x-1} = \underbrace{1 + \frac{2}{x-1}}$$

$$\forall \varepsilon > 0 \quad \exists M \;/\; \underline{\underline{\qquad\qquad}}_{M}$$

$$\forall x > M \qquad \left| \frac{x+1}{x-1} - 1 \right| < \varepsilon$$

$$\left| \cancel{1} + \frac{2}{x-1} - \cancel{1} \right| < \varepsilon \qquad \frac{2}{x-1} < \varepsilon$$

$$\frac{2}{x-1} < \varepsilon \qquad 2 < \varepsilon(x-1)$$

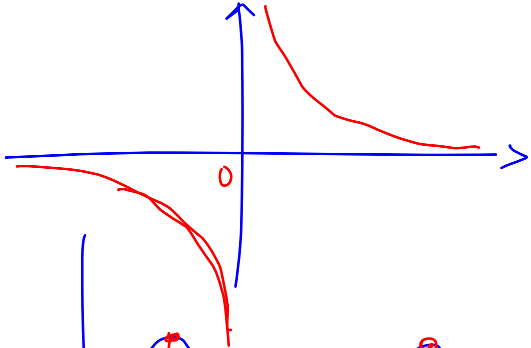
$$\frac{2}{\varepsilon} < x-1 \qquad x > 1+\frac{2}{\varepsilon}$$

Scelgo $M = 1+\frac{2}{\varepsilon}$

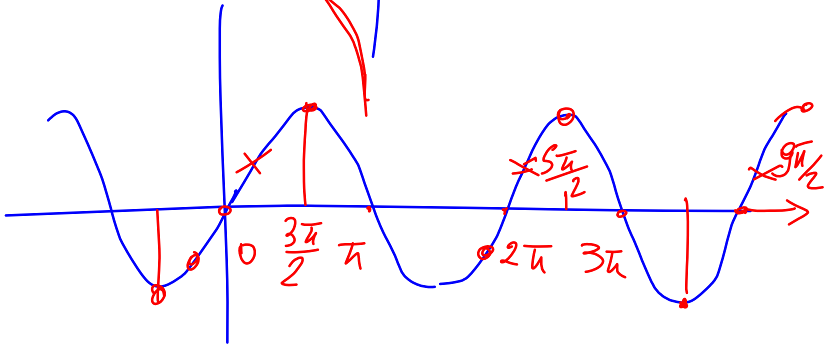
È vero che $\forall x > M$ (cioè $x > 1+\frac{2}{\varepsilon}$)

vale $|f(x)-\ell| < \varepsilon$? Sì

$$\lim_{x \to 0} \frac{1}{x} = ?$$



$$\lim_{x \to \infty} \operatorname{sen} x = ?$$



I limiti non esistono

$$\operatorname{sen}(n\pi) = 0 \quad n \in \mathbb{N}$$

$$\operatorname{sen}\left(\frac{\pi}{2} + 2\pi n\right) = 1 \quad \forall n \in \mathbb{N}$$

$$\operatorname{sen}\left(-\frac{\pi}{2} + 2\pi n\right) = -1 \quad \forall n \in \mathbb{N}$$

Limiti destro e sinistro:

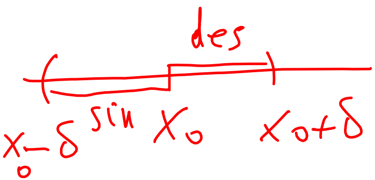
si ottengono restringendo l'intorno di x_0 a un intorno destro o sinistro:

x_0 :

$(x_0 - \delta, x_0 + \delta)$ intorno

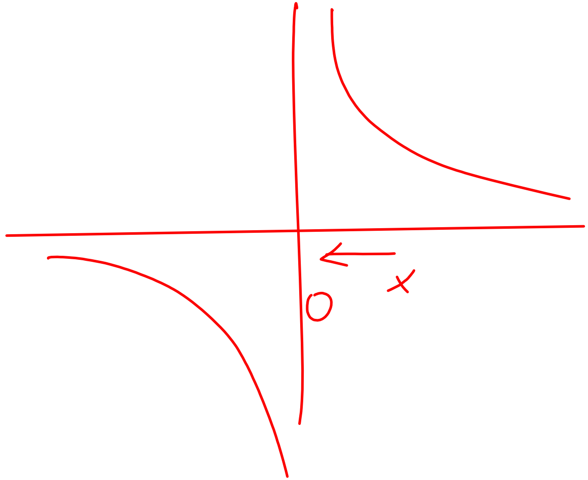
$(x_0, x + \delta)$ intorno destro

$(x_0 - \delta, x_0)$ intorno sinistro



$$\lim_{x \to 0^+} \frac{1}{x} = +\infty$$

limite destro



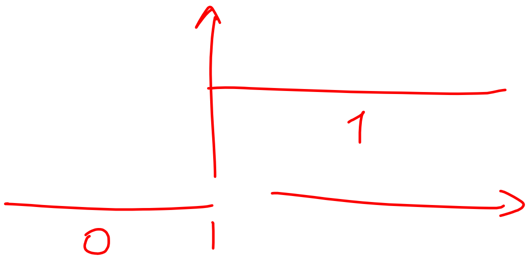
$$\lim_{x \to 0^-} \frac{1}{x} = -\infty$$

$$\frac{1}{x}$$

x	$\frac{1}{x}$
$1/10$	10
$1/100$	100
$\frac{1}{1000}$	1000

Se la funzione è continua in x_0

i limiti destro e sinistro coincidono

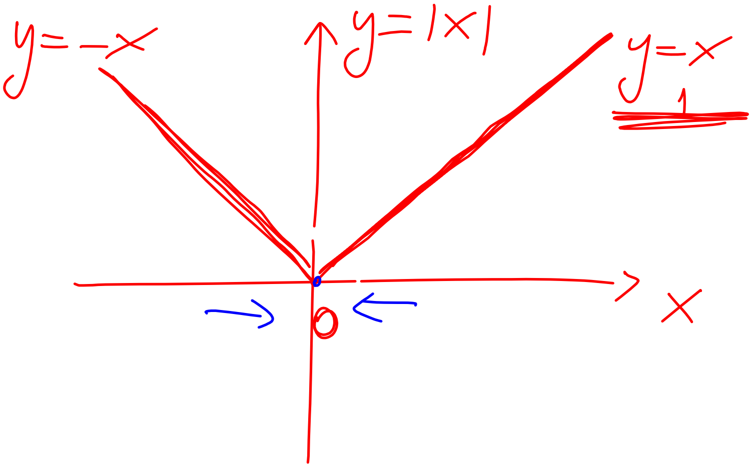


$$\theta(x) = \begin{cases} 1 & \text{per } x > 0 \\ 0 & \text{per } x < 0 \end{cases}$$

$$\lim_{x \to 0^+} \theta(x) = 1 \qquad \lim_{x \to 0^-} \theta(x) = 0$$

Funzione valore assoluto

$$y = f(x) = |x|$$



$$\lim_{x \to 0^+} |x| = 0$$

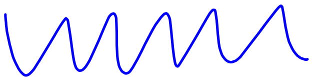
$$\lim_{x \to 0^-} |x| = 0$$

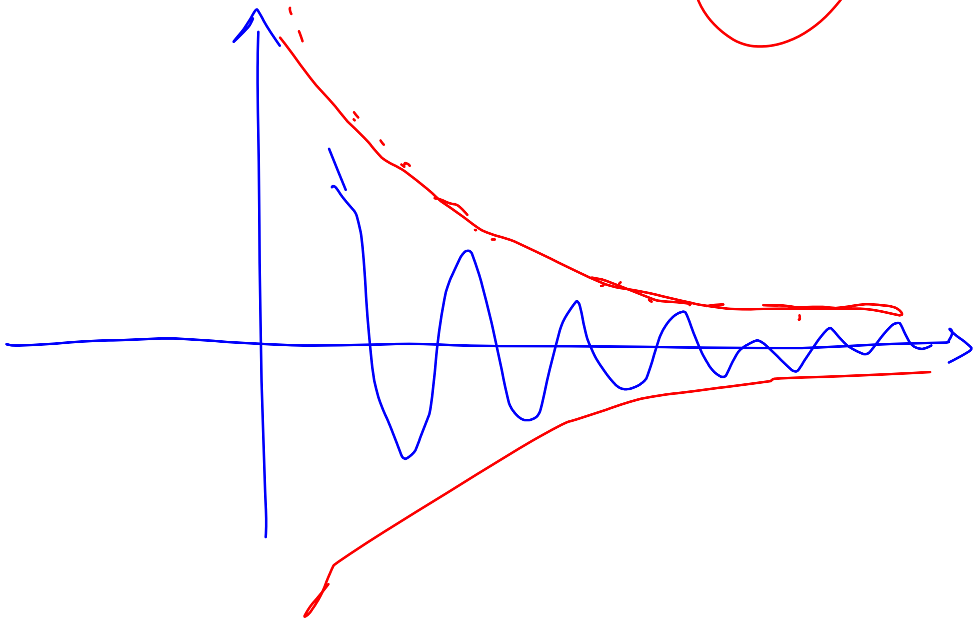
$$-(-2) = |-2| = 2$$

La funzione è continua in 0

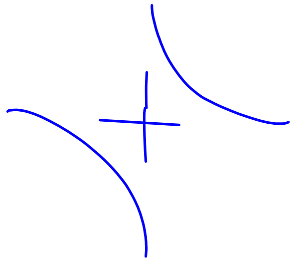
$$\lim_{x \to +\infty} \frac{\operatorname{sen} x}{x} = 0$$

$\operatorname{sen} x$





$\frac{1}{x}$



$$\lim_{x \to \infty} \frac{\operatorname{sen} x}{x} = 0$$

È vero che

$$\forall \varepsilon \ \exists M \ / \ \forall x > M \quad \left| \frac{\operatorname{sen} x}{x} \right| < \varepsilon \ ?$$

$$|\operatorname{sen} x| < 1 \qquad M = \frac{1}{\varepsilon} \ :$$

$$\forall x > M = \frac{1}{\varepsilon} \qquad \frac{|\operatorname{sen} x|}{|x|} < \varepsilon \ ? \quad \text{Sì}$$

$$x > M \qquad 1 > \frac{M}{x}$$

Infatti, $\frac{|\operatorname{sen} x|}{|x|} < \frac{1}{|x|} < \varepsilon$

$$\varepsilon = \frac{1}{M} > \frac{1}{x}$$

- $$\lim_{x \to x_0} f(x) + g(x) = $$

$$= \underbrace{\lim_{x \to x_0} f(x)}_{} + \underbrace{\lim_{x \to x_0} g(x)}_{}$$

se esistono entrambi

- $$\lim_{x \to x_0} f(x) g(x) = \lim_{x \to x_0} f(x) \cdot \lim_{x \to x_0} g(x)$$

[se esistono entrambi]

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{\lim_{x \to x_0} f(x)}{\lim_{x \to x_0} g(x)}$$

se esistono $\lim_{x \to x_0} f(x)$ e $\lim_{x \to x_0} g(x)$

e $\lim_{x \to x_0} g(x) \neq 0$

COMPOSIZIONE

- Se $\lim_{x \to x_0} g(x) = y_0$ e

$f(y)$ è continua in y_0 allora

$$\lim_{x \to x_0} f(g(x)) = f(y_0)$$

- Teorema del confronto (o dei carabinieri)

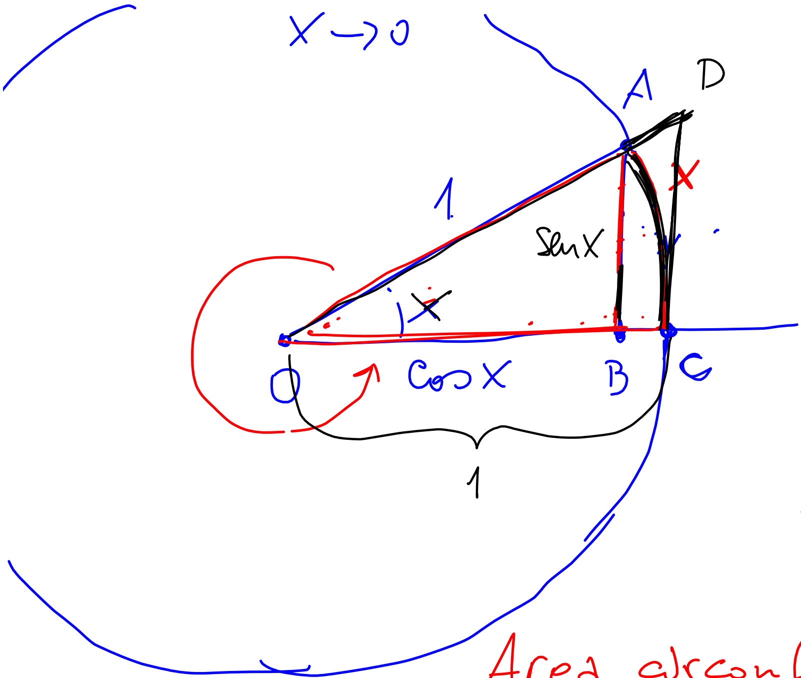
Se $f(x) \leq g(x) \leq h(x) \quad \forall x \in D$

e $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} h(x) = l \quad x_0 \in D$

allora $\lim_{x \to x_0} g(x) = l$

$$\lim_{x \to 0} \frac{\operatorname{sen} x}{x} = 1$$

$$\lim_{x \to 0} \operatorname{sen} x = 0 \qquad \lim_{x \to 0} x = 0$$

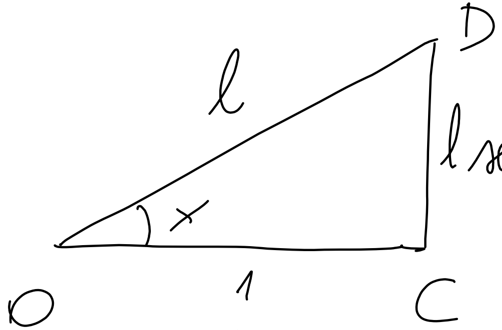


$$\text{Area } \widehat{AOB} = \frac{\cos x \operatorname{sen} x}{2} \leq$$

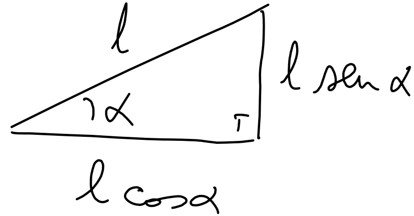
$$\leq \text{Area } AOC = \frac{x}{2\pi}\pi \leq$$

$$\leq \text{Area } \widehat{DOC} = \frac{1}{2}\frac{\operatorname{sen} x}{\cos x}$$

Area circonferenza di raggio $r = \pi r^2$



$l \operatorname{sen} x = h$



$$1 = l \cos x$$

$$l = \frac{1}{\cos x}$$

$$h = \frac{1}{\cos x} \operatorname{sen} x = \operatorname{tg} x$$

$$\text{Area di } \widehat{DOC} = \frac{1}{2} \cdot \frac{\operatorname{sen} x}{\cos x}$$

$$\frac{\cos x \operatorname{sen} x}{\not 2} \le \frac{x}{\not 2 \not \pi} \not\pi \le \frac{1}{\not 2} \frac{\operatorname{sen} x}{\cos x}$$

dividiamo per $\operatorname{sen} x$

$$\cos x \le \frac{x}{\operatorname{sen} x} \le \frac{1}{\cos x}$$

$a \le b \Rightarrow \frac{1}{b} \le \frac{1}{a}$

$$\begin{cases} \cos x \le \dfrac{x}{\operatorname{sen} x} \\[2ex] \dfrac{x}{\operatorname{sen} x} \le \dfrac{1}{\cos x} \end{cases}$$

molt. per $\frac{\operatorname{sen} x}{x \cos x}$

$$\frac{\operatorname{sen} x}{x} \le \frac{1}{\cos x}$$

$$\rightarrow \quad \cos x \le \frac{\operatorname{sen} x}{x}$$

$$f(x) \le g(x) \le h(x)$$

Teorema del confronto

$$\cos x \le \frac{\operatorname{sen} x}{x} \le \frac{1}{\cos x}$$

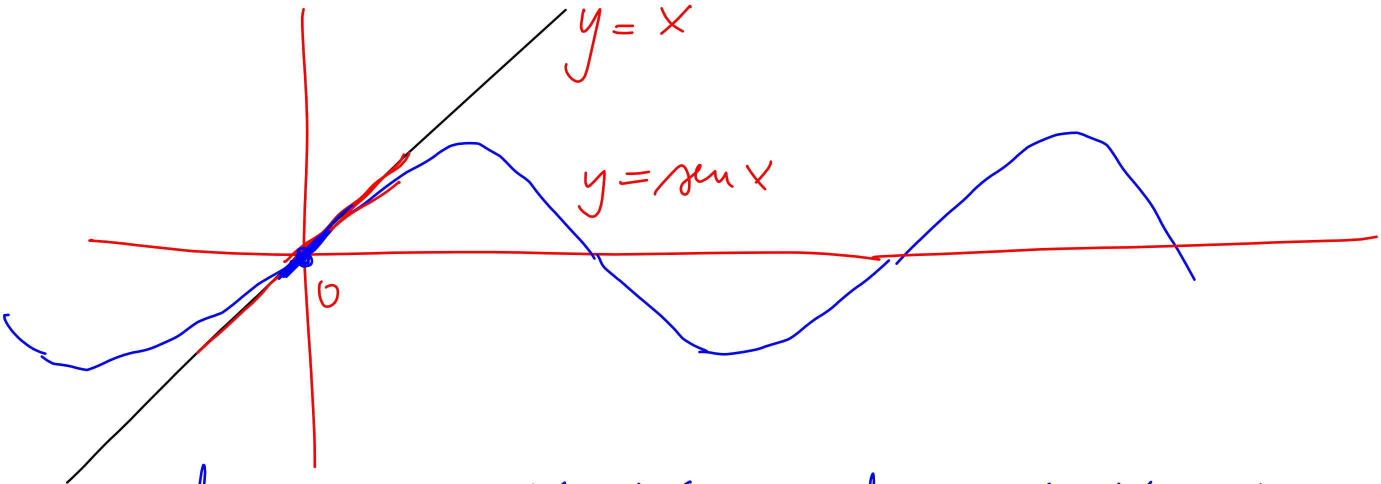
$$\cos x \le \frac{\operatorname{sen} x}{x} \le \frac{1}{\cos x}$$

$\cos x \to 1$ per $x \to 0$

$\frac{1}{\cos x} \to 1$ per $x \to 0$

Quindi

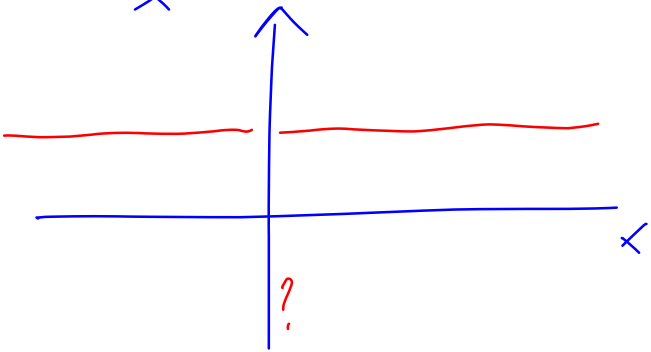
$$\lim_{x \to 0} \frac{\operatorname{sen} x}{x} = 1$$



$$\lim_{x \to 0} \frac{\operatorname{sen} x - x}{x} = \lim_{x \to 0} \frac{\operatorname{sen} x}{x} - \frac{x}{x} = 0$$

$$1 - 1$$

$$\frac{x}{x} = 1 \quad \forall x \neq 0 \qquad \frac{x}{x} = ? \text{ per } x = 0$$



$$\lim_{x \to 0} \frac{x}{x} = 1$$

Infinitesimo — Diciamo che $f(x)$ è infinitesima per $x \to x_0$ se $\lim_{x \to x_0} f(x) = 0$

Se $\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0$ e $\lim_{x \to x_0} f(x) = 0$ e $\lim_{x \to x_0} g(x) = 0$, allora si dice che f è un infinitesimo di ordine superiore a g : $f = o(g)$ f è o piccolo di g

Esempi $f(x) = x^2$ $g(x) = x$

$x \to 0$ $\lim_{x \to 0} f(x) = 0$ $\lim_{x \to 0} g(x) = 0$

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = 0$$

x	$f(x)$	$g(x)$
$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{100}$
$\frac{1}{100}$	$\frac{1}{100}$	$\frac{1}{10000}$
$\frac{1}{100}$	$\frac{1}{1000}$	$\frac{1}{1000000}$

$$\lim_{x \to 0} \frac{\operatorname{sen} x - x}{x} = 0$$

$$f(x) = \operatorname{sen} x - x \quad \to 0$$

$$g(x) = x \quad \to 0$$

$\operatorname{sen} x - x =$ inf. di ord. superiore a x

$$= o(x)$$

$$\operatorname{sen} x - x = o(x)$$

$$\boxed{\operatorname{sen} x = x + o(x)}$$

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$$

$$(1-\cos x)(1+\cos x) = 1-\cos^2 x$$
$$= \sin^2 x$$

$$\frac{1-\cos x}{x^2}\,\frac{1+\cos x}{1+\cos x} =$$

$$= \frac{\sin^2 x}{x^2}\,\frac{1}{1+\cos x} = \left(\frac{\sin x}{x}\right)^2 \frac{1}{1+\cos x}$$

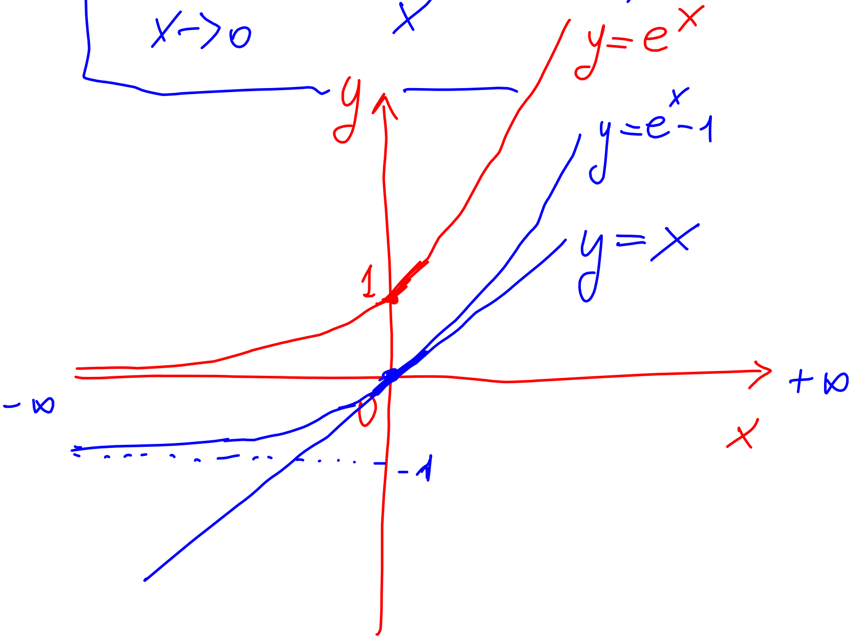
$$\lim_{x\to 0} \frac{x}{\sin x} = 1$$

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

$$\lim_{x \to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$$

$$\boxed{\lim_{x \to 0} \frac{e^x - 1}{x} = 1}$$

$$\lim_{x \to 0} \frac{\ln(1+x)}{x} = 1$$



$$\lim_{x \to \infty} e^x = \infty$$

$$\lim_{x \to -\infty} e^x = 0$$

$$\lim_{x\to 0} \frac{\ln(1+x)}{x} = \lim_{t\to 0} \frac{t}{e^t-1} = \lim_{t\to 0} \frac{1}{\frac{e^t-1}{t}} = 1$$

$$\boxed{1+x = e^t} \to \boxed{\ln(1+x) = t} \qquad x = e^t - 1$$

$$\ln e^t = t$$

$$\lim_{x\to 0} t(x) = 0$$

$$\log_e e^t = t$$

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{\lim_{x\to x_0} f(x)}{\lim_{x\to x_0} g(x)}$$

se esistono entrambi e il denominatore è non nullo

$$\lim_{x\to\infty} \frac{5x^2+4x+2}{2x^2-5x+7} \quad \frac{\infty}{\infty} = \frac{5}{2}$$

$$= \lim_{x\to\infty} \frac{5+\frac{4}{x}+\frac{2}{x^2}}{2-\frac{5}{x}+\frac{7}{x^2}} = \frac{5}{2}$$

$$\lim_{x\to\infty} \frac{x^3-2x^2+5x+1}{x^2-4x+5} = \lim_{x\to\infty} \frac{x^3}{x^2} = \infty$$

$$= \lim_{x\to\infty} \frac{x-2+\frac{5}{x}+\frac{1}{x^2}}{1-\frac{4}{x}+\frac{5}{x^2}} = \lim_{x\to\infty} \frac{x\left(1-\frac{2}{x}+\frac{5}{x^2}+\frac{1}{x^3}\right)}{1-\frac{4}{x}+\frac{5}{x^2}}$$

$\to \infty \leftarrow x$; $\left(1-\frac{2}{x}+\frac{5}{x^2}+\frac{1}{x^3}\right) \to 1$; $1-\frac{4}{x}+\frac{5}{x^2} \to 1$

Per $x \to 0$ x^3 è un infinitesimo di grado superiore a x^2

$$\lim_{x \to 0} x^3 = 0 \qquad \lim_{x \to 0} x^2 = 0$$

$$\lim_{x \to 0} \frac{x^3}{x^2} = 0$$

Per $x \to \infty$ x^3 è un infinito di grado superiore a x^2, cioè:

$$\lim_{x \to \infty} x^3 = \infty \qquad \lim_{x \to \infty} x^2 = \infty$$

$$\lim_{x \to \infty} \frac{x^3}{x^2} = \infty$$

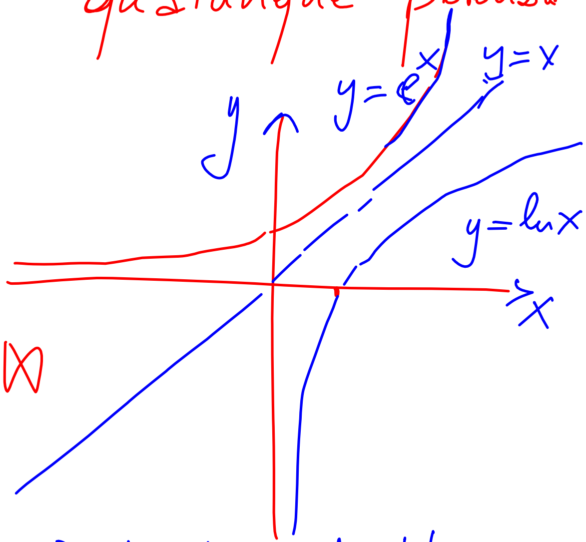
$$\lim_{x \to \infty} \frac{e^x}{x^n} = \infty$$

la crescita esponenziale è più veloce della crescita di qualunque potenza

$n = 1, 2, \ldots$

$e^x = t \qquad x = \ln t$

$$\lim_{t \to \infty} \frac{t}{(\ln t)^n} = \infty$$



la crescita logaritmica è più lenta della crescita a potenza

$$\lim_{x \to 0} \frac{\operatorname{sen} 3x}{x} = \lim_{x \to 0} 3 \cdot \left(\frac{\operatorname{sen} 3x}{3x} \right) = 3$$

$\left(\frac{\operatorname{sen} 3x}{3x} \right) \to 1$

$$\lim_{x \to 0} \frac{\operatorname{tg}(5x)}{x} = \lim_{x \to 0} \left(\frac{\operatorname{sen} 5x}{5x} \right) \frac{5}{\cos 5x} = 5$$

$\left(\frac{\operatorname{sen} 5x}{5x} \right) \to 1 \qquad \frac{5}{\cos 5x} \to 5$

$$\lim_{x \to 0} \frac{\operatorname{sen} x}{x} = 1 = \lim_{t \to 0} \frac{\operatorname{sen} 3t}{3t} = 1$$

$x = 3t$

Per $\alpha \sim 0$ $\quad \operatorname{sen} \alpha \sim \alpha$ $\quad \operatorname{tg} \alpha = \frac{\operatorname{sen} \alpha}{\cos \alpha}$

$\operatorname{tg} \alpha \sim \alpha$

$$\lim_{x\to\infty}\left(\underbrace{\frac{x^2-1}{x+2}}_{\to\infty}-\underbrace{x}_{\infty}\right)="\infty-\infty"=$$

$$=\lim_{x\to\infty}\frac{x^2-1-x(x+2)}{x+2}=$$

$$=\lim_{x\to\infty}\frac{\cancel{x^2}-1-\cancel{x^2}-2x}{x+2}=$$

$$=\lim_{x\to\infty}\frac{-2x-1}{x+2}=-2.$$

$$\lim_{x\to 0} \frac{\text{sen } 3x}{\text{tg } 5x} = \lim_{x\to 0} \frac{\text{sen } 3x}{\text{sen } 5x} (\cos 5x) =$$

$\hookrightarrow 1$

$$= \lim_{x\to 0} \left(\frac{\text{Sen } 3x}{3x}\right) \frac{3x}{\ } \left(\frac{5x}{\text{sen } 5x}\right) \frac{1}{5x} \cos 5x =$$

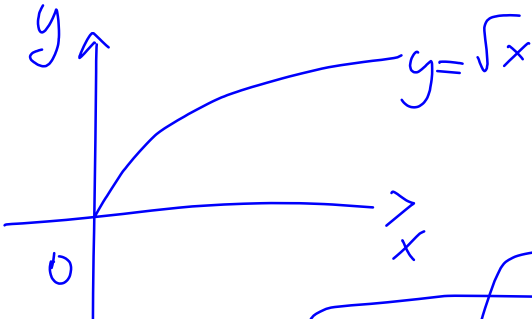
$$= \lim_{x\to 0} \left(\frac{\text{Sen } 3x}{3x}\right)\left(\frac{5x}{\text{sen } 5x}\right) \frac{3\cancel{x}}{5\cancel{x}} \cos 5x = \frac{3}{5}$$

$\hookrightarrow 1$ $\quad \hookrightarrow 1$ $\quad \hookrightarrow \frac{3}{5}$

$$\lim_{x \to \infty} \left(\sqrt{x+1} - \sqrt{x} \right)$$

$\infty - \infty$

$(a+b)(a-b) = a^2 - b^2$ ~~$-ab$~~ ~~$+ba$~~



moltiplicare e dividere per $\sqrt{x+1} + \sqrt{x}$

$$\lim_{x \to \infty} \frac{\left(\sqrt{x+1} - \sqrt{x}\right)\left(\sqrt{x+1} + \sqrt{x}\right)}{\sqrt{x+1} + \sqrt{x}} =$$

$$= \lim_{x \to \infty} \frac{x+1-x}{\sqrt{x+1} + \sqrt{x}} = \lim_{x \to \infty} \frac{1}{\sqrt{x+1} + \sqrt{x}} = 0$$

$$\lim_{x \to e} \frac{\ln(x) - 1) \to 0}{x^2 - x(1+e) + e} \to e^2 - e(1+e) + e$$

$$\ln e = \log_e e = 1$$

$$x^2 - x(1+e) + e = (x-e)(x+u) =$$

$$= \cancel{x^2} - x - e\cancel{x} + e = \cancel{x^2} - e\cancel{x} - eu + ux$$

$$e - x = u(x-e) \qquad u - 1$$

$$x^2 - x(1+e) + e = (x-e)(x-1)$$

$$\lim_{x \to e} \frac{\ln x - 1}{(x-e)(x-1)}$$

$t = x - e \qquad x = t + e$

$$\lim_{t \to 0} \frac{\overbrace{\ln(e+t)} - 1}{t(t+e-1)} \quad \parallel$$

$$\lim_{x \to 0} \frac{\ln(1+x)}{x} = 1 \qquad \ln ab = \ln a + \ln b$$

$$\ln(e+t) = \ln\left[\left(1+\frac{t}{e}\right)e\right] = \ln\left(1+\frac{t}{e}\right) + 1$$

$$\lim_{t \to 0} \frac{\ln\left(1+\frac{t}{e}\right)}{e\,\frac{t}{e}\,(t+e-1)} =$$

$$\underbrace{(t+e-1)}_{\to\, e-1}$$

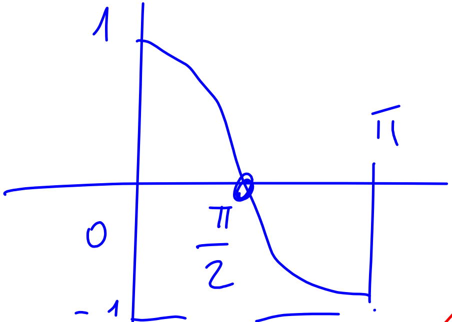
$$\lim_{x \to 0} \frac{\ln(1+x)}{x} = 1$$

$$= \lim_{t \to 0} \frac{\ln\left(1+\frac{t}{e}\right)}{\frac{t}{e}} \, \frac{1}{e} \, \frac{1}{t+e-1} =$$

$$x = \frac{t}{e}$$

$$= \frac{1}{e(e-1)}$$

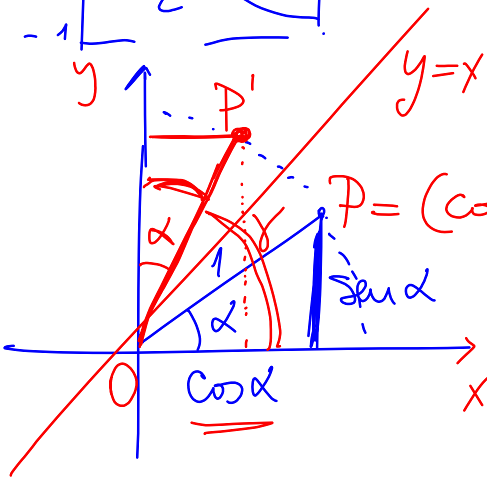
$$\lim_{x \to \frac{\pi}{2}} \frac{\frac{\pi}{2} - x}{\cos x} \qquad \frac{0}{0}$$



$$\text{sen}(\alpha+\beta) = \text{sen}\,\alpha\,\cos\beta + \cos\alpha\,\text{sen}\,\beta$$

$$\alpha + \gamma = \frac{\pi}{2}$$

$$\gamma = \frac{\pi}{2} - \alpha$$



$P = (\cos\alpha, \text{sen}\,\alpha)$

$P' = (\cos\gamma, \text{sen}\,\gamma)$

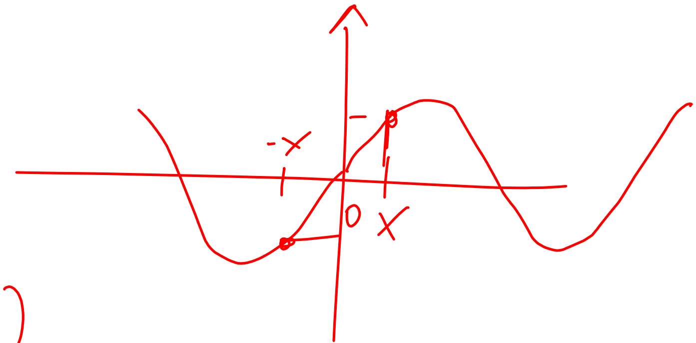
$P' = (\text{sen}\,\alpha, \cos\alpha)$

$$\begin{cases} \operatorname{sen}\left(\frac{\pi}{2}-\alpha\right) = \cos\alpha \\ \cos\left(\frac{\pi}{2}-\alpha\right) = \operatorname{sen}\alpha \end{cases}$$

$$\operatorname{sen}(\alpha+\beta) = \operatorname{sen}\alpha\cos\beta + \cos\alpha\operatorname{sen}\beta$$

$$\operatorname{sen}\left(\alpha-\frac{\pi}{2}\right) = -\operatorname{sen}\left(\frac{\pi}{2}-\alpha\right)$$

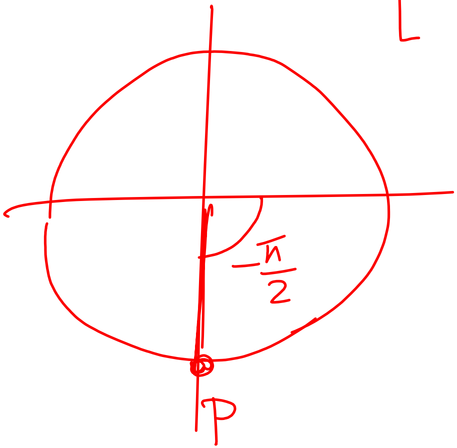
$$\beta = -\frac{\pi}{2}$$



$$\operatorname{sen}(x) = -\operatorname{sen}(-x)$$

$$\operatorname{sen}\left(\frac{\pi}{2}-\alpha\right) = -\operatorname{sen}\Big(\alpha \underbrace{-\frac{\pi}{2}}_{\beta}\Big) =$$

$$= -\Big[\operatorname{sen}\alpha \underbrace{\cos\frac{-\pi}{2}}_{=0} + \cos\alpha \underbrace{\operatorname{sen}\frac{-\pi}{2}}_{-1}\Big]$$



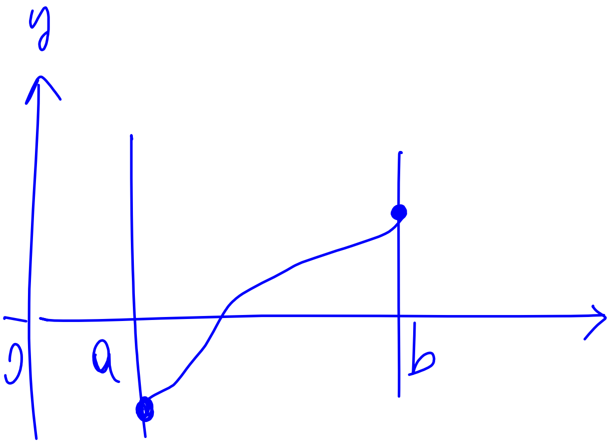
$$= \cos\alpha$$

$$t = \frac{\pi}{2} - x \qquad x = \frac{\pi}{2} - t$$

$$\lim_{x \to \frac{\pi}{2}} \frac{\frac{\pi}{2} - x}{\cos x} = \lim_{t \to 0} \frac{t}{\operatorname{sen} t} = 1$$

Esistenza degli zeri

Se f è continua in $[a, b]$ e ha valori di segno opposto agli estremi, allora $\exists\, x_0 \in [a, b]$ dove $f(x_0) = 0$

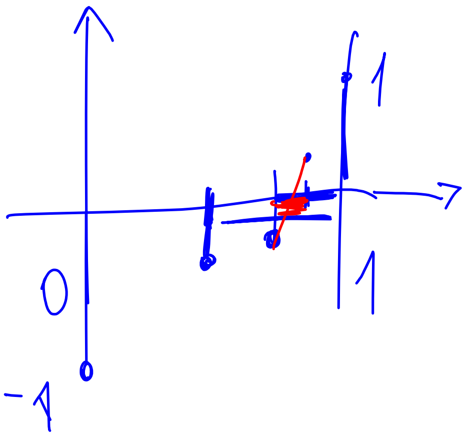


Sia $f(x) = x^5 + x - 1$

$$x^5 + x - 1 = 0$$

$$ax^2 + bx + c = 0 \qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

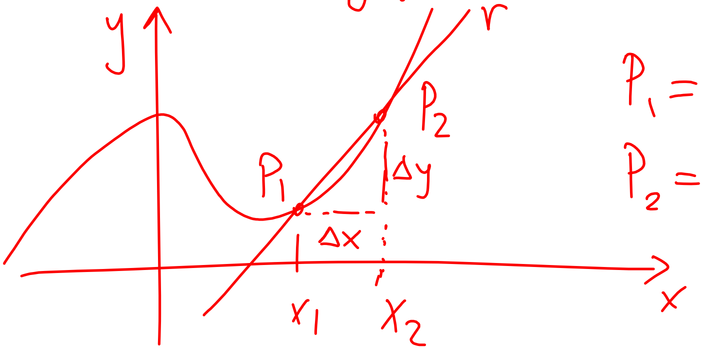
$f(0) = -1$ $\qquad f(1) = 1$



$$f\left(\frac{1}{2}\right) = \frac{1}{2^5} + \frac{1}{2} - 1 = \frac{1}{2^5} - \frac{1}{2}$$

$$= \frac{1}{2}\left(\frac{1}{2^4} - 1\right) < 0$$

Derivate



$$P_1 = (x_1, f(x_1))$$

$$P_2 = (x_2, f(x_2))$$

$$\Delta y = f(x_2) - f(x_1)$$

$$\Delta x = x_2 - x_1$$

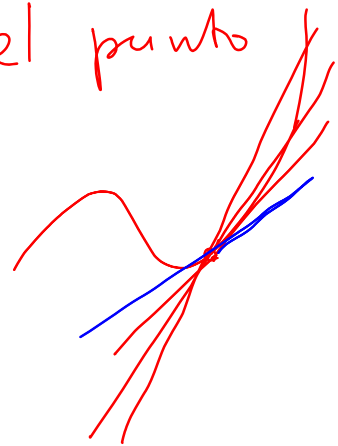
$$r: \quad y - y_1 = m(x - x_1)$$

$$m = \frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$r: \quad y = y_1 + \frac{f(x_2) - f(x_1)}{x_2 - x_1} \quad x - x_1$$

La derivata in P_1 è il limite di m per $P_2 \to P_1$

cioè la pendenza della funzione nel punto P_1



$$\lim_{X_2 \to X_1} \frac{f(X_2) - f(X_1)}{X_2 - X_1}$$

$X_2 = X_1 + \Delta X$

$X_1 \longrightarrow X$

rinominiamo

$$\lim_{\Delta X \to 0} \frac{f(X + \Delta X) - f(X)}{\Delta X} \equiv f'(X) \equiv \frac{df(X)}{dX}$$

"de f su de x"

derivata della funzione $f(x)$ nel punto x

Esempi $f(x) = c$ = costante

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{c - c}{\Delta x} = \lim_{\Delta x \to 0} 0 = 0$$

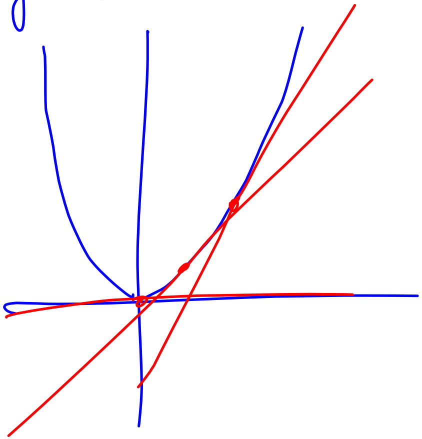
$$f(x) = ax + b$$

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{a(x+\Delta x) + b - (ax+b)}{\Delta x} =$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{ax} + a\Delta x + \cancel{b} - \cancel{ax} - \cancel{b}}{\Delta x} = a$$

la derivata di una retta è il suo coefficiente angolare

$f(x) = x^2$



$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - x^2}{\Delta x} =$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{x^2} + 2x\Delta x + \Delta x^2 - \cancel{x^2}}{\Delta x} = \lim_{\Delta x \to 0} 2x + \Delta x =$$

$$= 2x$$

$f'(x) = 2x$ $\quad f'(0) = 0 \quad f'(1) = 2 \quad f'(2) = 4$

$f'(-1) = -2 \quad \ldots$

$f(x) = x^n$ $\quad n$ intero > 0

$$f'(x) = \lim_{x \to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x}$$

$$(a+b)^n = a^n b^0 + \binom{n}{1} a^{n-1} b^1 + \dots \binom{n}{k} a^{n-k} b^k + \dots$$

$$k = 0, 1, \dots n$$

$$(x+\Delta x)^n = x^n + n x^{n-1} \Delta x + (\dots) \Delta x^2 + (\dots) \Delta x^3 + \dots$$

$$\frac{(x+\Delta x)^n - x^n}{\Delta x} = n x^{n-1} + \underbrace{(\dots)\Delta x}_{\to 0} + \underbrace{(\dots)\Delta x^2}_{\to 0} + \dots$$

$$f(x) = x^n \implies f'(x) = n x^{n-1}$$

$$\frac{dx^n}{dx} = n x^{n-1}$$

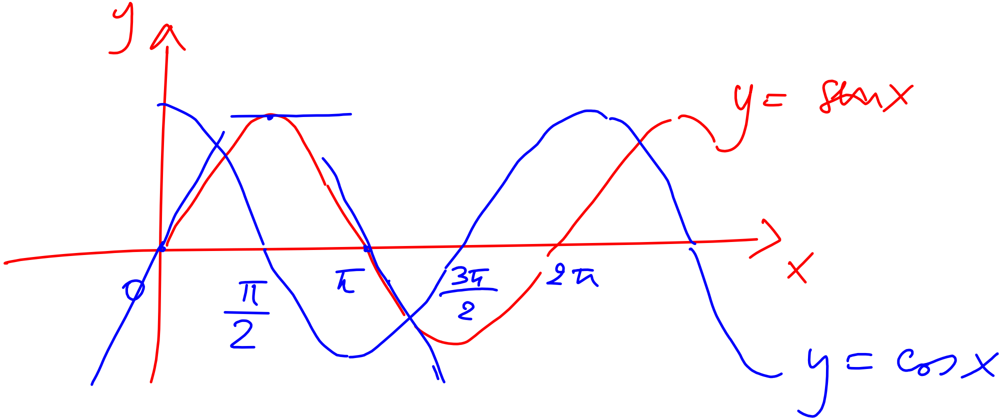
$f(x) = \operatorname{sen} x$ $\qquad \operatorname{sen}(\alpha+\beta) = \operatorname{sen}\alpha \cos\beta + \cos\alpha \operatorname{sen}\beta$

$$f'(x) = \lim_{\Delta x \to 0} \frac{\operatorname{sen}(x+\Delta x) - \operatorname{sen} x}{\Delta x} = \lim_{\Delta x \to 0} \frac{1}{\Delta x}\Big[\operatorname{sen} x \cos\Delta x$$

$$+ \cos x \operatorname{sen}\Delta x - \operatorname{sen} x\Big] = \lim_{x \to 0} \underbrace{\left(\frac{\operatorname{sen}\Delta x}{\Delta x}\right)}_{\to 1} \cos x +$$

$$+ \lim_{\Delta x \to 0} \underbrace{\frac{\cos\Delta x - 1}{\Delta x^2}}_{\to -\frac{1}{2}} \underbrace{\Delta x}_{\to 0} \operatorname{sen} x = \cos x$$

$$\lim_{\alpha \to 0} \frac{\operatorname{sen}\alpha}{\alpha} = 1 \qquad \lim_{\alpha \to 0} \frac{1-\cos\alpha}{\alpha^2} = \frac{1}{2} \qquad \frac{d \operatorname{sen} x}{dx} = \cos x$$



cos x è il coefficiente angolare della retta tangente a sen x in x

$$\frac{d \cos x}{dx} = -\operatorname{sen} x$$

$$\cos(\alpha+\beta) = \cos\alpha \cos\beta + \\ - \operatorname{sen}\alpha \operatorname{sen}\beta$$

$$f(x) = e^x \qquad \frac{de^x}{dx} = e^x \qquad a^{b+c} = a^b a^c$$

$$\lim_{\Delta x \to 0} \frac{e^{x+\Delta x} - e^x}{\Delta x} = \lim_{\Delta x \to 0} \frac{e^x e^{\Delta x} - e^x}{\Delta x} =$$

$$= e^x \lim_{\Delta x \to 0} \frac{e^{\Delta x} - 1}{\Delta x} = e^x \qquad \lim_{t \to 0} \frac{e^t - 1}{t} = 1$$

$$f(x) = \ln x \qquad \frac{d \ln x}{dx} = \frac{1}{x} \qquad \ln(ab) = \ln a + \ln b$$

$$\lim_{\Delta x \to 0} \frac{\ln(x+\Delta x) - \ln x}{\Delta x} = \lim_{\Delta x \to 0} \frac{\overbrace{\ln\left(x\left(1+\frac{\Delta x}{x}\right)\right)} - \ln x}{\Delta x} =$$

$$= \lim_{\Delta x \to 0} \frac{\ln x + \ln\left(1+\frac{\Delta x}{x}\right) - \ln x}{\Delta x} = \lim_{\Delta x \to 0} \frac{\ln\left(1+\frac{\Delta x}{x}\right)}{\frac{\Delta x}{x}} \frac{1}{x} = \frac{1}{x}$$

usando $\lim_{t \to 0} \frac{\ln(1+t)}{t} = 0$

$$\frac{d(f(x) + g(x))}{dx} = \frac{df(x)}{dx} + \frac{dg(x)}{dx}$$

$k =$ costante $\Rightarrow$

$$\frac{d}{dx}(k f(x)) = k \frac{df}{dx}$$

Regola di Leibniz:

$$\frac{d}{dx}(f(x)\, g(x)) = f'(x)\, g(x) + f(x)\, g'(x)$$

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x)\, g(x+\Delta x) - f(x) g(x)}{\Delta x} =$$

$$= \lim_{\Delta x \to 0} \underbrace{\frac{\left(f(x+\Delta x) - f(x)\right)}{\Delta x}}_{\to f'(x)} \underbrace{g(x+\Delta x)}_{\to g(x)} + f(x) \underbrace{\frac{\left(g(x+\Delta x) - g(x)\right)}{\Delta x}}_{\to g'(x)}$$

$$= f'(x)\, g(x) + f(x)\, g'(x)$$

Derivata del quoziente:

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)\, g(x) - g'(x) f(x)}{g^2(x)}$$

Derivata della funzione composta:

$$y = f(x) \qquad x = g(t)$$

$$y = f(g(t)) \qquad \frac{dy}{dt} = \frac{dy}{dx}\,\frac{dx}{dt} =$$

$$= f'(g(t))\, g'(t)$$

$$\lim_{\Delta t \to 0} \frac{f(g(t+\Delta t)) - f(g(t))}{\Delta t} =$$

$$= \lim_{\Delta t \to 0} \frac{f((g(t+\Delta t)) - f(g(t))}{g(t+\Delta t) - g(t)} \; \frac{g(t+\Delta t) - g(t)}{\Delta t}$$

$$\frac{g(t+\Delta t) - g(t)}{\Delta t} \longrightarrow g'(t)$$

$$\lim_{\Delta t \to 0} \frac{f(g(t+\Delta t)) - f(g(t))}{g(t+\Delta t) - g(t)} = \lim_{\Delta g \to 0} \frac{f(g+\Delta g) - f(g)}{\Delta g}$$

$g(t+\Delta t) - g(t) = \Delta g$ funzione di Δt $= f'(g(t))$

Se $\lim_{x \to x_0} g(x) = y_0$ e $f(y)$ è continua in y_0 allora $\lim_{x \to x_0} f(g(x)) = f(y_0)$

limite della funzione composta

Esempio $y = x^n$ $x = t^m$ $y = (t^m)^n = t^{mn}$

$$\frac{dy}{dt} = nm\, t^{nm-1} = \frac{dy}{dx}\frac{dx}{dt} = n x^{n-1} m t^{m-1} =$$

$$= nm (t^m)^{n-1} t^{m-1} = nm\, t^{mn - m + m - 1} = nm\, t^{nm-1}$$

ok

$$\frac{d}{dx}\,\text{sen}^2 x \qquad u = \text{sen}\, x \qquad y = u^2$$

$$\frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx} = 2u\,\cos x = 2\,\text{sen}\, x\,\cos x$$

Derivata della funzione inversa

$$y = f(x) \qquad x = f^{-1}(y)$$

$$f'(x) = \frac{dy(x)}{dx} \qquad \frac{dx(y)}{dy} = \frac{1}{\frac{dy}{dx}}$$

$y=f(x)$	c	x^n	$\operatorname{sen} x$	$\cos x$	e^x	$\ln x$
$f'(x)$	0	nx^{n-1}	$\cos x$	$-\operatorname{sen} x$	e^x	$\frac{1}{x}$

$f(x)+g(x)$	$k f(x)$	$f(x)g(x)$	$\frac{f(x)}{g(x)}$
$f'(x)+g'(x)$	$k f'(x)$	$f'(x)g(x)+f(x)g'(x)$	$\frac{f'(x)g(x)-f(x)g'(x)}{g^2(x)}$

$f(g(t))$	$x=f^{-1}(y)$	$y=\operatorname{arcsen} x$
$\frac{df}{dg}\frac{dg}{dt}$	$\frac{dx}{dy}=\frac{1}{\frac{dy}{dx}}$	$\frac{1}{\sqrt{1-x^2}}$

Esempi

$$y = e^x = f(x) \qquad \frac{dy}{dx} = e^x = f'(x)$$

$$x = \ln y \qquad \frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{e^x} = \frac{1}{y}$$

$$\frac{d \ln y}{dy} = \frac{1}{y}$$

$$x \leftrightarrow y$$

$$y = \ln x \qquad \frac{dy}{dx} = \frac{1}{x} = \frac{d \ln x}{dx} = \frac{1}{x}$$

$$\boxed{y = \operatorname{sen} x} \qquad x = \operatorname{arcsen} y \qquad \frac{dy}{dx} = \cos x$$

$$\frac{dx(y)}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{\cos x(y)} = \frac{1}{\sqrt{1-y^2}}$$

$$\operatorname{sen}^2 x + \cos^2 x = 1 \qquad y = \operatorname{sen} x$$

$$y^2 + \cos^2 x = 1 \qquad \cos x = \sqrt{1-y^2}$$

$$\frac{d \operatorname{arcsen} y}{dy} = \frac{1}{\sqrt{1-y^2}}$$

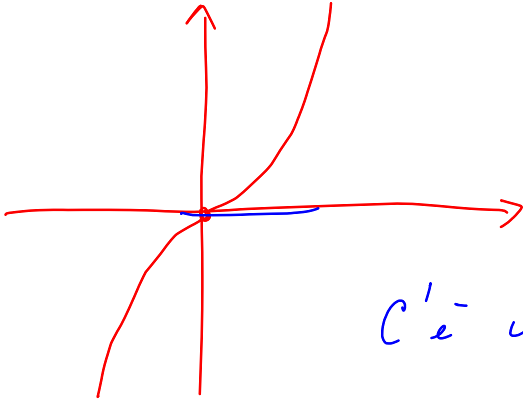
$$\frac{d \operatorname{arcsen} x}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$f'(x) > 0$: f cresce

$f'(x) < 0$: f decresce

$f'(x) = 0$: f ha un punto stazionario (min o max locali o flesso)

$$y = x^3$$



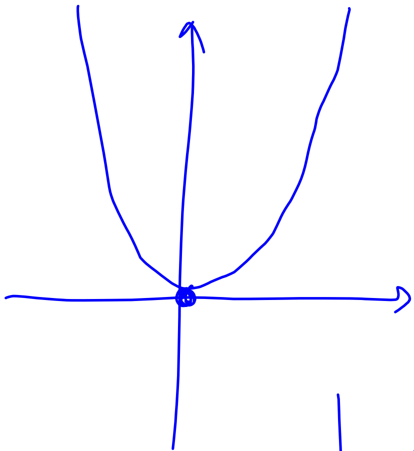
$$\frac{dx^n}{dx} = n x^{n-1}$$

$$\frac{dy}{dx} = 3x^2 \geq 0$$

C'è un flesso in $x = 0$

Se $f'(x_0) = 0$, x_0 è un minimo locale se $f''(x_0) > 0$, un massimo locale se $f''(x_0) < 0$ o un flesso

Se $f''(x_0) = 0$ bisogna andare alla terza derivata ...

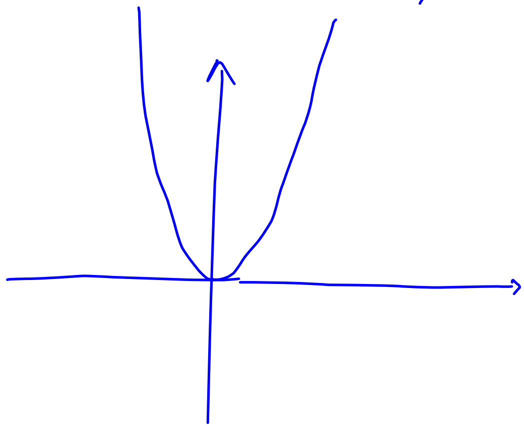


$$y = x^2 \qquad \frac{dy}{dx} = 2x$$

si annulla solo per $x=0$

$$\frac{d^2y}{dx^2} = 2 > 0$$

$$y = x^4$$



$$\frac{dy}{dx} = 4x^3$$

si annulla in $x=0$

$$\frac{d^2y}{dx^2} = 12x^2$$

si annulla in $x=0$

$$\frac{d^3y}{dy^3} = 24x$$

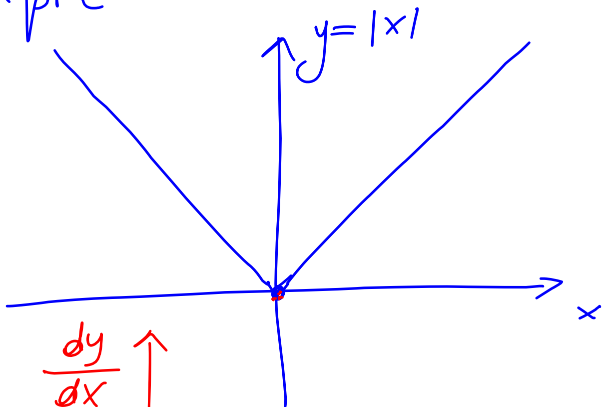
si annulla in $x=0$

$$\frac{d^4y}{dx^4} = 24 > 0$$

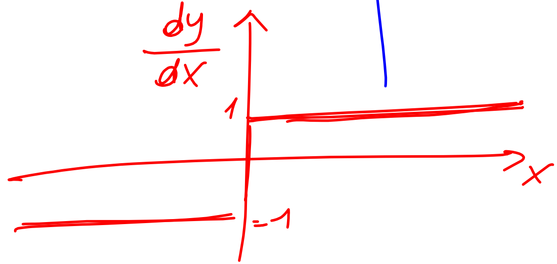
minimo locale

La derivata non esiste sempre

$$y = |x| = \begin{cases} x & \text{se } x > 0 \\ 0 & \text{se } x = 0 \\ -x & \text{se } x < 0 \end{cases}$$



$$\frac{dy}{dx} = \begin{cases} 1 & \text{per } x > 0 \\ ?? & \text{per } x = 0 \\ -1 & \text{per } x < 0 \end{cases}$$



$y = |x|$ NON è derivabile per $x = 0$

$$\frac{d^2y}{dx^2} = \begin{cases} 0 & \text{per } x > 0 \\ \text{non esiste} & \text{per } x = 0 \\ 0 & \text{per } x < 0 \end{cases}$$

Teorema di de l'Hôpital

Se f e g sono continue in I_{x_0} (intorno di x_0)

e derivabili in $I_{x_0} \setminus \{x_0\}$

e $g(x) \neq 0$ $g'(x) \neq 0$ in $I_{x_0} \setminus \{x_0\}$

Se $\frac{f(x)}{g(x)} = \frac{0}{0}$ o $\frac{\infty}{\infty}$ per $x = x_0$

Allora $\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}$

$$\lim_{x\to 0} \frac{\operatorname{sen} x}{x} = \lim_{x\to 0} \frac{\cos x}{1} = 1$$

f, g

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0} \frac{\operatorname{sen} x}{2x} = \lim_{x\to 0} \frac{\cos x}{2} = \frac{1}{2}$$

$f = 1-\cos x$, $g = x^2$, $\frac{0}{0}$

$$\lim_{x\to\infty} \frac{e^x}{x^m} = \infty$$

$$\lim_{x\to\infty} \frac{e^x}{x} = \lim_{x\to\infty} \frac{e^x}{1} = \infty$$

$\frac{\infty}{\infty}$

m intero positivo

$$\lim_{x\to\infty} \frac{e^x}{x^m} = \lim_{x\to\infty} \frac{e^x}{m x^{m-1}} = \lim_{x\to\infty} \frac{e^x}{m(m-1)x^{m-2}} \ldots$$

$$= \dots = \lim_{x \to \infty} \frac{e^x}{m!} = \infty$$

$$\frac{e^x}{x^m}$$

~~$\frac{\infty}{\infty}$~~ ~~$\frac{0}{0}$~~

qui non posso applicare il teorema

Se f è derivabile in x_0, allora è continua in x_0

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x}$$

$$\lim_{\Delta x \to 0} g(\Delta x) = 0$$

$$\frac{f(x_0+\Delta x) - f(x_0)}{\Delta x} = g(\Delta x)$$

$$f(x_0+\Delta x) = f(x_0) + \Delta x \, g(\Delta x)$$

$$f(x_0+\Delta x) = f(x_0) + \Delta x \, g(\Delta x)$$

Allora $\lim_{\Delta x \to 0} f(x_0+\Delta x) = f(x_0)$

cioè la funzione è continua in x_0

Esercizi

$$\frac{0}{0}$$

$$\lim_{x \to 0} \frac{e^x - 1}{x} = \lim_{x \to 0} \frac{e^x}{1} = 1$$

$$\lim_{x \to 0} \frac{\ln(1+x)}{x} = \lim_{x \to 0} \frac{\frac{1}{1+x}}{1} = 1$$

$$\frac{d \ln u}{du} = \frac{1}{u} \qquad u = 1 + x$$

$$\frac{d \ln(1+x)}{dx} = \frac{d \ln u}{dx} = \frac{d \ln u}{du} \frac{du}{dx} =$$

$$= \frac{1}{u} \, 1 = \frac{1}{1+x}$$

$$\lim_{x \to e} \frac{\ln x - 1}{x^2 - (1+e)x + e} = \frac{1}{e(e-1)}$$

$$\frac{0}{0} \qquad \ln_e e = 1 \qquad \to e^2 - (1+e)e + e = 0$$

$$= \lim_{x \to e} \frac{\frac{1}{x} + 0}{2x - (1+e)} = \frac{\frac{1}{e}}{2e - 1 + e} = \frac{1}{e(e-1)}$$

$$\frac{d \ln x}{dx} = \frac{1}{x} \qquad \frac{d(kf)}{dx} = k\frac{df}{dx}$$

$$\frac{dx^n}{dx} = n x^{n-1} \qquad \frac{d(f+g)}{dx} = \frac{df}{dx} + \frac{dg}{dx}$$

$$f(x) = 3x^3 + 9x^{②} - x - 4 \qquad \frac{dx^n}{dx} = nx^{n-1}$$

$$f'(x) = 9x^{②} + 18x - 1$$

$$f''(x) = 18x + 18 \qquad f'''(x) = 18 \qquad f^{IV}(x) = 0$$

$$f(x) = 10^x \qquad \frac{de^x}{dx} = e^x$$

$$a = e^{\ln a} \qquad a = 10^x \qquad \ln c^{d} = d \ln c$$

$$10^x = e^{\ln 10^x} = e^{x \ln 10} = e^{kx} \qquad k = \ln 10$$

$$\frac{de^u}{du} = e^u \qquad u = kx \qquad \frac{du}{dx} = k$$

$$\frac{de^{kx}}{dx} = \frac{de^u}{dx} = \frac{de^u}{du}\,\frac{du}{dx} = e^u\,k =$$

$$= k\,e^{kx}$$

$$k = \ln 10 \qquad \frac{d10^x}{dx} = \ln 10 \cdot 10^x$$

$$e^{kx} = 10^x$$

$$\frac{da^x}{dx} = \ln a \cdot a^x \qquad a = e \qquad \ln e = 1$$

$$10^x = e^{x \ln 10}$$

$$u = x \ln 10$$

$$\frac{de^x}{dx} = e^x$$

$$10^x = e^u$$

$$\frac{d10^x}{dx} = \frac{de^{u(x)}}{dx} = \frac{de^u}{du}\frac{du}{dx} = e^u \ln 10 =$$

$$= \ln 10 \cdot 10^x$$

$$f(x) = \ln x^2 = \ln u \qquad u = x^2 = u(x)$$

$$\frac{df}{dx} = \frac{d\ln x^2}{dx} = \frac{d\ln u(x)}{dx} = \frac{d\ln u}{du}\frac{du}{dx} =$$

$$= \frac{1}{u} 2x = \frac{2x}{x^2} = \frac{2}{x}$$

$$f(x) = \ln x^2 \qquad x > 0$$

$$= 2 \ln x \qquad \frac{df}{dx} = 2\,\frac{d \ln x}{x} = \frac{2}{x}$$

$$f(x) = x \cos x = u(x)\, v(x)$$

$$u(x) = x \qquad v(x) = \cos x$$

$$\frac{d(u \cdot v)}{dx} = \frac{du}{dx} v + u \frac{dv}{dx} =$$

$$= 1 \cdot v + u \cdot (-\operatorname{sen} x) = \cos x - x \operatorname{sen} x$$

$$f(x) = \frac{1}{x} = x^{-1}$$

$$\frac{dx^n}{dx} = n\, x^{n-1} \quad \forall n \in \mathbb{R}$$

$n = -1$ $\quad f'(x) = -\frac{1}{x^2}$

$$= \frac{u(x)}{v(x)} \qquad u(x) = 1 \quad v(x) = x$$

$$u' = 0 \qquad v' = 1$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v - v'u}{v^2} = \frac{\cancel{0 \cdot x} - 1 \cdot 1}{x^2} =$$

$$= -\frac{1}{x^2}$$

$$f(x) = \cos(e^{2x}) = \cos u(x) = \cos(e^{v})$$

$$u(x) = e^{2x} = e^{v(x)} \qquad v(x) = 2x$$

$$\frac{df}{dx} = \frac{d\cos u(x)}{dx} = \left(\frac{d\cos u}{du}\right)\left(\frac{du}{dx}\right) = \frac{d\cos u}{du}\,\frac{de^{v}}{dx} =$$

$$= \left(\frac{d\cos u}{du}\right)\frac{de^{v}}{dv}\,\frac{dv}{dx} =$$

$$= -\operatorname{sen} u \cdot e^{v} \cdot 2 = -2\operatorname{sen}(e^{2x})\,e^{2x}$$

$$f(x) = \sqrt{x} = x^{1/2}$$

$$\frac{dx^n}{dx} = n\,x^{n-1}$$

$$y = \sqrt{x}$$

$$n = \frac{1}{2}$$

$$f'(x) = \frac{1}{2} x^{\frac{1}{2}-1} = \frac{1}{2} \frac{1}{\sqrt{x}}$$

$$y = \sqrt{x} \qquad x = y^2 \qquad \frac{dx}{dy} = 2y^{2-1} = 2y$$

$$f'(x) = \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{2y} = \frac{1}{2\sqrt{x}}$$

$$f(x) = \begin{cases} \dfrac{x^3}{1 + x - \cos x} & \text{per } x \leq 0 \\ & \text{per } x > 0 \end{cases}$$

dire dove è continua e dove è derivabile

$$\lim_{x \to 0^+} 1 + x - \cos x \overset{?}{=} 0$$

sì la funzione è continua

$$\lim_{x \to 0^-} x^3 = 0$$

$$f'(x) = \begin{cases} 3x^2 & x \leq 0 \\ 1 + \operatorname{sen} x & x > 0 \end{cases}$$

$$\lim_{x \to 0^-} f'(x) = 0 \qquad \lim_{x \to 0^+} f'(x) = 1$$

la funzione non è derivabile in $x=0$

$$\lim_{x \to 0} \frac{1-\cos(x^2)}{x^3 \operatorname{sen} x} \quad \frac{0}{0} \quad = \lim_{x \to 0} \frac{1-\cos(x^2)}{(x^2)^2} \cdot \frac{x}{\operatorname{sen} x} = \frac{1}{2}$$

$$\frac{1-\cos(x^2)}{(x^2)^2} \to \frac{1}{2} \qquad \frac{x}{\operatorname{sen} x} \to 1$$

$$\lim_{x \to 0} \frac{1-\cos x}{x^2} = \frac{1}{2} \qquad x \to x^2$$

Oppure: uso de l'Hopital

$f(x) = 1 - \cos(x^2)$

$g(x) = x^3 \operatorname{sen} x$

$$\lim_{x \to 0} \frac{f}{g} = \lim_{x \to 0} \frac{f'}{g'}$$

$f(x) = \underline{1 - \cos u}$
$u = x^2$

$f'(x) = 2x \operatorname{sen} x^2$

$$\frac{df}{dx} = \frac{df}{du}\frac{du}{dx} = (\operatorname{sen} u)\, 2x = 2x \operatorname{sen} x^2$$

$$g' = x^3 \cos x + 3x^2 \operatorname{sen} x$$

$$\lim_{x\to 0} \frac{f'}{g'} = \lim_{x\to 0} \frac{2\cancel{x} \operatorname{sen} x^2}{x^{\cancel{3}\,2} \cos x + 3x^{\cancel{2}} \operatorname{sen} x} =$$

$$= \lim_{x\to 0} \frac{2 \operatorname{sen}(x^2)}{x^2 \cos x + 3x \operatorname{sen} x} =$$

$$= \lim_{x\to 0} \frac{2 \cos x^2 \cdot 2x}{2x \cos x - \operatorname{sen} x \cdot x^2 + 3 \operatorname{sen} x + 3x \cos x} =$$

divido sopra e sotto per x

$$= \lim_{x\to 0} \frac{4 \cos(x^2)}{5 \cos x - x \operatorname{sen} x + 3 \frac{\operatorname{sen} x}{x}} = \frac{4}{8} = \frac{1}{2}$$

5   0   3

$$\lim_{x \to 0} \frac{2x+1}{x+1} \overset{\not\frac{0}{0}}{=} 1 \neq \lim_{x \to 0} \frac{f'}{g'} = \lim_{x \to 0} \frac{2}{1} = 2$$

$f = 2x+1$

$g = x+1$

Non posso applicare de l'Hopital perchè non ho una forma indeterminata

$$\lim_{x \to \infty} \frac{\ln x}{x} \overset{\frac{\infty}{\infty}}{=} \lim_{x \to \infty} \frac{\frac{1}{x}}{1} = 0$$

$$\lim_{x \to \infty} \frac{\ln^2 x}{x} \underset{\frac{\infty}{\infty}}{=} \lim_{x \to \infty} \frac{\frac{2}{x} \ln x}{1} = 0$$

$$f(x) = \ln^2 x = u^2 \qquad \underline{u = \ln x}$$

$$f'(x) = \frac{du^2}{dx} = \frac{du^2}{du}\frac{du}{dx} = 2u\frac{1}{x} =$$

$$= \frac{2}{x}\ln x$$

$$\lim_{x\to\infty} \frac{x}{e^x} = \lim_{x\to\infty} \frac{1}{e^x} = 0$$

$$\frac{\infty}{\infty}$$

$$\lim_{x\to\infty} \frac{x^2}{e^x} = \lim_{x\to\infty} \frac{2x}{e^x} = 0$$

Sviluppo di Taylor

$\frac{0}{0}$

Data $f(x)$ continua e derivabile (infinite volte) in $x=0$

Considero $\lim_{x \to 0} \frac{f(x)-f(0)}{x} =$

$=$ (applico la regola di de l'Hopital)

$= \lim_{x \to 0} \frac{f'(x)}{1} = f'(0)$

$$\lim_{x \to 0} \frac{f(x)-f(0)-xf'(0)}{x} = \lim_{x \to 0} \left(\frac{f(x)-f(0)}{x} \right) - f'(0) =$$

$\frac{0}{0}$

$= 0$

per $x \to 0$ $f(x) - f(0) - x f'(0)$ tende a 0

più velocemente di x

$$\lim_{x \to 0} \frac{f(x) - f(0) - x f'(0)}{x^2} = \text{de l'Hopital}$$

$\frac{0}{0}$

$$= \lim_{x \to 0} \frac{f'(x) - f'(0)}{2x} = \lim_{x \to 0} \frac{f''(x)}{2} = \frac{f''(0)}{2}$$

$$\lim_{x \to 0} \frac{f(x) - f(0) - x f'(0) - \frac{x^2}{2} f''(0)}{x^2} = 0$$

$f(x) - f(0) - x f'(0) - \frac{x^2}{2} f''(0)$ tende a 0 più velocemente di x^2

$$f(x) - f(0) - x f'(0) - \frac{x^2}{2} f''(0) - \frac{x^3}{3!} f'''(0) - \frac{x^4}{4!} f^{iv}(0) \ldots\ldots - \frac{x^n}{n!} f^{(n)}(0)$$

tende a zero più velocemente di x^n

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2} f''(0) + \ldots\ldots \frac{x^n}{n!} f^{(n)}(0) + \cancel{+\ o(x^n)}$$

$x = \frac{1}{10}$ $\quad n = 3$ $\quad x^3 = \frac{1}{1000}$

$f(x) = e^x \qquad f'(x) = e^x \qquad f^{(n)}(x) = e^x$

$f^{(n)}(0) = 1$

sviluppo in serie di Taylor

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots \frac{x^n}{n!} + \dots$$

$=$

$n = 10$

$x = \frac{1}{10}$

$$\frac{x^n}{n!} = \frac{\frac{1}{10^{10}}}{10!}$$

$$f(x) = \operatorname{sen} x = 0 + x + \frac{x^2}{2} \cdot 0 + \frac{x^3}{3!}(-1) \dots$$

$$f'(x) = \cos x$$

$$f''(x) = -\operatorname{sen} x \qquad \operatorname{sen} x = x - \frac{x^3}{3!} \dots = x + o(x)$$

$$f'''(x) = -\cos x$$

$$\lim_{x \to 0} \frac{\operatorname{sen} x}{x} = \lim_{x \to 0} \frac{x + o(x)}{x} =$$

$$= 1 + \lim_{x \to 0} \frac{o(x)}{x} = 1$$

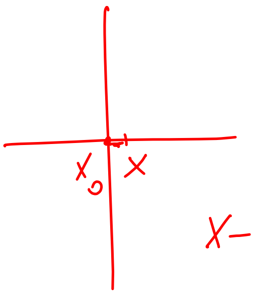
attorno a x_0 :

$$f(x) = f(x_0) + (x-x_0)f'(x_0) + $$

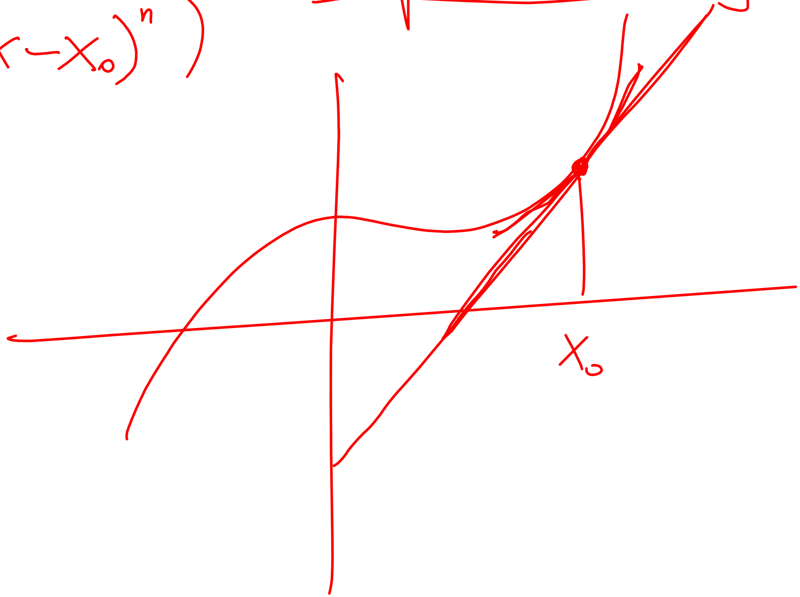
$$+ \frac{1}{2}(x-x_0)^2 f''(x_0) + \cdots \frac{1}{n!}(x-x_0)^n f^{(n)}(x_0) +$$

polinomio di Taylor

$$+ o\left((x-x_0)^n\right)$$



$x - x_0$



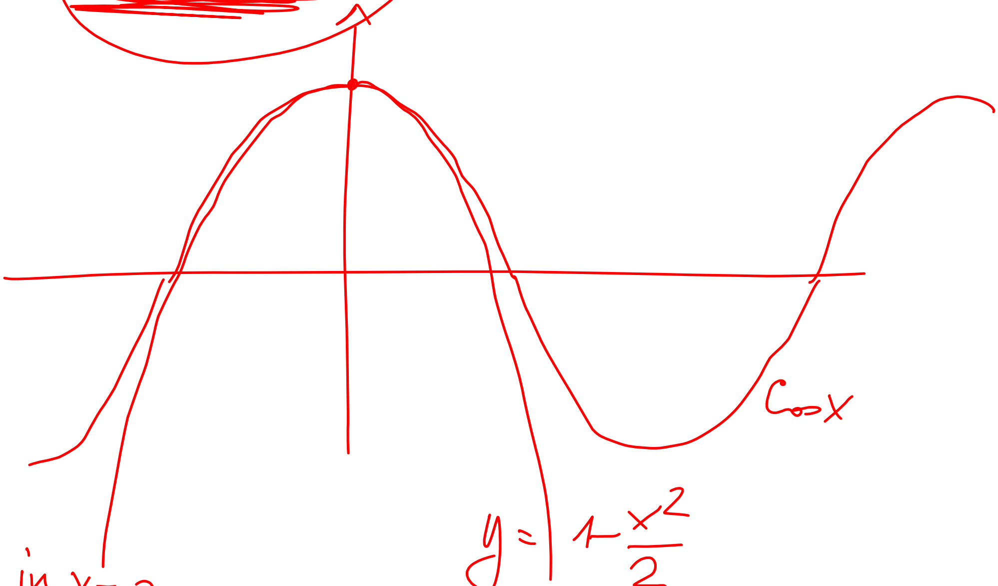
$f(x) = \cos x = 1 + x \cdot 0 + \frac{x^2}{2}(-1) + \frac{x^3}{3!} \cdot 0 + \frac{(\ldots)x^4}{4!}$

$x_0 = 0$

$= 1 - \frac{x^2}{2} + o(x^3)$

$f'(x) = -\operatorname{sen} x$

$f''(x) = -\cos x$



$y = 1 - \frac{x^2}{2}$

$y' = -x \qquad y' = 0$ in $x = 0$

$y'' = -1 < 0$

$$\lim_{x \to 0} \frac{1-\cos x}{x^2} = \frac{1}{2} =$$

$$= \lim_{x \to 0} \frac{1 - \left(1 - \frac{x^2}{2} + o(x^3)\right)}{x^2} =$$

$$= \frac{1}{2} + \lim_{x \to 0} \frac{o(x^3)}{x^2} = \frac{1}{2}$$

Integrali

$f(x)$ $\qquad f'(x)$ è la derivata di $f(x)$

$f(x)$ è una primitiva di $f'(x)$

$g(x)$ è la primitiva di $f(x)$ se

$g'(x) = f(x)$ $\qquad f(x) = 0 \quad g(x) = c$

$f(x) = 1 \qquad g'(x) = 1 \qquad g(x) = x + c$

$g(x) = x + 3$

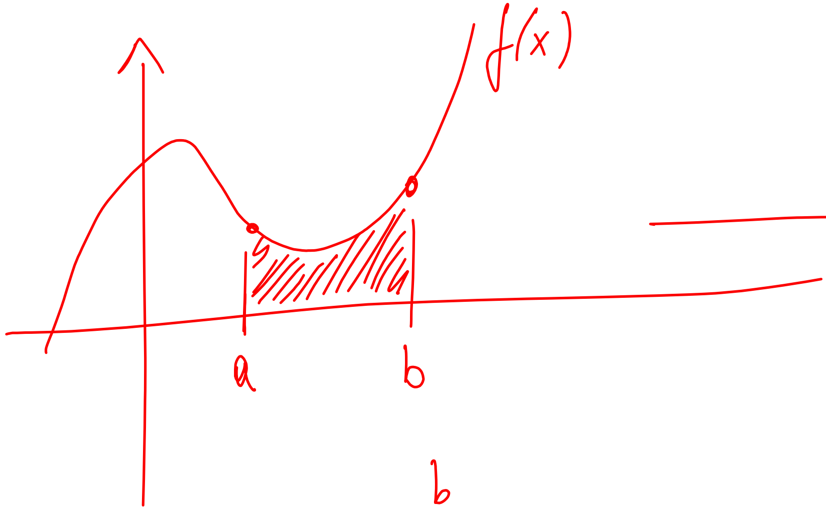
$f(x)$	0	d	$x^n \quad n \neq -1$	$\operatorname{sen} x$	$\cos x$	e^x	$\frac{1}{x}$
$g(x)$	c	$dx + c$	$\frac{x^{n+1}}{n+1} + c$	$-\cos x + c$	$\operatorname{sen} x + c$	$e^x + c$	$\ln x + c$

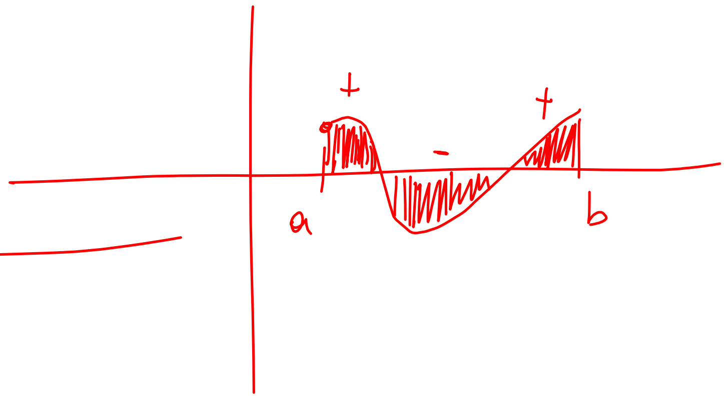
d, c costanti

$$f(x) = x^n \qquad g(x) = a\, x^{n+1}$$

$$g'(x) = \underbrace{a\,(n+1)}_{=1}\, x^n = f(x)$$

$$a = \frac{1}{n+1}$$

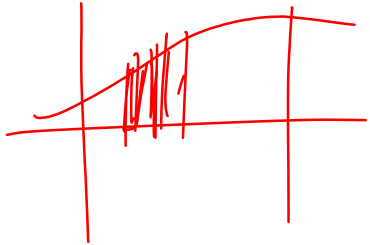




Integrale: $\int_a^b f(x)\,dx = g(b) - g(a)$

Se $g(x)$ è una primitiva di $f(x)$.

$$\int_a^b f(x)\,dx = g(b) - g(a)$$



a, b estremi di integrazione

$f(x)$ integrando

dx misura di integrazione

$$\int_a^x f(t)\,dt = g(x) - g(a) \equiv F(x)$$

$$F'(x) = g(x)$$

$$F'(x) = \lim_{\Delta x \to 0} \frac{F(x+\Delta x) - F(x)}{\Delta x} =$$

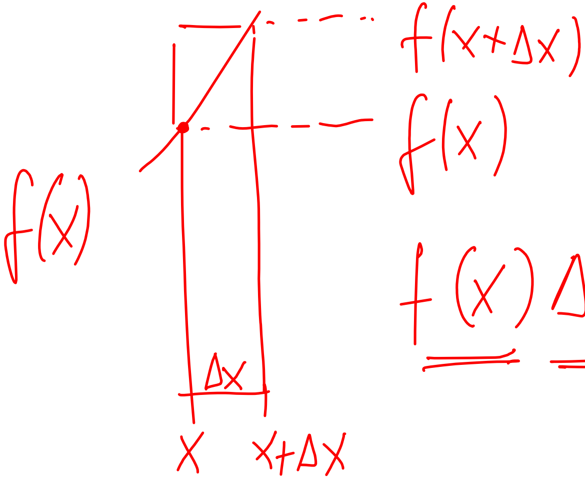
$$= \lim_{\Delta x \to 0} \frac{\int_a^{x+\Delta x} f(t)\,dt - \int_a^{x} f(t)\,dt}{\Delta x}$$

//// : $F(x+\Delta x)$

\\\\ : $F(x)$

$F(x+\Delta x) +$

$- F(x)$



$$f(x)\,\Delta x \le \text{Area} \le \Delta x\, f(x+\Delta x)$$

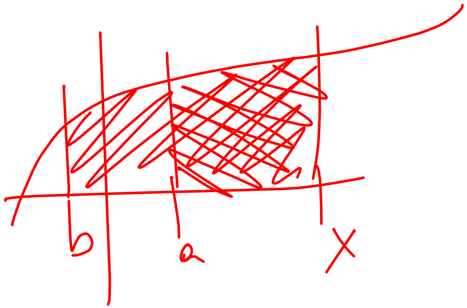
$$f(x)\,\Delta x \le F(x+\Delta x) - F(x) \le f(x+\Delta x)\,\Delta x$$

divido per Δx

$$f(x) \le \frac{F(x+\Delta x) - F(x)}{\Delta x} \le f(x+\Delta x)$$

$\Delta x \to 0$ ↪ $F'(x) = f(x)$

$$F_a(x) = \int_a^x f(t)\,dt$$ è una primitiva di $f(x)$



$$F_b(x) = \int_b^x f(t)\,dt$$

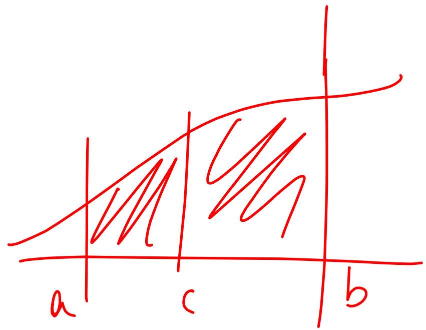
$$F_b(x) - F_a(x) = \int_b^a f(t)\,dt = c$$

non dipende da x

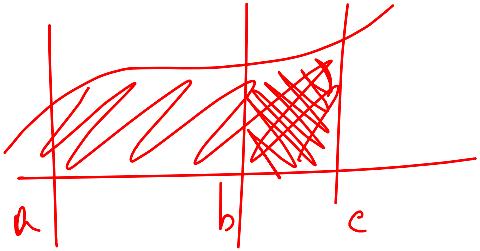
$$\int_a^x f(t)\,dt + \int_a^x g(t)\,dt = \int_a^x \big(f(t)+g(t)\big)\,dt$$

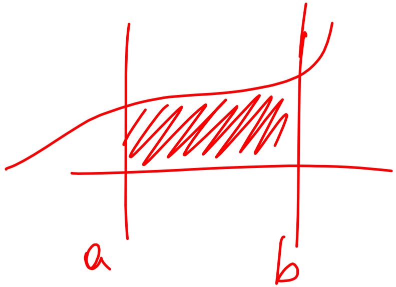
$$\int_a^x k\,f(t)\,dt = k\int_a^x f(t)\,dt$$

$$\int_a^a f(t)\,dt = 0$$

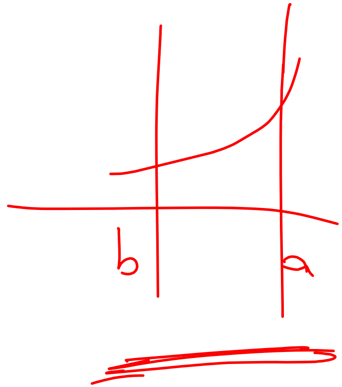


$$\int_a^b f(t)\,dt = \underbrace{\int_a^c f(t)\,dt}_{} + \int_c^b f(t)\,dt$$





$$\int_a^b f(t)\,dt = -\int_b^a f(t)\,dt$$



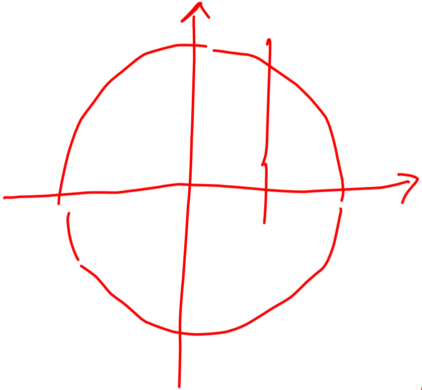
$$\Delta x = x_1 - x_2$$

$f(x) \qquad x = g(t) \qquad f(g(t))$

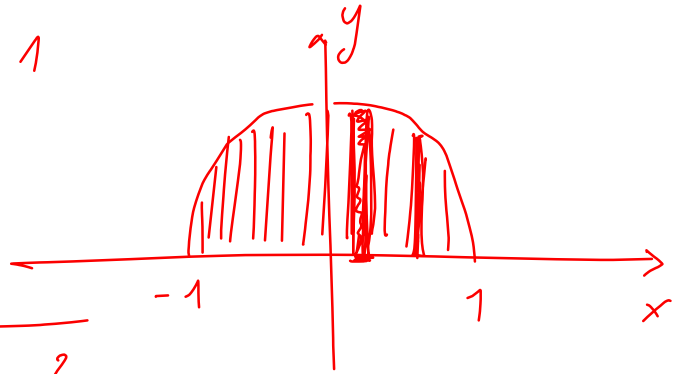
$$\frac{df}{dt} = \frac{df}{dx}\,\frac{dx}{dt} \qquad x = a = g(t)$$

$$\int_a^b f(x)\,dx = \int f\,\frac{dx}{dt}\,dt =$$

$$= \int_{g^{-1}(a)}^{g^{-1}(b)} f(g(t))\,\frac{dg(t)}{dt}\,dt$$



$$x^2+y^2=1$$



$$y=\sqrt{1-x^2}$$

$$-1 \le x \le 1$$

$$\text{Area cerchio} = 2\int_{-1}^{+1} \sqrt{1-x^2}\, dx$$

$$x = \operatorname{sen} \alpha$$

$$-1 = \operatorname{sen} \alpha \qquad = 2\int_{?}^{?} \sqrt{1-x^2}\, \frac{dx}{d\alpha}\, d\alpha =$$

$$\operatorname{sen}\left(-\frac{\pi}{2}\right) = -1$$

$$= 2 \int_{-\frac{\pi}{2}}^{\pi/2} \sqrt{1 - \operatorname{sen}^2\alpha}\ \cos\alpha \, d\alpha =$$

$$= 2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \underline{\underline{\cos^2\alpha}} \, d\alpha = \not{2} \int_{-\frac{\pi}{2}}^{\pi/2} \frac{1 + \cos 2\alpha}{\not{2}} \, d\alpha$$

$$\operatorname{sen}^2\alpha + \cos^2\alpha = 1 \qquad \operatorname{sen}^2\alpha = 1 - \cos^2\alpha$$

$$\cos(\alpha + \beta) = \cos\alpha \cos\beta - \operatorname{sen}\alpha \operatorname{sen}\beta$$

$\beta = \alpha$

$$\cos(2\alpha) = \cos^2\alpha - \operatorname{sen}^2\alpha =$$

$$\cos^2\alpha = \frac{1 + \cos 2\alpha}{2}$$

$$= \cos^2\alpha - (1 - \cos^2\alpha) = 2\cos^2\alpha - 1$$

$$\cancel{2}\int_{-\frac{\pi}{2}}^{\pi/2} \frac{1+\cos 2\alpha}{\cancel{2}}\,d\alpha = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1\cdot d\alpha + \int_{-\pi/2}^{\pi/2} \cos(2\alpha)\,d\alpha$$

$$= 0$$

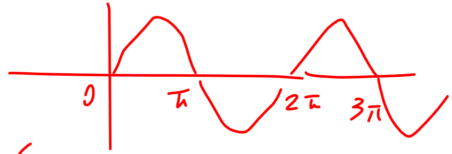
$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1 \cdot d\alpha = F\left(\frac{\pi}{2}\right) - F\left(-\frac{\pi}{2}\right) =$$

$$\alpha + C = F(\alpha) \quad \text{primitiva di } f(\alpha) = 1$$

$$= \frac{\pi}{2} + \cancel{C} - \left(-\frac{\pi}{2} + \cancel{C}\right) =$$

$$= \pi$$

$$\int_{-\pi/2}^{\pi/2} \cos(2\alpha)\, d\alpha = \int_{-\pi}^{\pi} \cos\gamma \ \frac{1}{2}\, d\gamma =$$

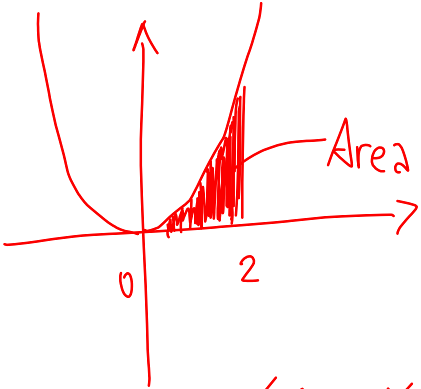


$$2\alpha = \gamma \qquad d\alpha = \frac{d\alpha}{d\gamma}\, d\gamma \qquad \frac{d\alpha}{d\gamma} = \frac{1}{2}$$

$$\alpha = \frac{\gamma}{2}$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} \cos\gamma \, d\gamma = \frac{1}{2}\Big[\operatorname{sen}\gamma\Big]_{-\pi}^{\pi} = \frac{1}{2}(0-0) = 0$$

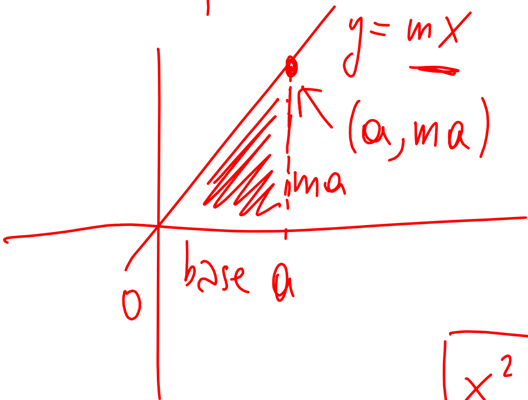
$$\Big[F(x)\Big]_a^b = F(b) - F(a)$$



$$y = x^2$$

$$x^n \xrightarrow{\text{Primitiva}} \frac{x^{n+1}}{n+1}$$

$$= \int_0^2 x^2\,dx = \left[\frac{x^3}{3}\right]_0^2 = \frac{8}{3}$$



h altezza
$\parallel$
ma base

$$y = mx$$

$$\int_0^a mx\,dx = m\left[\frac{x^2}{2}\right]_0^a = \frac{ma \cdot a}{2}$$

$$\boxed{x^2 = y}$$

$$dx = \frac{dx}{dy}\,dy$$

$$\int_2^7 x e^{x^2}\,dx = \int_4^{49} \not{x}\, e^y \frac{dy}{2\not{x}} =$$

$$dx = \left(\frac{dx}{dy}\right) dy = \frac{dy}{\left(\frac{dy}{dx}\right)} = \frac{dy}{2x}$$

$$y = x^2 \qquad \frac{dy}{dx} = 2x$$

$$= \frac{1}{2}\int_4^{49} e^y \, dy = \frac{1}{2}\left[e^y\right]_4^{49} = \frac{1}{2}\left(e^{49} - e^4\right)$$

$$e = 2.718\ldots$$

Integrale per parti

$$\int_a^b x \operatorname{sen} x \, dx$$

$$\int_a^b f(x)\,g'(x)\,dx = \int_a^b \frac{d}{dx}\big(f(x)\,g(x)\big)\,dx - \int_a^b f'(x)\,g(x)\,dx$$

$$\frac{d}{dx}\big(f(x)\,g(x)\big) = f'(x)\,g(x) + f(x)\,g'(x)$$

$$f(x)\,g'(x) = \frac{d}{dx}\big(f(x)\,g(x)\big) - f'(x)\,g(x)$$

$$\int_a^b \frac{df(x)}{dx}\,dx = \Big[f(x)\Big]_a^b = f(b) - f(a)$$

la primitiva è $f(x) + c$

$$\int_a^b f(x)\, g'(x)\, dx = f(b)\,g(b) - f(a)\,g(a) - \int_a^b f'(x)\, g(x)\, dx$$

$$\int_a^b x \operatorname{sen} x \, dx = -b\cos b + a\cos a + \int_a^b \cos x \, dx$$

$\operatorname{sen} x = g'(x)$ $\qquad g(x) = -\cos x$

$f(x) = x$ $\qquad f'(x) = 1$

$$= a\cos a - b\cos b + \left[\operatorname{sen} x\right]_a^b =$$

$$= a\cos a - b\cos b + \operatorname{sen} b - \operatorname{sen} a$$

$$\int_a^b x e^x \, dx = b e^b - a e^a - \int_a^b e^x \, dx =$$

$$f(x) = x \qquad f'(x) = 1$$

$$g'(x) = e^x \qquad g(x) = e^x$$

$$= b e^b - a e^a +$$

$$- \left[e^x \right]_a^b =$$

$$= b e^b - a e^a - e^b + e^a$$

$$f(x) = e^x \qquad f'(x) = e^x$$

$$g'(x) = x \qquad g(x) = \frac{x^2}{2}$$

ESERCIZI DI RIEPILOGO

$$y = x^2 e^{-x} \qquad \text{dominio} = \mathbb{R} \qquad y \geq 0$$

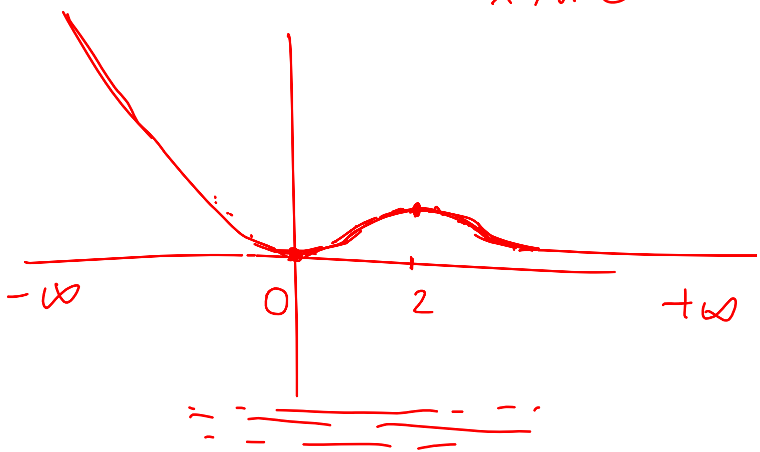
$$\lim_{x \to +\infty} x^2 e^{-x} = \lim_{x \to \infty} \frac{x^2}{e^x} \;\; \frac{\infty}{\infty} = \lim_{x \to \infty} \frac{2x}{e^x} \;\; \frac{\infty}{\infty} =$$

$$= \lim_{x \to \infty} \frac{2}{e^x} = 0$$

$$\lim_{x \to -\infty} x^2 e^{-x} = +\infty$$

$$e^{-x} \qquad x \to -\infty \qquad e^{-x}$$

$$e^{y} \qquad y \to +\infty \qquad y = -x$$



$$y = x^2 e^{-x^2}$$

Esercizi su primitive e integrali

$$f(x) = (x+3)^2 \qquad u = x+3 \qquad x = u-3$$

$$f(x)\,dx = (x+3)^2\,dx = u^2 \frac{dx}{du}\,du = u^2\,du =$$

$$\text{primitiva} = \frac{u^3}{3} + C = \frac{(x+3)^3}{3} + C$$

$$f(x) = \frac{1}{(x+12)^3} \qquad u = x+12$$

$$y = x^2 e^{-x} \qquad y' = 2x e^{-x} - x^2 e^{-x} =$$

$$= x(2-x)e^{-x}$$

Si annulla per $x = 2$

$x = 0$

$$\frac{de^{-x}}{dx} = -e^{-x}$$

$$y = -x$$

$$\frac{dy}{dx} = -1$$

$$y(0) = 0$$

$$y(2) = 4e^{-2}$$

$$y' = (2x - x^2)e^{-x}$$

$$y'' = (2-2x)e^{-x} + (2x - x^2)(-e^{-x}) =$$

$$= \left[2 - 2x - 2x + x^2\right] e^{-x} = (x^2 - 4x + 2)e^{-x}$$

$y''(0) = 2 > 0$ minimo rel. $\quad y''(2) = (4 - 8 + 2)e^{-2} < 0$

$$f(x)\,dx = \frac{1}{u^3}\,\frac{dx}{du}\,du = \frac{du}{u^3} \qquad u^{-3}$$

$$x^n \to \frac{x^{n+1}}{n+1}$$

$$\text{primitiva} = \frac{u^{-2}}{-2} + C = -\frac{1}{2}\,\frac{1}{(x+12)^2} + c$$

$$n = -3$$

$$f(x) = \sqrt{x-7} \qquad u = x-7$$

$$f(x)\,dx = \sqrt{u}\,du \qquad \text{primitiva} = \frac{2}{3}u^{3/2} + c =$$

$$= \frac{2}{3}(x-7)^{3/2} + c$$

$$f(x) = 5x^4 \cos(x^5+4)$$

$$f(x)\,dx = \cos(x^5+4)\,5x^4\,dx =$$

$$u = x^5+4 \qquad = \cos u \,\frac{du}{\cancel{dx}}\,\cancel{dx} =$$

$$\frac{du}{dx} = 5x^4 \qquad = \cos u \, du$$

$$\text{primitiva} = \operatorname{sen} u = \operatorname{sen}(x^5+4)$$

$$f(x) = \frac{3x^2}{x^3+4}$$

$$u = x^3+4 \qquad \frac{du}{dx} = 3x^2$$

$$f(x)\,dx = \frac{du}{\cancel{dx}} \cdot \frac{1}{u}\,\cancel{dx} = \frac{du}{u}$$

$$\text{primitiva} = \ln u + C = \ln(x^3+4) + C$$

$$f(x) = \frac{x}{1+x}$$

$1+x=u$

$\frac{dx}{du} = 1$

$$f(x)\,dx = \frac{u-1}{u}\,du = \left(1 - \frac{1}{u}\right)du$$

$$\text{primitiva} = u - \ln u + C =$$

$$= x+1 - \ln(x+1) + C$$

$$f(x) = \frac{1}{1+e^x}$$

$u = 1+e^x$

$e^x = u-1$

$$\frac{du}{dx} = e^x$$

$$f(x)\,dx = \frac{1}{u}\left(\frac{dx}{du}\right)du =$$

$$= \frac{1}{u}\,\frac{1}{e^x}\,du$$

$$= \frac{1}{u(u-1)}\,du$$

$$\frac{1}{u(u-1)}\,du = \frac{1}{u}\,\frac{1}{u-1}\,du$$

$$\frac{1}{u(u-1)} = -\frac{1}{u} + \frac{1}{u-1} = \frac{-\cancel{u}+1+\cancel{u}}{u(u-1)}$$

$$\left(\frac{1}{u-1} - \frac{1}{u}\right) du$$

$$\text{primitiva} = \ln(u-1) - \ln u + C =$$

$$= \ln e^x - \ln(1+e^x) + C =$$

$$= x - \ln(1+e^x) + C$$

$$\frac{1}{1+e^x}dx = \frac{1+e^x-e^x}{1+e^x}dx = \left(1-\frac{e^x}{1+e^x}\right)dx$$

$$= \dots$$

$$f(x) = \frac{1}{(x-3)(x-2)} = \frac{a}{x-3} + \frac{b}{x-2} =$$

$$= \frac{ax-2a+bx-3b}{(x-3)(x-2)}$$

$$ax - 2a + bx - 3b = 1 \quad \text{per ogni } x$$

$$(a+b)x - 2a - 3b = 1$$

$$\begin{cases} a+b=0 \\ -2a-3b=1 \end{cases}$$

$b = -a \qquad -2a - 3(-a) = 1$

$a = 1 \qquad b = -1$

$$\frac{1}{x-3} - \frac{1}{x-2} = \frac{1}{(x-3)(x-2)}$$

$$\text{primitiva} = \ln(x-3) - \ln(x-2) + C =$$

$$= \ln\frac{x-3}{x-2} + C$$

$$\ln a + \ln b = \ln ab$$
$$\ln a - \ln b = \ln\frac{a}{b}$$

$$f(x) = \frac{x}{x^2-3x+2} = \frac{x}{(x-1)(x-2)}$$

$$\frac{1}{(x-1)(x-2)} = -\left(\frac{1}{x-1} - \frac{1}{x-2}\right) =$$

$$\frac{\cancel{x}-2-\cancel{x}+1}{(x-1)(x-2)} = \frac{-1}{(x-1)(x-2)}$$

$$= \frac{1}{x-2} - \frac{1}{x-1}$$

$$f(x) = \frac{x}{x-2} - \frac{x}{x-1} = \frac{(x-2)+2}{(x-2)} - \frac{(x-1)+1}{(x-1)} =$$

$$= \not{1} + \frac{2}{x-2} - \not{1} - \frac{1}{x-1} =$$

$$= \frac{2}{x-2} - \frac{1}{x-1}$$

primitiva $= 2\ln(x-2) - \ln(x-1) + C =$

$$= \ln\frac{(x-2)^2}{x-1} + C$$

$$f(x) = \frac{x}{(x-1)^2} = \frac{(x-1)+1}{(x-1)^2} = \frac{1}{x-1} + \frac{1}{(x-1)^2}$$

$$\text{primitiva} = \ln(x-1) - \frac{1}{x-1} + C$$

$$x^n \cdots > \frac{x^{n+1}}{n+1} \qquad x^{-2} \cdots > \frac{x^{-1}}{-1} = -\frac{1}{x}$$

$$n = -2$$

$$f(x) = (\cos x)^2$$

$$(\operatorname{sen} x)^2 + (\cos x)^2 = 1$$

$$\operatorname{sen} x = \sqrt{1 - (\cos x)^2}$$

$$f(x)\,dx = (\cos x)^2\,dx$$

$$u = \cos x$$

$$= u^2 \frac{dx}{du}\,du$$

$$\frac{du}{dx} = \operatorname{sen} x$$

$$= \frac{u^2\,du}{\operatorname{sen} x} = \frac{u^2\,du}{\sqrt{1-u^2}}$$

.... metodo non conveniente

$$x e^x = f(x) g'(x) \qquad f'(x) = 1$$

$$f(x) = x \qquad g'(x) = e^x \qquad g(x) = e^x$$

$$\big(f(x) g(x)\big)' = f' g + f g' \Longleftarrow$$

regola di Leibniz

$$x e^x = f g' = (f g)' - f' g =$$

$$= (x e^x)' - e^x$$

primitiva di $x e^x = x e^x - e^x + C$

$$x \ln x = f(x)\, g'(x) = (f g)' - f' g$$

$$g'(x) = x \qquad f(x) = \ln x$$

$$g(x) = \frac{x^2}{2} \qquad f'(x) = \frac{1}{x}$$

$$x \ln x = \left(\frac{x^2}{2} \ln x\right)' - \frac{1}{x} \frac{x^2}{2}$$

primitiva di $x \ln x = \frac{x^2}{2} \ln x - \frac{x^2}{4} + C$

$$\ln x = f(x)\, g'(x) = (fg)' - f'g$$

$$f(x) = \ln x \qquad g'(x) = 1 \qquad \left(\begin{aligned} g(x) &= x \\ f'(x) &= \frac{1}{x} \end{aligned} \right)$$

$$\ln x = (x \ln x)' - \frac{1}{x}\, x$$

primitiva di $\ln x = x \ln x - x + C$

verifica: $1 \cdot \ln x + x \frac{1}{x} - 1 = \ln x$

$$(\cos x)^2 = f(x)\,g'(x) = (fg)' - f'g$$

$$f(x) = \cos x \qquad g'(x) = \cos x$$

$$f'(x) = -\operatorname{sen} x \qquad g(x) = \operatorname{sen} x$$

$$(\cos x)^2 = (\operatorname{sen} x \cos x)' - (-\operatorname{sen} x)\operatorname{sen} x =$$

$$= (\operatorname{sen} x \cos x)' + (\operatorname{sen} x)^2 =$$

$$= (\operatorname{sen} x \cos x)' + 1 - (\cos x)^2$$

$$2(\cos x)^2 = (\text{sen}x \cos x)' + 1$$

$$(\cos x)^2 = \frac{1}{2}(\text{sen}x \cos x)' + \frac{1}{2} = \left(\frac{\text{sen}x \cos x}{2}\right)' + \frac{1}{2}$$

primitiva di $(\cos x)^2 = \frac{\text{sen}x \cos x}{2} + \frac{x}{2} + C$

Verifica : $\frac{1}{2}[\cos x \cos x + (-\text{sen}x)\text{sen}x] + \frac{1}{2} = \frac{1}{2}[(\cos x)^2 - (\text{sen}x)^2 + 1] =$

$= (\cos x)^2$

$-(\text{sen}x)^2 + 1 = (\cos x)^2$

$$(kf)' = kf'$$

k costante

$$(\cos x)^2 = \cos^2 x \qquad \cos(x^2)$$

$f(x) = x^2 e^{-x^2}$ studiare questa funzione

dominio $= \mathbb{R}$ $\quad f(0) = 0 \quad f(-x) = f(x)$

funzione pari

$$\lim_{x \to +\infty} x^2 e^{-x^2} = 0$$

$$\lim_{x \to \infty} \frac{x^2}{e^{x^2}} \stackrel{\frac{\infty}{\infty}}{=} \lim_{x \to \infty} \frac{2x}{e^{x^2} 2x} = 0$$

de l'Hopital

$$f'(x) = 2x e^{-x^2} + x^2 \left(e^{-x^2} \cdot (-2x) \right) =$$

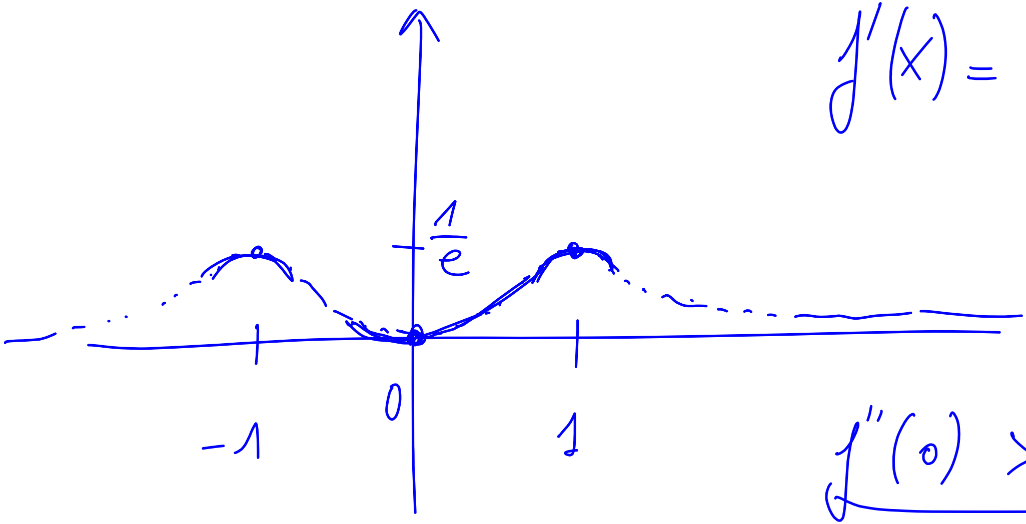
$$= 2x e^{-x^2} (1 - x^2)$$

$f'(x) = 0$ per

$x = 1$

$x = -1$

$x = 0$



$f'(x) = e^{-x^2}(2x - 2x^3)$

$f''(0) = 2 > 0$

$f''(1) = \frac{1}{e}(-4) < 0$

$f''(0) > 0$

$f''(1) < 0$

$f(1) = \frac{1}{e} = f(-1)$.

$$f''(x) = e^{-x^2}(2 - 6x^2) + (2x - 2x^3)e^{-x^2}(-2x) =$$

$$= e^{-x^2}(2 - 6x^2 - 4x^2 + 4x^4) =$$

$$= e^{-x^2}(2 - 10x^2 + 4x^4)$$

damiano.anselmi@gmail.com