Stability criteria of ideal magnetohydrodynamic plasmas with flows

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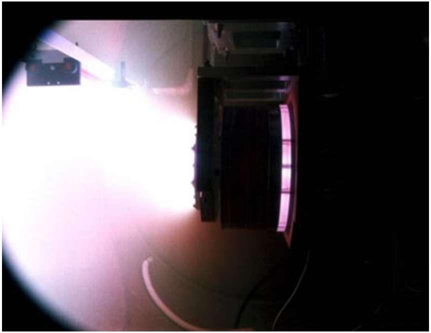
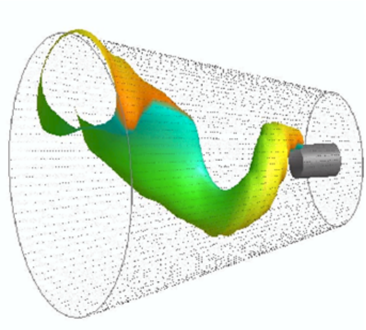
Introduction





MPDT plasma (left) and tomographic reconstruction of the flow (right)[1]

[1] F. Bonomo et al., Phys. Plasmas 12, 093301 (2005)

Introduction

- Ideal magnetohydrodynamics (MHD) is an important tool for assessing the design and interpretation of laboratory plasma experiments and for understanding phenomena in naturally occurring plasmas.
- Variational principles for equilibria have been discovered over a period of many years.
- δW energy principles[2] and other energy-like principles, based on Lagrangian displacements or Eulerian quantities, have been discovered and effectively utilized.

[2] I. B. Bernstein, E. A. Frieman, M. D. Kruskal, and R. M. Kulsrud, Proc. R. Soc. London, Ser. A 244, 17 (1958)

- All of these variational principles for equilibria and all of the energy principles, both Lagrangian and Eulerian, are a consequence of the fact that ideal MHD is a Hamiltonian field theory[3,4].
- The existence of variational principles for equilibrium states follows from the fact that equilibria are extremal points of Hamiltonian functionals.

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[3] W. A. Newcomb, Nucl. Fusion Suppl. Part 2, 451–463 (1962)

[4] P. J. Morrison and J. M. Greene, Phys. Rev. Lett. 45, 790-794 (1980) and Phys. Rev. Lett. 48, 569 (1982)

Introduction

- Stability conditions can be directly derived from the Hamiltonian formulation:
  - the δW energy principle for static equilibria is an infinite-dimensional version of Lagrange's stability condition of mechanics;
  - all of the sufficient conditions for stability of equilibria are infinite-dimensional versions of Dirichlet's stability condition.

MHD Equilibria

- MHD equilibria are solutions to the equations

$$\begin{aligned}
&\rho_e \mathbf{v}_e \cdot \nabla \mathbf{v_e} = -\nabla p_e + \mathbf{J}_e \times \mathbf{B}_e + \rho_e \nabla \Phi \\
&\nabla \times (\mathbf{v}_e \times \mathbf{B}_e) = 0 \\
&\nabla \cdot (\rho_e \mathbf{v}_e) = 0 \\
&\mathbf{v}_e \cdot \nabla s_e = 0
\end{aligned}$$

for the equilibrium velocity field $\mathbf{v}_e$, magnetic field $\mathbf{B}_e$, density field ρ_e, and entropy/mass field s_e. Here Φ represents and external gravitational potential.

MHD Equilibria

- In terms of Hamiltonian formulations, the equilibrium equations derive from:
  - the extremization of the MHD Hamiltonian in Lagrangian variables, which gives not only static equilibria in a natural way but also, introducing a canonical time-dependent relabeling transformation, stationary equilibria.
  - the extremization of the MHD Hamiltonian in Eulerian variables constrained by Casimir invariants, which are associated with the non-canonical variables.
  - the extremization of the MHD Hamiltonian in Eulerian variables with dynamically accessible variations.

Lagrangian Formulae

- The Hamiltonian for MHD in Lagrangian variables is

$$H[\mathbf{q},\pi] = \int d^3a \left[\frac{\pi_i \pi^i}{2\rho_0} + \rho_0 U(s_0, \rho_0/\mathscr{J}) + \frac{\partial q_i}{\partial a^k} \frac{\partial q^i}{\partial a^\ell} \frac{B_0^k B_0^\ell}{8\pi \mathscr{J}} + \rho_0 \Phi \right] \tag{1}$$

  where $(\mathbf{q},\pi)$ are the conjugate fields with $\boldsymbol{q}(\mathbf{a},t) = (q^1, q^2, q^3)$ denoting the position of a fluid element at time t labeled by $\mathbf{a} = (a^1, a^2, a^3)$ and π being its momentum density.

Lagrangian Formulae

- The Hamiltonian together with the canonical Poisson bracket

$$\{F, G\} = \int d^3a \left(\frac{\delta F}{\delta q^i} \frac{\delta G}{\delta \pi_i} - \frac{\delta G}{\delta q^i} \frac{\delta F}{\delta \pi_i} \right), \qquad (2)$$

renders the equations of motion in the form

$$\dot{\pi}_i = \{\pi_i, H\} = -\frac{\delta H}{\delta q^i} \quad \text{and} \quad \dot{q}^i = \left\{q^i, H\right\} = \frac{\delta H}{\delta \pi_i}. \qquad (3)$$

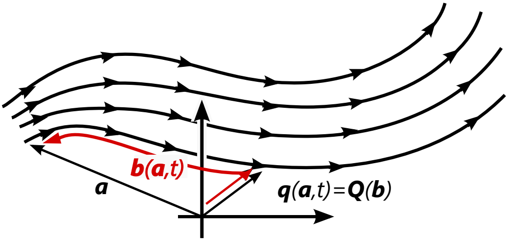
Relabeling Transformation

- Lagrangian equilibrium states correspond to static conditions

$$\frac{\delta H}{\delta \pi} = 0 \quad \rightarrow \quad \pi = 0. \tag{4}$$

- To accommodate stationary equilibria, a relabeling transformation can be adopted

$$\mathbf{a} = \mathfrak{A}(\mathbf{b}, t) \quad \leftrightarrow \quad \mathbf{b} = \mathfrak{B}(\mathbf{a}, t) \tag{5}$$



Relabeling Transformation

- The general time-dependent relabeling transformation give rise to the new dynamical variables

$$\mathbf{\Pi}(\mathbf{b},t) = \mathfrak{J}\pi(\mathbf{a},t), \qquad \mathbf{Q}(\mathbf{b},t) = \mathbf{q}(\mathbf{a},t) \tag{6}$$

and the new Hamiltonian

$$\tilde{H}[\mathbf{Q},\mathbf{\Pi}] = H - \int d^3b\,\mathbf{\Pi}\cdot(\mathbf{V}\cdot\nabla_b\mathbf{Q}), \tag{7}$$

where $\mathfrak{J} := \det(\partial a^i/\partial b^j)$ and

$$\mathbf{V}(\mathbf{b},t) := \dot{\mathfrak{B}}\circ\mathfrak{B}^{-1} = \dot{\mathfrak{B}}(\mathfrak{A}(\mathbf{b},t),t) \tag{8}$$

- Extremization of Hamiltonians give equilibrium equations: for the Hamiltonian $H[\mathbf{q}, \pi]$ this gives static equilibria, while for $\tilde{H}[\mathbf{Q}, \mathbf{\Pi}]$ one obtains stationary equilibria.
- The relabeling allows us to express stationary equilibria in terms of Lagrangian variables, which would ordinarily be time dependent, as time-independent orbits with the moving labels.

Relabeling Transformation

- The equilibrium equations are

$$
\begin{aligned}
0 =& \partial_t \mathbf{Q}_e = \frac{\mathbf{\Pi}_e}{\tilde{\rho}_0} - \mathbf{V}_e \cdot \nabla_b \mathbf{Q}_e, \\
0 =& \partial_t \mathbf{\Pi}_e = -\nabla_b \cdot (\mathbf{V}_e \otimes \mathbf{\Pi}_e) + \mathbf{F}_e
\end{aligned}
$$

- From these it follows the equation

$$
\nabla_b \cdot (\tilde{\rho}_0 \mathbf{V}_e \mathbf{V}_e \cdot \nabla_b \mathbf{Q}_e) = \mathbf{F}_e
$$

Relabeling Transformation

- From the equation

$$\nabla_b \cdot (\tilde{\rho}_0 \mathbf{V}_e \mathbf{V}_e \cdot \nabla_b \mathbf{Q}_e) = \mathbf{F}_e,$$

using $\mathbf{b} = \mathbf{Q}_e(\mathbf{b}) = \mathbf{q}_e(\mathfrak{A}_e(\mathbf{b}, t), t) = \mathfrak{B}_e(\mathbf{a}, t)$ and the definition of $\mathbf{V}(\mathbf{b}, t) = \dot{\mathfrak{B}}_e(\mathfrak{A}_e(\mathbf{b}, t), t) = \mathbf{v}_e(\mathbf{b})$, where $\mathbf{v}_e(\mathbf{b})$ denotes an Eulerian equilibrium state, we obtain the usual stationary equilibrium equation

$$\nabla \cdot (\rho_e \mathbf{v}_e \mathbf{v}_e) = \mathbf{F}_e.$$

- For stability, we expand as follows

$$\mathbf{Q} = \mathbf{Q}_r(\mathbf{b}, t) + \eta(\mathbf{b}, t), \quad \mathbf{\Pi} = \mathbf{\Pi}_r(\mathbf{b}, t) + \pi_\eta(\mathbf{b}, t)$$

- The second variation of the Hamiltonian results

$$\delta^2 H_{\mathrm{la}}[Z_e; \eta, \pi_\eta] = \frac{1}{2}\int d^3x \frac{1}{\rho_e}\left|\pi_\eta - \rho_e \mathbf{v}_e \cdot \nabla \eta\right|^2 + \delta^2 W_{\mathrm{la}}[Z_e; \eta]$$

Lagrangian Stability

- The functional $\delta^2 W_{\mathrm{la}}$ is identical to that obtained by Frieman and Rotenberg[5]

$$\delta^2 W_{\mathrm{la}}[Z_e;\eta] := \frac{1}{2}\int d^3x\, \eta\cdot\mathfrak{V}_e\cdot\eta$$
$$= \frac{1}{2}\int d^3x \left[\rho_e\left(\mathbf{v}_e\cdot\nabla\mathbf{v}_e\right)\cdot\left(\eta\cdot\nabla\eta\right) - \rho_e\left|\mathbf{v}_e\cdot\nabla\eta\right|^2\right] + \delta^2 W[\eta]$$

[5] E. A. Frieman and M. Rotenberg, Rev. Mod. Phys. 32, 898 (1960)

Eulerian formulae

- The Hamiltonian for MHD can be written in terms of the Eulerian variables $Z = (\rho, s, \mathbf{v}, \mathbf{B})$ as

$$H[Z] = \int d^3x \left[\frac{\rho}{2} |\mathbf{v}|^2 + \rho U(s, \rho) + \frac{\mathbf{B}^2}{8\pi} + \rho \Phi \right]$$

- Eulerian variables are non-canonical and the corresponding Poisson bracket has degeneracy that gives rise to Casimir invariants C_i.
- Eulerian equilibria Z_e satisfy $\delta \mathfrak{F} = 0$, where $\mathfrak{F} = H + \sum C_i$ is the Energy-Casimir functional.

Eulerian formulae

- For MHD with no symmetry the Casimirs are

$$C_s = \int d^3x \, \rho f(s)$$

and the magnetic and cross helicities

$$C_B = \int d^3x \, \mathbf{A} \cdot \mathbf{B}, \quad \text{and} \quad C_v = \int d^3x \, \mathbf{v} \cdot \mathbf{B}.$$

Eulerian formulae

- If translational symmetry is assumed, all variables are independent of the coordinate z and

$$\begin{aligned}\mathbf{B} &= B_z\hat{\mathbf{z}} + \nabla\psi \times \hat{\mathbf{z}} \\ \mathbf{M} &= M_z\hat{\mathbf{z}} + \nabla\chi \times \hat{\mathbf{z}} + \nabla\Upsilon\end{aligned}$$

  where $\mathbf{M} = \rho\mathbf{v}$ and $\hat{\mathbf{z}}$ is the unit vector in the symmetry direction.
- With this symmetry assumption, the set of Casimir is expanded and is sufficient to obtain a variational principle for the equilibria considered here.

Casimir Invariants

- The Casimir invariants of the case with translational symmetry are

$$C_s = \int d^3x\rho\, \mathscr{J}\Big(s, \psi, [s,\psi]/\rho, \Big[[s,\psi]/\rho, \psi\Big]/\rho, \Big[s, [s,\psi]/\rho\Big]/\rho, \ldots\Big) \tag{9}$$

$$C_{B_z} = \int d^3x\, B_z \mathscr{H}(\psi)\,, \tag{10}$$

$$C_{v_z} = \int d^3x\, \rho\, v_z \mathscr{G}(\psi)\,, \tag{11}$$

Casimir Invariants

- If the entropy is assumed to be a flux function, i.e., $[\psi, s] = 0$ where $[f,g] = \hat{\mathbf{z}} \cdot \nabla f \times \nabla g$, then

$$C_s = \int d^3x \rho \, \mathscr{J}(\psi),$$

and there is the additional cross helicity Casimir

$$\begin{aligned} C_v &= \int d^3x \left(v_z B_z \mathscr{F}' + \frac{1}{\rho} \nabla \mathscr{F} \cdot \nabla \chi + \frac{[\Upsilon, \mathscr{F}]}{\rho} \right) \\ &= \int d^3x \, \mathbf{v} \cdot \mathbf{B} \mathscr{F}'. \end{aligned}$$

Energy-Casimir Stability

- For equilibria Z_e a sufficient condition for stability follows from the positiveness of[6]

$$\delta^2 \mathfrak{F}[Z_e;\delta Z_e] = \int d^3x \left[a_1 |\delta \mathbf{S}|^2 + a_2 (\delta Q)^2 \right. \\ \left. + a_3 (\delta R_z)^2 + a_4 |\delta \mathbf{R}_\perp|^2 + a_5 (\delta \psi)^2 \right]$$

  where the variations $(\delta \mathbf{S}, \delta \mathbf{R}, \delta Q, \delta \psi)$ are linear combinations of $(\delta \mathbf{v}, \delta \mathbf{B}, \delta \rho, \delta \psi)$.

[6] T. Andreussi, P. J. Morrison, and F. Pegoraro, Phys. Plasmas 20, 092104 (2013) and Phys. Plasmas 22, 039903 (2015)

Energy-Casimir Stability

- Upon extremizing over all variables except $\delta\psi$ we obtain

$$\delta^2\mathfrak{F}[Z_e;\delta\psi] = \int d^3x \left[b_1\,|\nabla\delta\psi|^2 + b_2\,(\delta\psi)^2 + b_3\left|\mathbf{e}_\psi\times\nabla\delta\psi\right|^2\right]$$

where $\mathbf{e}_\psi = \nabla\psi/|\nabla\psi|$ and

$$b_1 = \frac{1-\mathcal{M}^2}{4\pi}\frac{c_s^2-\mathcal{M}^2\left(c_s^2+c_a^2\right)}{c_s^2-\mathcal{M}^2\left(c_s^2+c_a^2\right)+\frac{\mathcal{M}^2}{4\pi\rho}|\nabla\psi|^2}$$

$$b_2 = \nabla\cdot\left[\frac{\partial}{\partial\psi}\left(\frac{\mathcal{M}^2}{4\pi}\right)\nabla\psi\right] - \frac{\partial}{\partial\psi^2}\left(p+\frac{B_z^2}{8\pi}+\frac{\mathcal{M}^2}{4\pi}|\nabla\psi|^2\right)$$

$$b_3 = \frac{1-\mathcal{M}^2}{4\pi} - b_1$$

Dynamically Accessible Formulae

- With energy-Casimir, the constraints are incorporated essentially by using Lagrange multipliers.
- Dynamically accessible variations restrict the variations to be those generated by the noncanonical Poisson bracket

$$\delta\rho_{\mathrm{da}} = \nabla\cdot(\rho\mathbf{g}_1)\,, \tag{12}$$

$$\begin{aligned}\delta\mathbf{v}_{\mathrm{da}} &= \nabla g_3 + s\nabla g_2 + (\nabla\times\mathbf{v})\times\mathbf{g}_1 \\ &\quad + \mathbf{B}\times(\nabla\times\mathbf{g}_4)/\rho\end{aligned} \tag{13}$$

$$\delta s_{\mathrm{da}} = \mathbf{g}_1\cdot\nabla s\,, \tag{14}$$

$$\delta\mathbf{B}_{\mathrm{da}} = \nabla\times(\mathbf{B}\times\mathbf{g}_1)\,, \tag{15}$$

Dynamically Accessible Formulae

- Using the dynamically accessible variations in the variation of the Eulerian Hamiltonian gives

$$
\begin{aligned}
\delta H_{\rm da} &= \int d^3x \left[\left(v^2/2 + (\rho U)_\rho \right) \delta\rho_{\rm da} + \rho \mathbf{v} \cdot \delta \mathbf{v}_{\rm da} \right. \\
&\qquad \left. + \rho U_s \delta s_{\rm da} + \mathbf{B} \cdot \delta \mathbf{B}_{\rm da}/4\pi \right], \\
&= \int d^3x \left[\mathbf{g}_1 \cdot \Big(\rho \mathbf{v} \times (\nabla \times \mathbf{v}) - \rho \nabla v^2/2 \right. \\
&\qquad - \rho \nabla h + \rho T \nabla s + \mathbf{J} \times \mathbf{B} \Big) - g_2 \nabla \cdot (\rho s \mathbf{v}) \\
&\qquad \left. - g_3 \nabla \cdot (\rho \mathbf{v}) + \mathbf{g}_4 \cdot \nabla \times (\mathbf{v} \times \mathbf{B}) \right] = 0
\end{aligned}
$$

Dynamically Accessible Stability

- Stability is assessed by expanding the Hamiltonian to second order using the dynamically accessible variations

$$\delta^2 H_{\mathrm{da}}[Z_e;\mathbf{g}] = \int d^3x\, \rho\, |\delta \mathbf{v}_{\mathrm{da}} - \mathbf{g}_1 \cdot \nabla \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{g}_1|^2 + \delta^2 W_{\mathrm{la}}[Z_e;\mathbf{g}_1]$$

- If $\delta \mathbf{v}_{\mathrm{da}}$ were independent and arbitrary we could use it to nullify the first term and then upon setting $\mathbf{g}_1 = -\eta$, we would see that dynamically accessible stability is identical to Lagrangian stability.

MHD Stability

- The second variation of the three variational principles can be used to determine sufficient (and, in some case, necessary) conditions for stability.
- Different perturbations are associated with the three approaches and, in terms of Eulerian variables, these perturbations can be written as

$$\begin{cases} \delta\rho_{\mathrm{la}} & = -\nabla\cdot(\rho\eta) \\ \delta\mathbf{v}_{\mathrm{la}} & = \frac{\partial\eta}{\partial t} + \mathbf{v}\cdot\nabla\eta - \eta\cdot\nabla\mathbf{v} \\ \delta s_{\mathrm{la}} & = -\eta\cdot\nabla s \\ \delta\mathbf{B}_{\mathrm{la}} & = -\nabla\times(\mathbf{B}\times\eta) \end{cases} \qquad \begin{cases} \delta\rho_{\mathrm{ec}} \\ \delta\mathbf{v}_{ec} \\ \delta s_{\mathrm{ec}} \\ \delta\mathbf{B}_{ec} \end{cases} \qquad \begin{cases} \delta\rho_{\mathrm{da}} & = -\nabla\cdot(\rho\mathbf{g}_1) \\ \delta\mathbf{v}_{\mathrm{da}} & = \mathbf{X} + \mathbf{v}\cdot\nabla\mathbf{g}_1 - \mathbf{g}_1\cdot\nabla\mathbf{v} \\ \delta s_{\mathrm{da}} & = -\mathbf{g}_1\cdot\nabla s \\ \delta\mathbf{B}_{\mathrm{da}} & = -\nabla\times(\mathbf{B}\times\mathbf{g}_1) \end{cases}$$

where

$$\mathbf{X} = 2(\mathbf{v}\cdot\nabla)\mathbf{g}_1 + \mathbf{v}\times(\nabla\times\mathbf{g}_1) + s\nabla g_2 + \nabla g_3 + \frac{1}{\rho}\mathbf{B}\times(\nabla\times\mathbf{g}_4)$$

Comparison Formulae

- We established the inclusions[7]

$$\mathfrak{P}_{\mathrm{da}} \subset \mathfrak{P}_{\mathrm{la}} \subset \mathfrak{P}_{\mathrm{ec}}\,,$$

  which led to the conclusions

$$\mathfrak{stab}_{ec} \Rightarrow \mathfrak{stab}_{\mathrm{la}} \Rightarrow \mathfrak{stab}_{\mathrm{da}}\,.$$

- Dynamically accessible perturbations are the most constrained, while energy-Casimir stability is the most general, when it exists, for its perturbations are not constrained at all.

[7] T. Andreussi, P. J. Morrison, and F. Pegoraro, Phys. Plasmas 20, 092104 (2013) and Phys. Plasmas 22, 039903 (2015)

Comparison Formulae

- If $\delta\mathbf{v}_{\mathrm{da}}$ is arbitrary, independently of $\mathbf{g}_1$, then $\delta^2 H_{\mathrm{da}}$ is reduced to the energy expression obtained for Lagrangian stability, making the two kinds of stability equivalent.
- Given that $\mathbf{g}$ has in addition to $\mathbf{g}_1$, the five components of g_2, g_3 and $\mathbf{g}_4$, one might think that this is always possible. However, this is not always possible and whether or not it is depends on the state or equilibrium under consideration[8].

[8] T. Andreussi, P. J. Morrison, and F. Pegoraro, Phys. Plasmas 23, 102112 (2016)

Cross Helicity

- Consider first a static equilibrium state that has entropy as a flux function. Thus, for this case, the cross helicity C_v vanishes. For a dynamically accessible perturbation

$$\begin{aligned} \delta C_v &= \int d^3x\, \delta \mathbf{v}_{\mathrm{da}} \cdot \mathbf{B} = \int d^3x\, (\nabla g_3 + s \nabla g_2) \cdot \mathbf{B} \\ &= - \int d^3x\, g_2\, \mathbf{B} \cdot \nabla s = 0\,. \end{aligned}$$

- The last equality assumes g_3 is single-valued and the vanishing of surface terms, as well as s being a flux function.

Cross Helicity

- The fact that $\delta C_v = 0$ for this case is not a surprise since it is a Casimir, but we do see clearly that if s were not a flux function, then a perturabtion $\delta \mathbf{v}_{\mathrm{da}}$ could indeed create cross helicity.
- Because of the term $\partial \eta / \partial t$ which can be chosen arbitrarily, it is clear that $\delta \mathbf{v}_{\mathrm{la}}$ can create cross helicity for any equilibrium state, supplying clear evidence that $\delta \mathbf{v}_{\mathrm{da}}$ is not completly general.

Convection Stability

- A clear comparison between the three approaches is possible for the convection in static equilibria, both with and without a magnetic field.
- For the case $B = 0$, the Lagrangian and dynamically accessible approaches both give the simple necessary and sufficient condition for stability, $ds/dy > 0$, or equivalently the inequality

$$\frac{d\rho}{dy} < -\frac{\rho g}{c_s^2} < 0.$$

- The Eulerian energy- Casimir approach gives this same result, but only as a sufficient condition for stability and only applicable to the case with the imposed translational symmetry.

Convection Stability

- For the case $B \neq 0$, the situation is different, although it again must be true that the Lagrangian and dynamically accessible approaches must give the same necessary and sufficient condition for stability

$$\frac{d\rho}{dy} < -\frac{\rho g}{c_s^2 + c_a^2} < 0.$$

- The Eulerian energy-Casimir approach gives more complex inequalities

$$\frac{dp}{ds} + \frac{p_s}{c_s^2} = \Delta < 0, \qquad \frac{d\left(J/\rho\right)/dy}{d\psi/dy} + \frac{J^2}{p^2 c_s^2 \Delta} \frac{d\rho/dy}{ds/dy} < 0,$$

which again represent sufficient conditions for stability and are only applicable to the case with the imposed translational symmetry.

Convection Stability

- The energy-Casimir inequalities depend on an extra derivative with respect to y of at least one of the equilibrium profiles, e.g. a derivative of the current J.
- This dJ/dy term can be removed by inserting into the second variation of the energy-Casimir functional the Lagrangian variations, adapted to the convection example

$$\delta \psi_{\mathrm{la}} = \eta \cdot \nabla \psi$$

- Such a correspondence by constraining the Eulerian variations in general connects energy-Casimir and Lagrangian stability.

Rotating Pinch Stability

- Within the Lagrangian, Energy-Casimir (Eulerian) and Dynamically accessible frameworks, a second comparison was then performed on the stability of an azimuthally symmetric rotating pinch.
- We consider rigidly rotating pinch equilibria and, to compare the Lagrangian and the dynamically accessible stability conditions with those obtained in the energy-Casimir framework, we restrict our analysis to perturbations η that do not depend on z (no "sausage" or kink instabilities).

Rotating Pinch Stability

- The results of the stability analysis for such perturbations can be expressed as stability bounds on the normalized rotation frequency w.
- These bounds are modified by the presence of an equilibrium magnetic field along the symmetry direction, B_z, that couples the component η_z to the other components of the displacement leading in general to stricter bounds.

Rotating Pinch Stability

- Comparing the Lagrangian and the dynamically accessible stability conditions, we observe that the constraints obeyed by the dynamically accessible perturbations in the presence of flows lead to an additional stabilizing term.
- This additional term cannot be made to vanish for azimuthally symmetric perturbations, however, this term does not modify the stability analysis since azimuthally symmetric perturbations are found to be stable even within the Lagrangian framework.
- For more general equilibria than the ones considered here, this need not be the case.

- The minimization of $\delta^2 W_{\mathrm{la}}$ for our pinch case reduced to the study of 4×4 matrix (for $B_z \neq 0$) for $|m| = 1$ perturbations

$$\begin{bmatrix} m^2\left(\hat{\Pi}/r^2 - \hat{p}w^2\right) & im\hat{p}w^2 & -im\hat{\Pi}/r^2 & -m^2\hat{B}/r \\ -im\hat{p}w^2 & m^2\varpi & 0 & 0 \\ im\hat{\Pi}/r^2 & 0 & 1+\hat{\Pi}/r^2 & -im\hat{B}/r \\ -m^2\hat{B}/r & 0 & im\hat{B}/r & m^2\varpi \end{bmatrix}$$

where $\varpi = 1 - \hat{p}w^2$ and $\hat{\Pi} = \hat{p} + \hat{B}^2$.

Rotating Pinch Stability

- A necessary and sufficient condition for the positivity of this matrix is provided by the Sylvester criterion which yields

$$\begin{aligned} w^2 < 1/2 \quad &\text{for} \quad B_z = 0 \\ w^2 B_z^2 < 1 \quad &\text{for} \quad B_z \neq 0 \textit{ and } w^2 \rightarrow 0. \end{aligned}$$

- A partial minimization procedure with respect to η_ϕ and η_z leads to less restrictive conditions

$$\begin{aligned} w^2 \lesssim 0.62 \quad &\text{for} \quad B_z = 0 \\ w^2 \lesssim 0.46 \quad &\text{for} \quad B_z \neq 0 \textit{ and } B_z^2 = 1. \end{aligned}$$

Rotating Pinch Stability

- Extremization of the energy-Casimir functional over all variables except $\delta\psi$ leads to a reduced energy-Casimir functional and to sufficient stability bounds on w^2 that, similar to the Lagrangian case, become stricter as B_z^2 increases.
- These bounds are in general more restrictive than those found within the Lagrangian framework, for example we obtain

$$w^2 \lesssim 0.31 \quad \text{for} \quad B_z \neq 0 \textit{ and } B_z^2 = 1.$$

Rotating Pinch Stability

- Sharper stability conditions could be obtained by solving the Euler-Lagrange equation associated with this reduced energy-Casimir functional subject to a normalization constraint on $\delta\psi$.
- Stability criteria obtained from dynamically accessible perturbations and Lagrangian perturbations are in general different when s is not a flux function. However, this pinch example demonstrates that, even though s is a flux function, dynamically accessible and Lagrangian stability can be equivalent.