Reciprocity and Adjoint Methods Applied to Charged Particle Dynamics

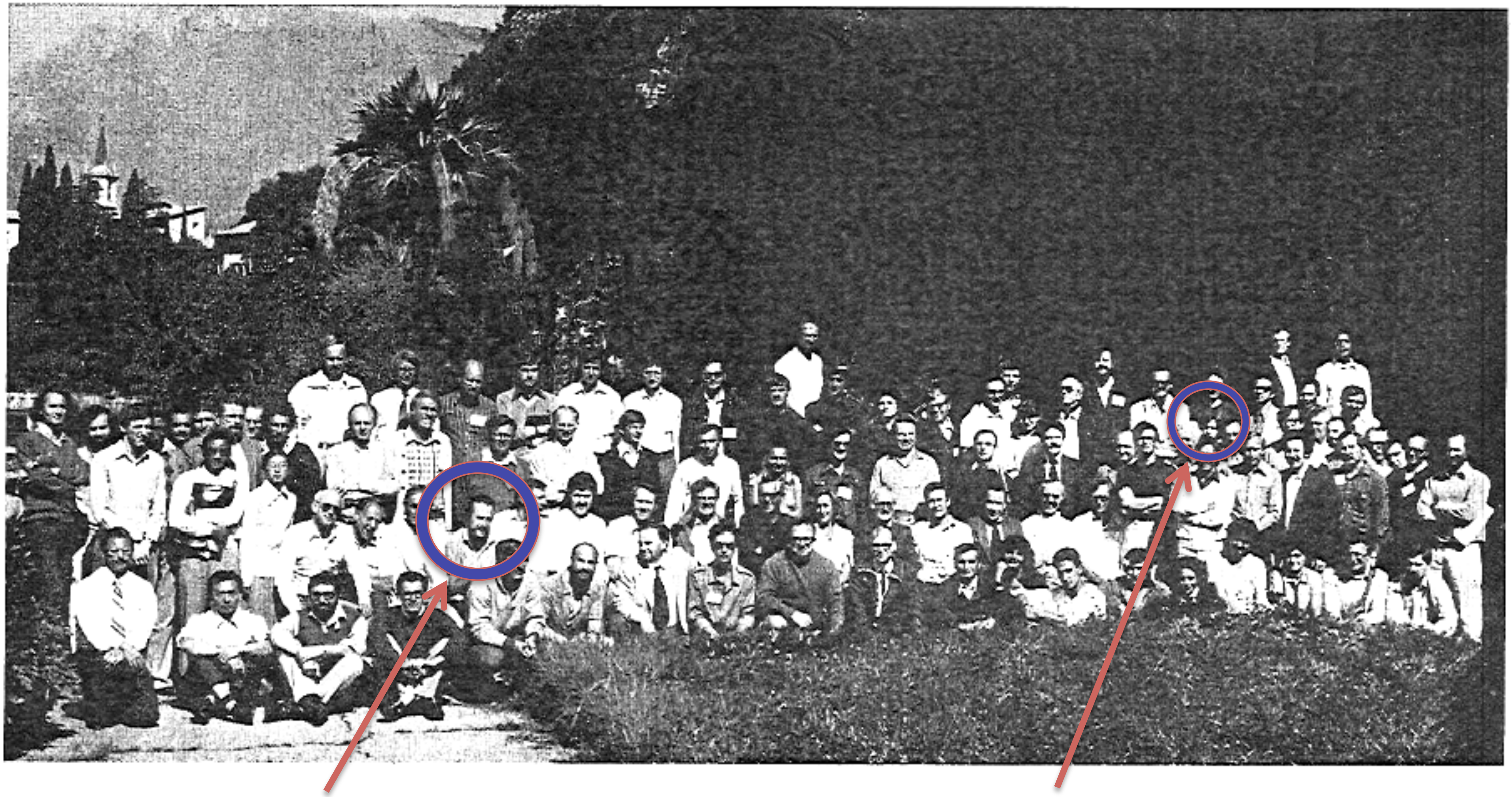
T.M. Antonsen Sept 18, 2017

Happy Birthday Francesco

International School of Plasma Physics

Course on: Physics of Plasmas Close to Thermonuclear Conditions

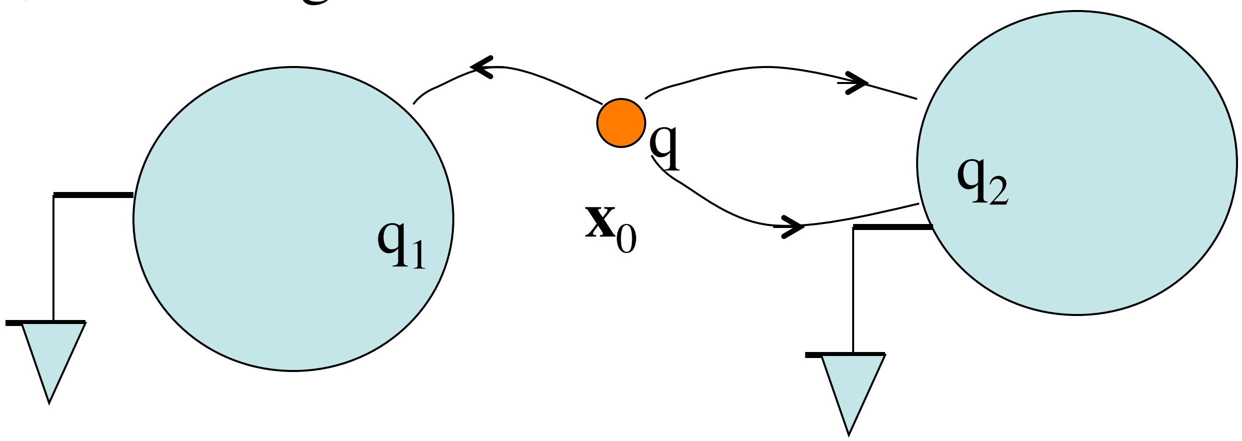
Villa Monastero, Varenna (Italy 27) August – 8 September 1979



Example

Our reading from the book of Jackson, Problems 1.12 and 1.13

A charge q is placed at an arbitrary point, $\mathbf{x}_o$, relative to two grounded, conducting electrodes.



What is the charge q_1 on the surface of electrode 1?

<u>Repeat for different $\mathbf{x}_0$</u>

Solution - Green's Reciprocation Theorem

<u>Prob #1</u>
Your
Problem

$$\nabla^2\phi = -q\delta(\mathbf{x}-\mathbf{x}_0) \qquad \text{BC:} \quad \phi|_{B1} = \phi|_{B2} = \phi(x\to\infty) = 0$$

$$q_1 = \int_{B1} d^2x\, \mathbf{n}\cdot\nabla\phi$$

<u>Prob #2</u>
N.Y. P.

$$\nabla^2\psi = 0 \qquad \text{BC:} \quad \psi|_{B1} = 1, \quad \psi|_{B2} = \psi(x\to\infty) = 0$$

Green's Theorem

$$\int_V d^3x\left(\psi\nabla^2\phi - \phi\nabla^2\psi\right) = \ldots$$

When the dust settles:

$$q_1 = -q\psi(\mathbf{x}_0)$$

George Green 1793-1841

The Green of Green Functions

In 1828, an English miller from Nottingham published a mathematical essay that generated little response. George Green's analysis, however, has since found applications in areas ranging from classical electrostatics to modern quantum field theory.

Lawrie Challis and Fred Sheard

his family built a house next to the mill, Green spent most of his days and many of his nights working and indeed living in the mill. When he was 31, Jane Smith bore him a daughter. They had seven children in all but never married. It was said that Green's father felt that Jane was not a suitable wife for the son of a prosperous tradesman and landowner and threatened to disinherit him.

Physics Today Dec. 2003

- Born in Nottingham (Home of Robin Hood)
- Father was a baker
- At age 8 enrolled in Robert Goodacre's Academy
- Left after 18 months (extent of formal education pre 40)
- Worked in bakery for 5 years
- Sent by father to town mill to learn to be a miller

- Fell in love with Jane, the miller's daughter.
- Green's father forbade marriage.
- Had 7 children with Jane.
- Self published work in 1828
- With help, entered Cambridge 1833, graduated 1837.
- "Discovered" by Lord Kelvin in 1840.
- Theory of Elasticity, refraction, evanescence
- Died of influenza, 1841

Green's Mill: still functions

Features of Problems Suited to Adjoint Approach

1. Many computations need to be repeated.
   (many different locations of charge, q)

2. Only a limited amount of information about the solution is required.
   (only want to know charge on electrode #1)

Basic Formulation – Linear Algebra

We wish to solve : $\underline{\underline{A}} \cdot \underline{x} = \underline{B}$ for many B's.

And then evaluate for each B: $D = \underline{C} \cdot \underline{x}^{\dagger}$ D(B) is the answer.

Instead solve for $\underline{y}$ *once*: $\underline{\underline{A}}^{\dagger} \cdot \underline{y} = \underline{C}$

Then: $D = \underline{B}^{\dagger} \cdot \underline{y}$

Other Examples of Reciprocity

Electrostatics	Symmetry of the Capacitance Matrix
Electromagnetics	Symmetry of the Inductance Matrix Symmetry of Scattering Matrix

Collisional Transport: Onsager Symmetry of off-diagonal elements of transport matrices.

Temperature gradient	➔	Electric current
Electric field	➔	Heat flux

RESEARCH | OPEN ACCESS

Courtesy, Elizabeth Paul

Adjoint methods for car aerodynamics

Carsten Othmer

Journal of Mathematics in Industry 2014 4:6 | DOI: 10.1186/2190-5983-4-6 | © Othmer; licensee Springer. 2014

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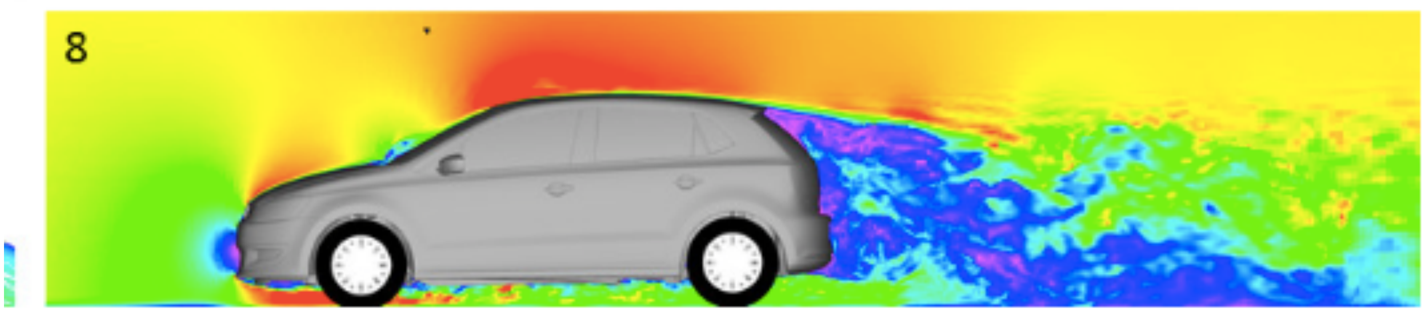
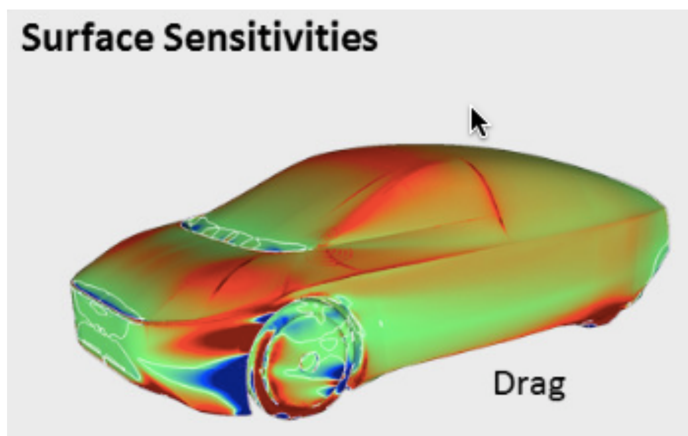


Optimize shape to minimize drag.



Result is also aesthetically appealing.

Super Computer

1985 Volvo 240 DL

2017 Porche Panamera

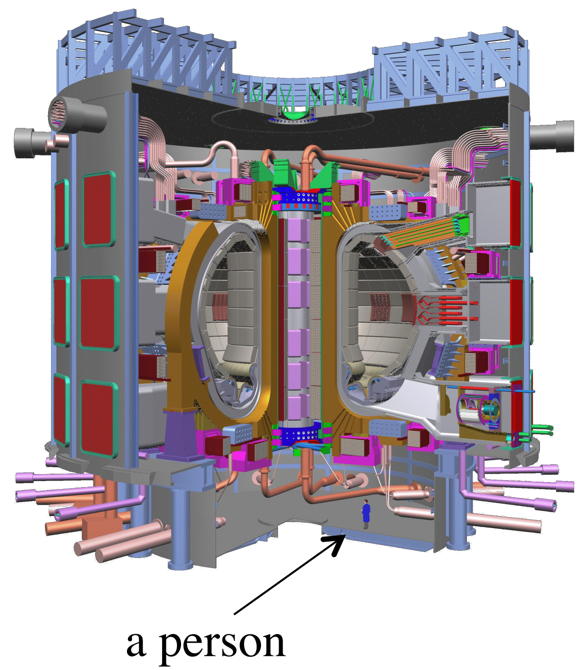
That's more like it !!!

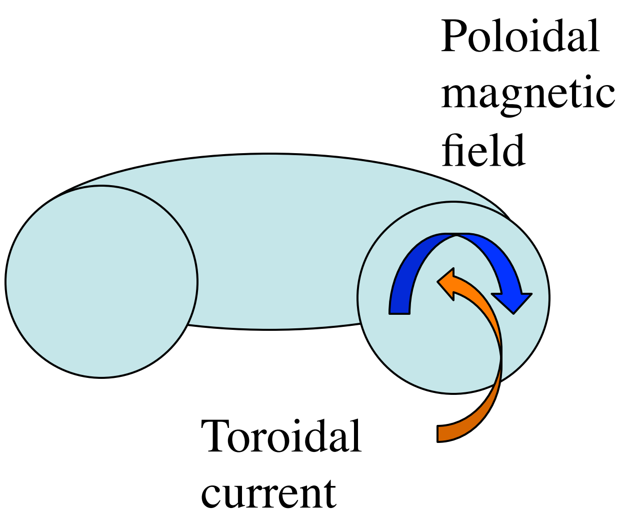
RF Current Drive in Fusion Plasmas

Magnetic Confinement: ITER

US-EU-Russia-Japan-India Collaboration
Will be built in Cardarache France
Completion 2016??

http://www.iter.org/





Injecting RF waves can drive a toroidal current. N. Fisch

RF Current Drive Efficiency

Original Langevin Treatment: Nat Fisch, PRL (1978)

RF pushes particles to higher energy.

Collisions relax distribution back to equilibrium. J/P_D

Γ = RF induced velocity space particle flux

$$J = \int d^3v \, \Gamma \cdot \frac{\partial}{\partial v} \Psi \qquad P_D = \int d^3v \, \Gamma \cdot \frac{\partial}{\partial v} \varepsilon$$

ψ inversely proportional to collision rate

Adjoint Problem: distribution function driven by a DC electric field.
TMA and KR Chu, PoF 25, (1982)

Adjoint Approach

For a Homogeneous Plasma, we want to solve steady state kinetic equation

$$\frac{\partial f}{\partial t} = 0 = C(f) - \frac{\partial}{\partial \mathbf{v}} \cdot \Gamma$$

Problem #1

Linearized collision operator

RF induced velocity space flux

Then find parallel current

$$J_{||} = -e \int d^3v \, v_{||} f$$

Adjoint problem: Spitzer-Harm

$$-ev_{||} f_M = C(g)$$

Problem #2

Parallel current

$$J_{||} = \int d^3v \, \Gamma \cdot \frac{\partial}{\partial \mathbf{v}} \left(\frac{g}{f_M} \right)$$

RF Current Drive Efficiency

RF pushes particles to higher energy.

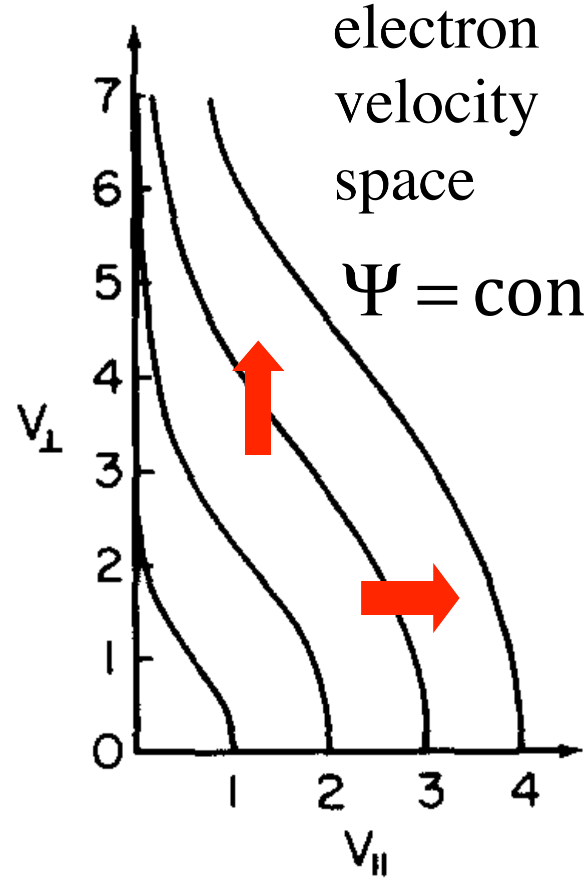
Collisions relax distribution back to equilibrium. J/P_D

$$J = \int d^3v\, \Gamma \cdot \frac{\partial}{\partial v}\Psi$$

Adjoint Problem: distribution function driven by a DC electric field.
TMA and KR Chu, PoF 25, (1982)

electron velocity space

$\Psi = \text{const.}$



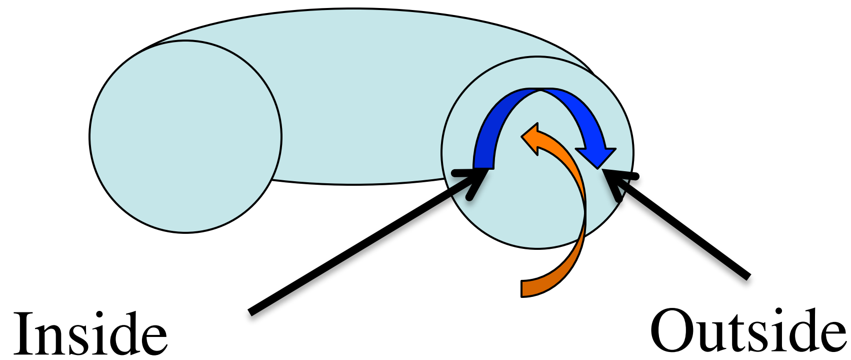
Toroidal Geometry Makes a Difference

(TMA and KR Chu, PoF 25, (1982))

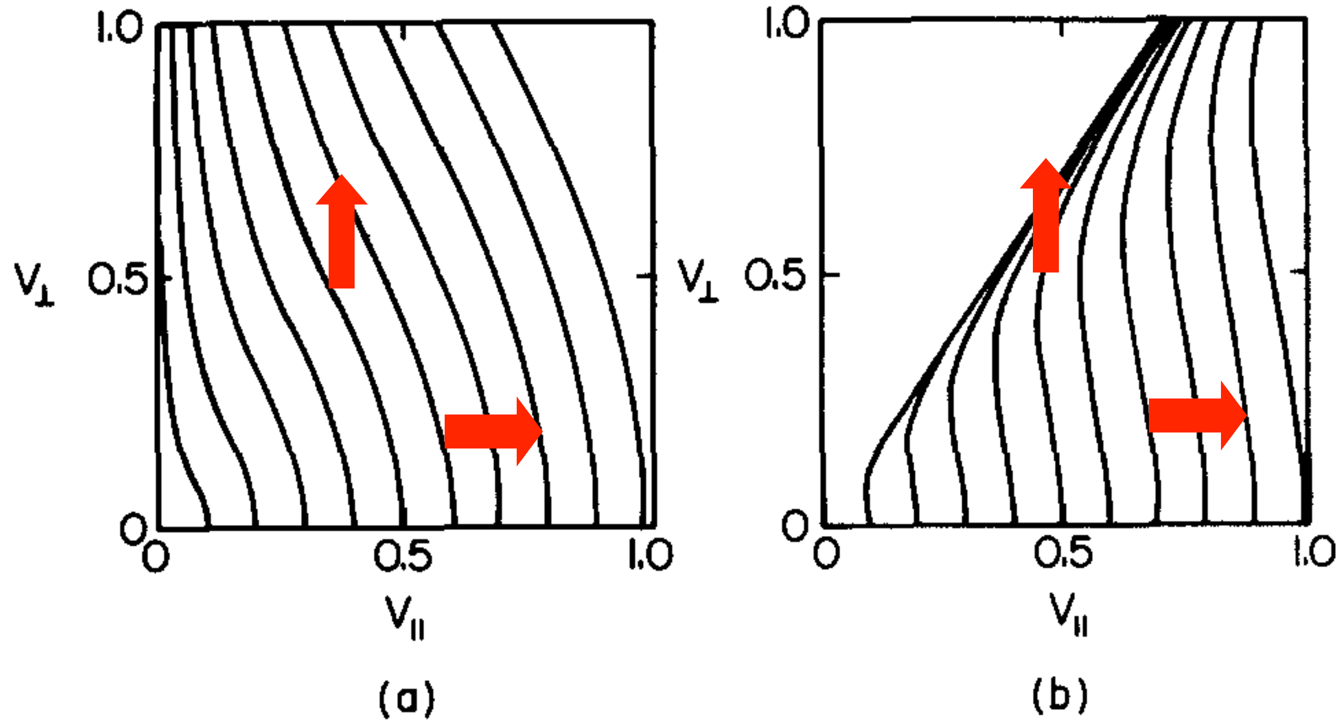
streaming

$$v_{\|}\mathbf{b}\cdot\nabla g - e v_{\|} f_M = C(g)$$

$$J = \int d^3v\,\Gamma\cdot\frac{\partial}{\partial v}\,{g}\big/{f_M}$$



$\Psi = \text{const.}$



Global Beam Sensitivity Function for Electron Guns

Goal

Derive and Calculate a function that gives the variation of specific beam parameters to

- variations in electrode potential/position
- variations in magnet current/position

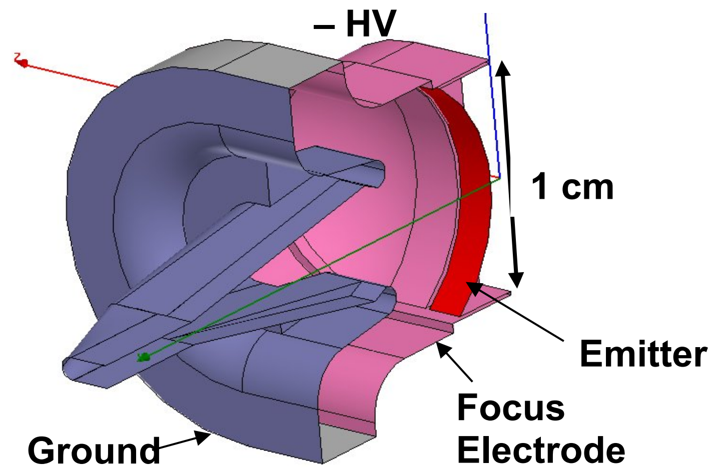
Can be used to

- establish manufacturing tolerances
- optimize gun designs

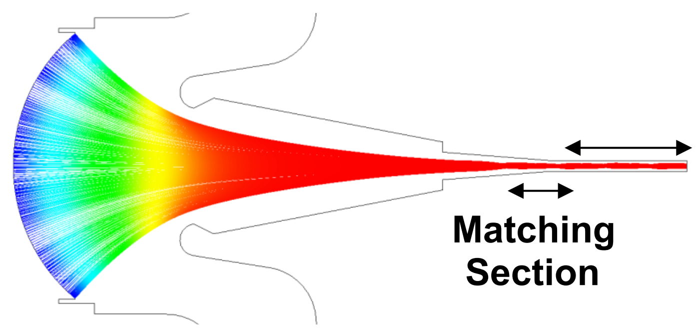
Should be embedded in gun code (e.g. Michelle)

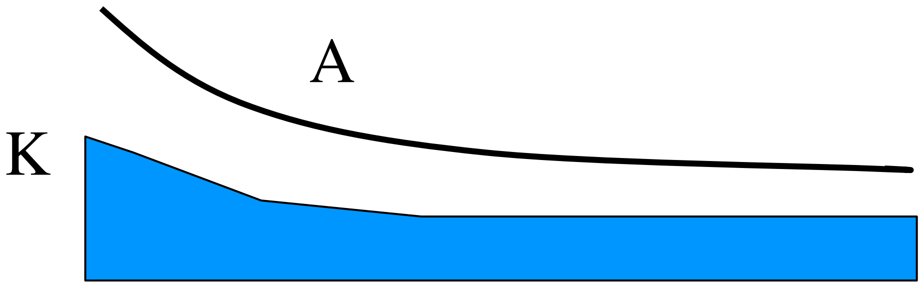
Michelle: Petillo, J; Eppley, K; Panagos, D; et al., IEEE TPS 30, 1238-1264 (2002).

Solid Model of Electrode



Cut away view of trajectories





Code (Michelle) solves the following equations:

Equations of motion for N particles j=1,N

$$\frac{d\boldsymbol{x}_j}{dt} = \frac{\partial H}{\partial \boldsymbol{p}} \qquad \frac{d\boldsymbol{p}_j}{dt} = -\frac{\partial H}{\partial \boldsymbol{x}}$$

Accumulates a charge density

$$\rho(\boldsymbol{x}) = \sum_j I_j \int_0^{T_j} dt\, \delta(\boldsymbol{x} - \boldsymbol{x}_j(t))$$

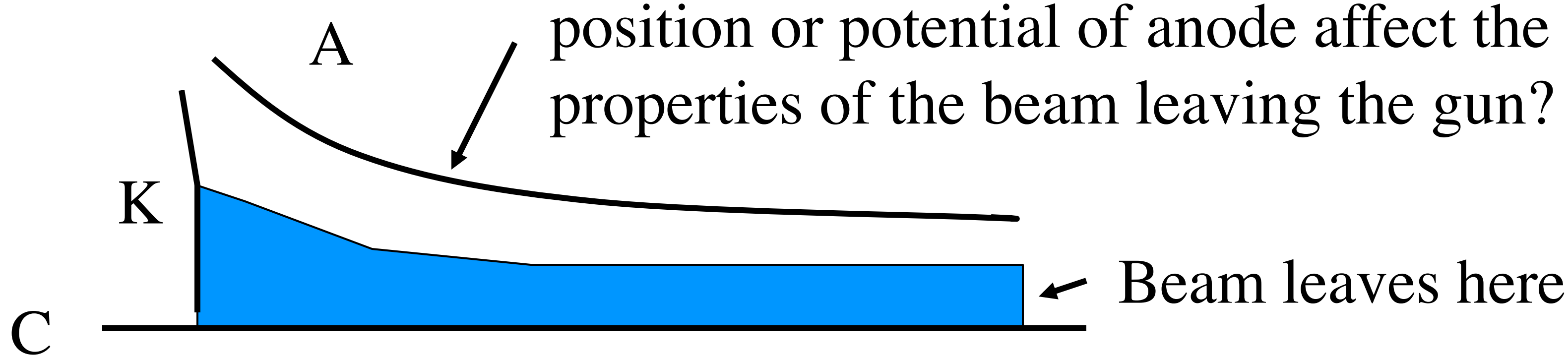
Solves Poisson E

$$-\nabla^2 \Phi = \rho / \varepsilon_0$$

Iterates until converged

Sensitivity Function Example

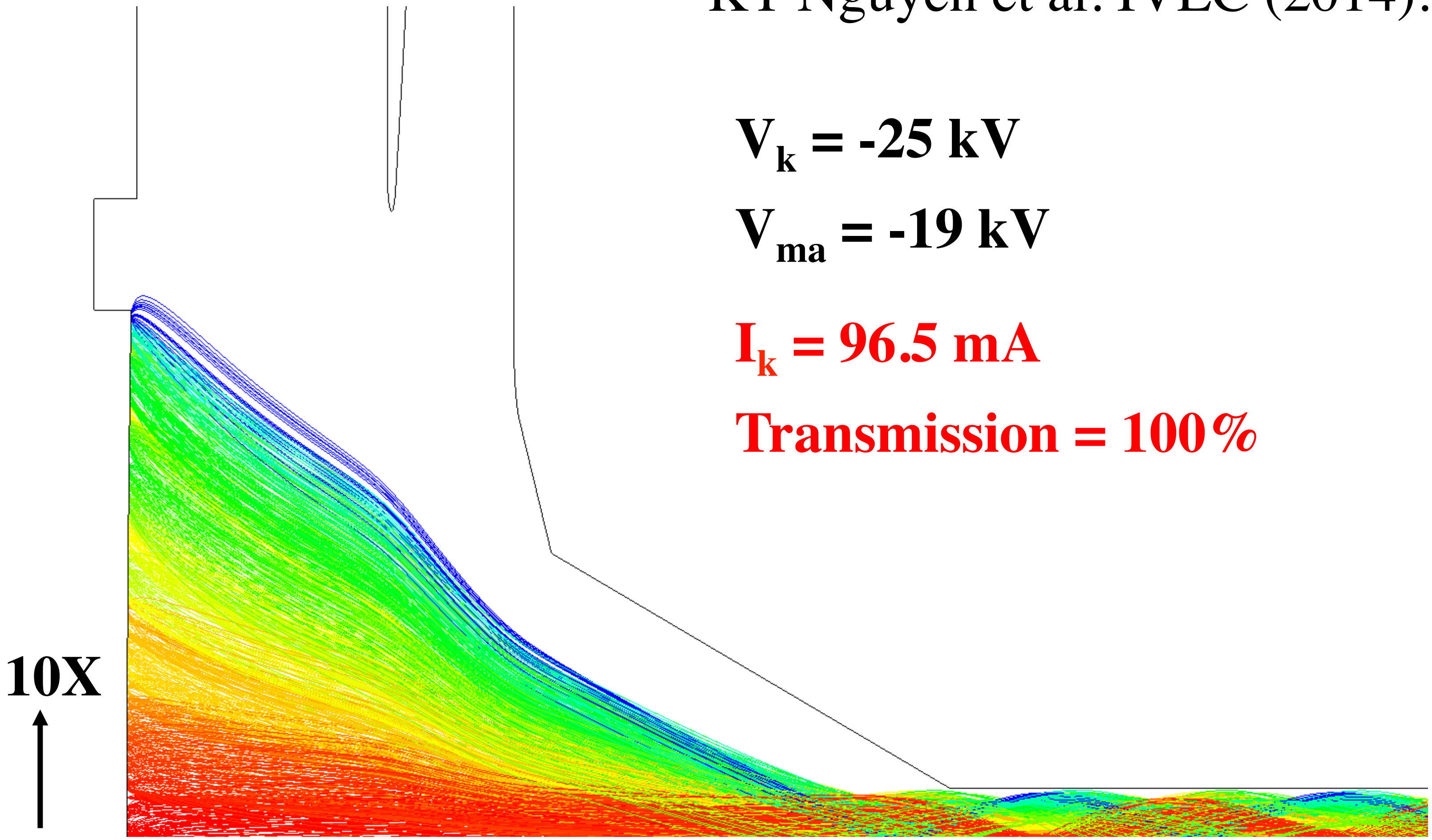
<u>Basic question:</u> How do small changes in position or potential of anode affect the properties of the beam leaving the gun?



<u>Conventional approach</u>: trial and error. Do many simulations with different anode potentials or positions select the best based on some metric measured at the exit.

Beamstick: *Gun Baseline Design*
Thermal Beam

KT Nguyen et al. IVEC (2014).



V_k = -25 kV

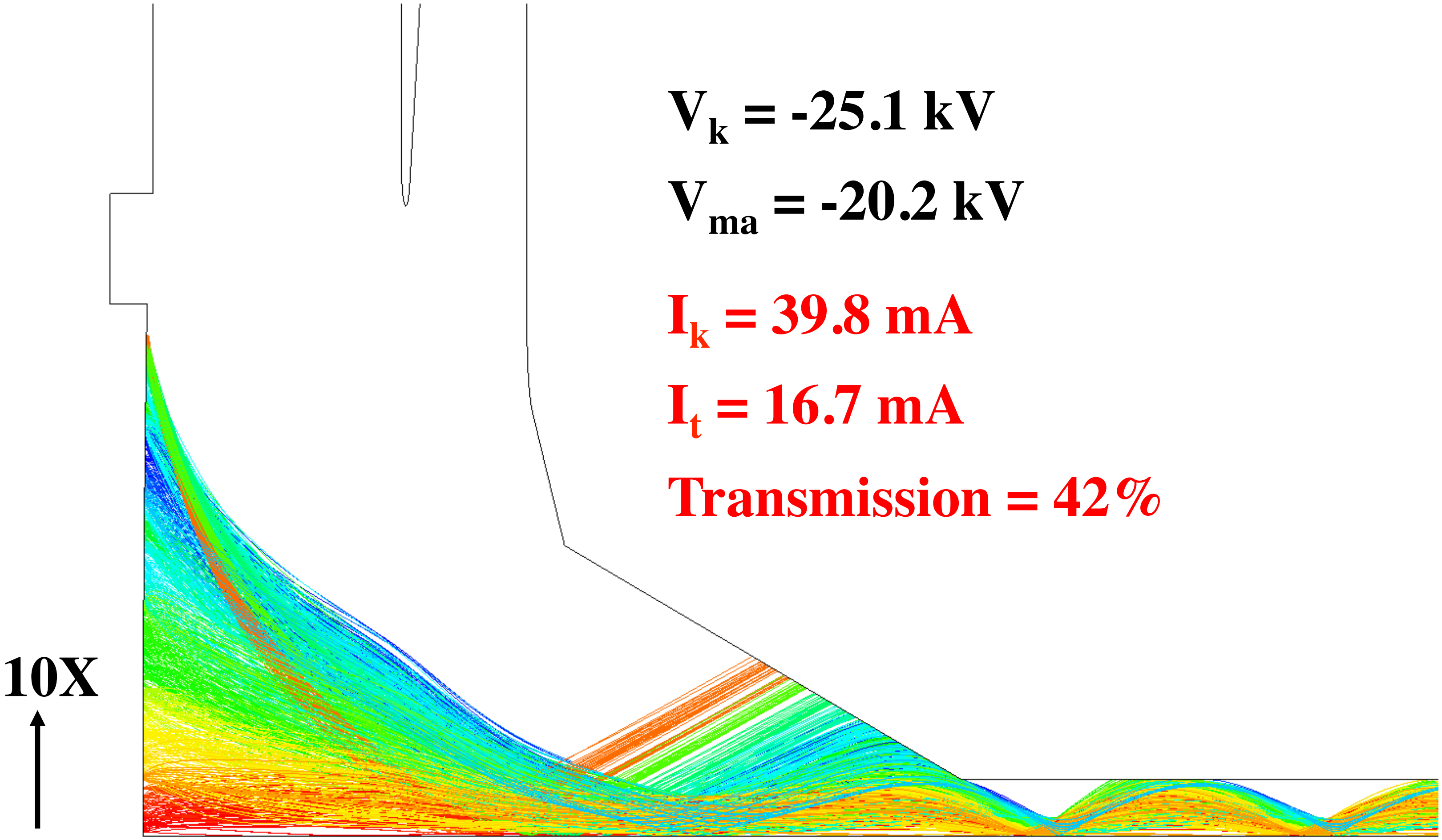
V_{ma} = -19 kV

I_k = 96.5 mA

Transmission = 100%

Approved for public release; Distribution unlimited

Beamstick: *Gun Baseline Geometry*
Particle Trajectories at Actual Voltages



V_k = -25.1 kV

V_{ma} = -20.2 kV

I_k = 39.8 mA

I_t = 16.7 mA

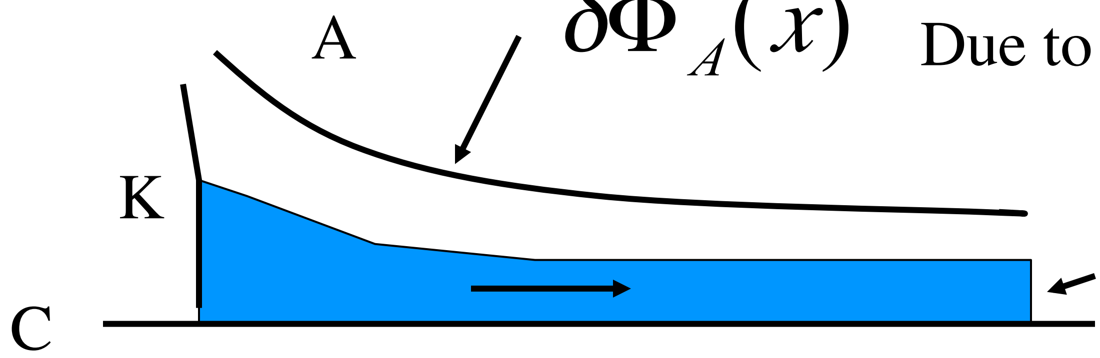
Transmission = 42%

Approved for public release; Distribution unlimited

It can be shown …

Problem #1

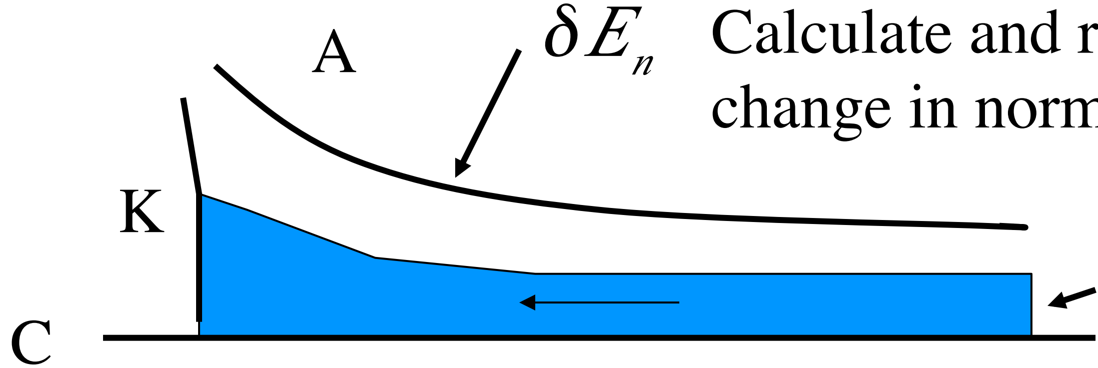
$\delta\Phi_A(x)$ Due to wall displacement



Change in beam radius

Problem #2

δE_n Calculate and record change in normal E.

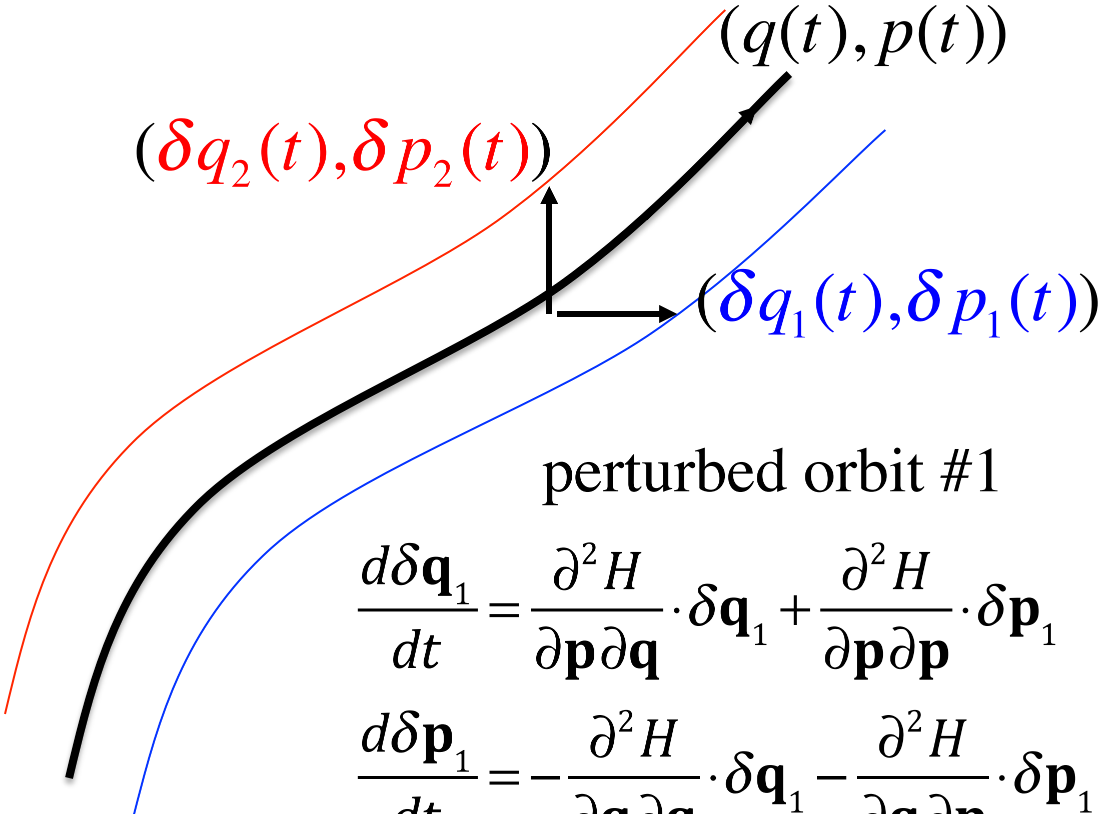


Reverse and perturb electron coordinates

Electrons run backwards

δE_n Is the sensitivity function

Hamilton's Equations *H(**p**,**q**,t)* Conserve Symplectic Area



$$\frac{d\mathbf{q}}{dt} = \frac{\partial H}{\partial \mathbf{p}}$$

$$\frac{d\mathbf{p}}{dt} = -\frac{\partial H}{\partial \mathbf{q}}$$

perturbed orbit #1

$$\frac{d\delta\mathbf{q}_1}{dt} = \frac{\partial^2 H}{\partial\mathbf{p}\partial\mathbf{q}} \cdot \delta\mathbf{q}_1 + \frac{\partial^2 H}{\partial\mathbf{p}\partial\mathbf{p}} \cdot \delta\mathbf{p}_1$$

$$\frac{d\delta\mathbf{p}_1}{dt} = -\frac{\partial^2 H}{\partial\mathbf{q}\partial\mathbf{q}} \cdot \delta\mathbf{q}_1 - \frac{\partial^2 H}{\partial\mathbf{q}\partial\mathbf{p}} \cdot \delta\mathbf{p}_1$$

perturbed orbit #2

$$\frac{d\delta\mathbf{q}_2}{dt} = \ldots$$

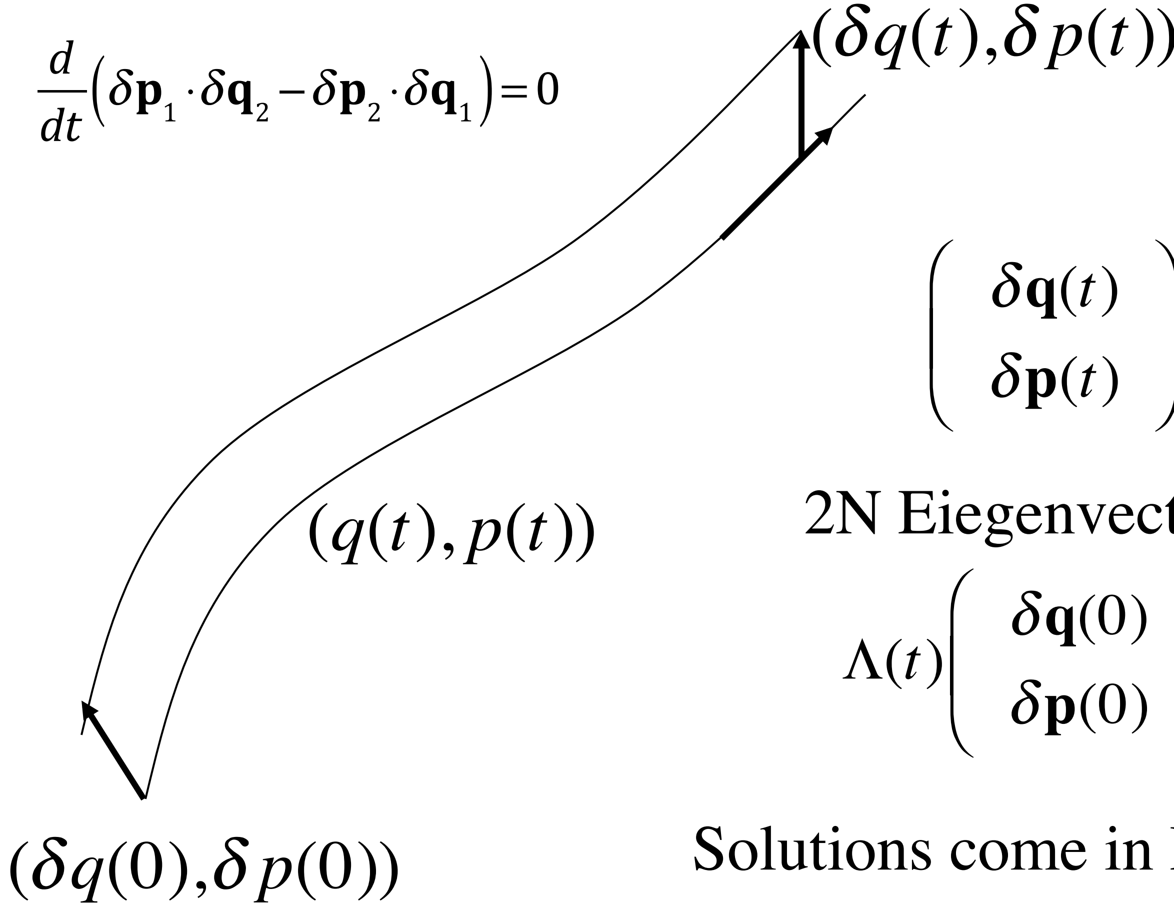
$$\frac{d\delta\mathbf{p}_2}{dt} = -\ldots$$

$$\frac{d}{dt}\left(\delta\mathbf{p}_1 \cdot \delta\mathbf{q}_2 - \delta\mathbf{p}_2 \cdot \delta\mathbf{q}_1\right) = 0$$

Area conserved for any choice of 1 and 2

Jacobian Matrix – M(t)

$$\frac{d}{dt}\left(\delta\mathbf{p}_1\cdot\delta\mathbf{q}_2-\delta\mathbf{p}_2\cdot\delta\mathbf{q}_1\right)=0$$



$$\begin{pmatrix}\delta\mathbf{q}(t)\\ \delta\mathbf{p}(t)\end{pmatrix}=\underline{\underline{M}}^{2N\mathrm{x}2N}(t)\cdot\begin{pmatrix}\delta\mathbf{q}(0)\\ \delta\mathbf{p}(0)\end{pmatrix}$$

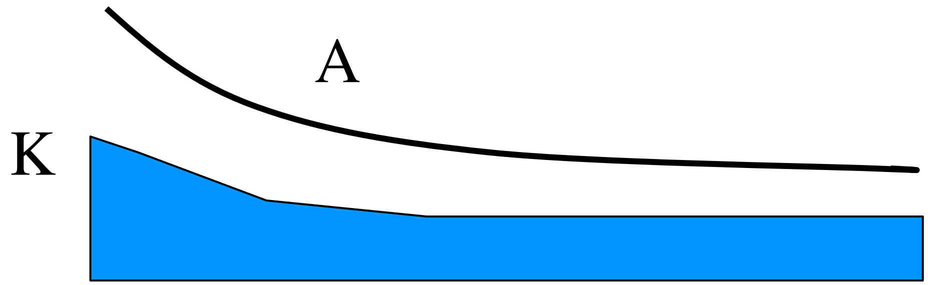
2N Eiegenvectors and Eigenvalues of M

$$\Lambda(t)\begin{pmatrix}\delta\mathbf{q}(0)\\ \delta\mathbf{p}(0)\end{pmatrix}=\underline{\underline{M}}(t)\cdot\begin{pmatrix}\delta\mathbf{q}(0)\\ \delta\mathbf{p}(0)\end{pmatrix}$$

Solutions come in N pairs- $\Lambda_1\Lambda_2=1$

Eigenvectors from different pairs orthogonal

$$\left(\delta\mathbf{p}_1\cdot\delta\mathbf{q}_2-\delta\mathbf{p}_2\cdot\delta\mathbf{q}_1\right)=0$$



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Accumulates a charge density

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Solves Poisson Equation

$$-\nabla^2 \Phi = \rho / \varepsilon_0$$

Iterates until converged

Reference Solution + Two Linearized Solutions

$$\left(\mathbf{x}_j,\mathbf{p}_j\right) \rightarrow \left(\mathbf{x}_j,\mathbf{p}_j\right) + \left(\delta\mathbf{x}_j,\delta\mathbf{p}_j\right)$$

$$\rho(\mathbf{x}) \rightarrow \rho(\mathbf{x}) + \delta\rho(\mathbf{x})$$

$$\Phi(\mathbf{x}) \rightarrow \Phi(\mathbf{x}) + \delta\Phi(\mathbf{x})$$

Reference Solution Perturbation

<u>Two Linearized Solutions</u>

$[\delta x_j(t),\delta p_j(t)]$ true

$[\delta\hat{x}_j(t),\delta\hat{p}_j(t)]$ adjoint

subject to different BC's

Can show

$$\sum_j I_j \left(\delta\hat{\mathbf{p}}_j\cdot\delta\mathbf{x}_j - \delta\mathbf{p}_j\cdot\delta\hat{\mathbf{x}}_j\right)\Big|_0^{T_j} = -q\varepsilon_0 \int_S da\,\mathbf{n}\cdot\left[\delta\Phi\nabla\delta\hat{\Phi} - \delta\hat{\Phi}\nabla\delta\Phi\right]$$

Can show

$$\sum_j I_j \left(\delta \hat{\mathbf{p}}_j \cdot \delta \mathbf{x}_j - \delta \mathbf{p}_j \cdot \delta \hat{\mathbf{x}}_j \right) \Big|_0^{T_j} = -q\varepsilon_0 \int_S da \mathbf{n} \cdot \left[\delta\Phi \nabla \delta\hat{\Phi} - \delta\hat{\Phi} \nabla \delta\Phi \right]$$

Problem #1 (true problem) Unperturbed trajectories at cathode, Perturbed potential on boundary.

$$\delta p_j \Big|_0 = 0, \quad \delta q_j \Big|_0 = 0, \ \delta\Phi(\mathbf{x}) \neq 0$$

Problem #2 (adjoint problem) Perturbed trajectories at exit, Unperturbed potential on boundary.

$$\delta \hat{p}_j \Big|_T = \lambda \mathbf{x}_{\perp j}, \quad \delta q_j \Big|_T = 0, \ \delta\hat{\Phi}(\mathbf{x}) = 0$$

$$\lambda I R_{RMS} \delta R_{RMS} = \lambda \sum_j I_j \left(\mathbf{x}_j \cdot \delta \mathbf{x}_j \right) \Big|_{T_j} = -q\varepsilon_0 \int_S da \, \delta\Phi \left(\mathbf{n} \cdot \nabla \delta\hat{\Phi} \right)$$

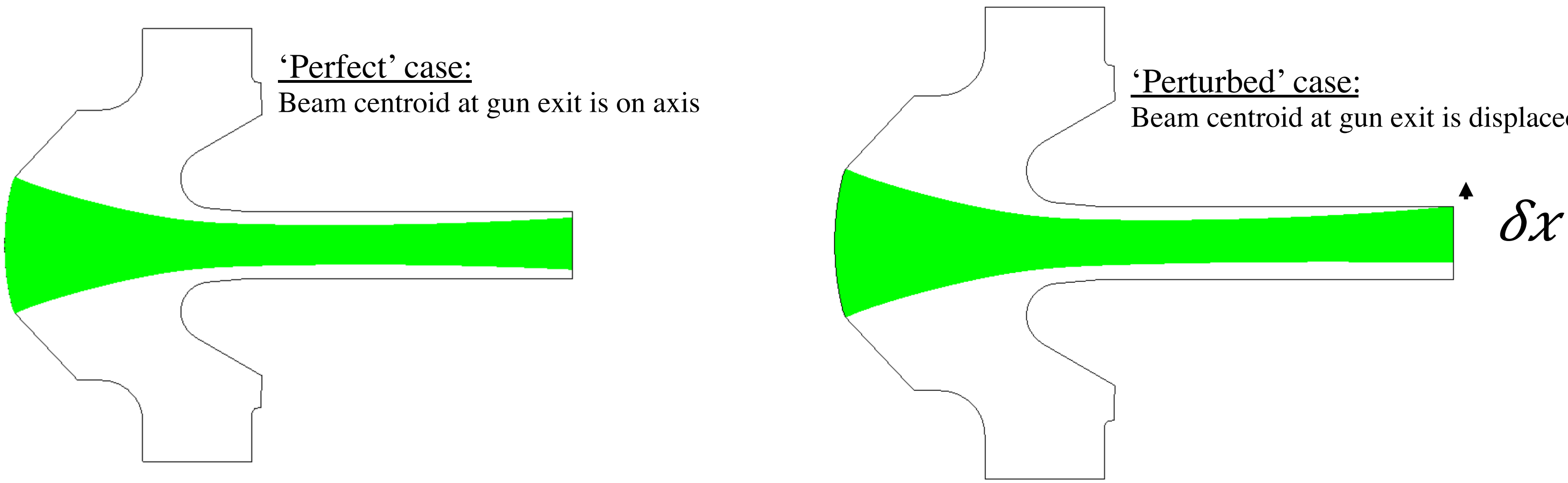
Sensitivity Function

Theoretical Study of Statistical Variations

Example of the Adjoint Method in Action

Problem: Compute the displacement of the beam in a sheet beam gun due to a small change in anode potential or a small displacement of the anode:

MICHELLE Simulations of Sheet Beam Gun



The adjoint method gives us a way to compute the displacement of the beam *without* re-running MICHELLE:

$$\delta x = -\frac{q}{4\pi\lambda I}\int_S da\,\mathbf{n}\cdot\delta\Phi\,\nabla\,\delta\hat{\Phi}$$

δx = Beam centroid displacement at gun exit

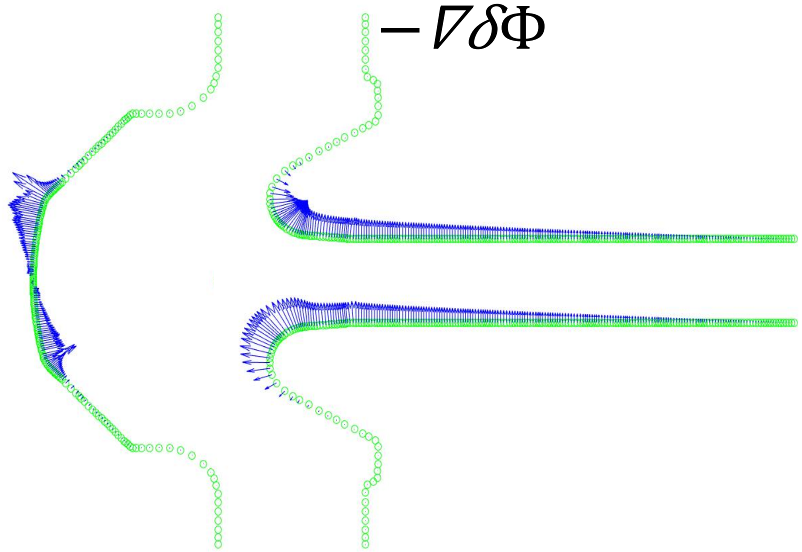
$\delta\Phi$ = Small change or error in anode or other electrode potential

$-\boldsymbol{n}\cdot\nabla\Phi$ = Sensitivity (Green's) function

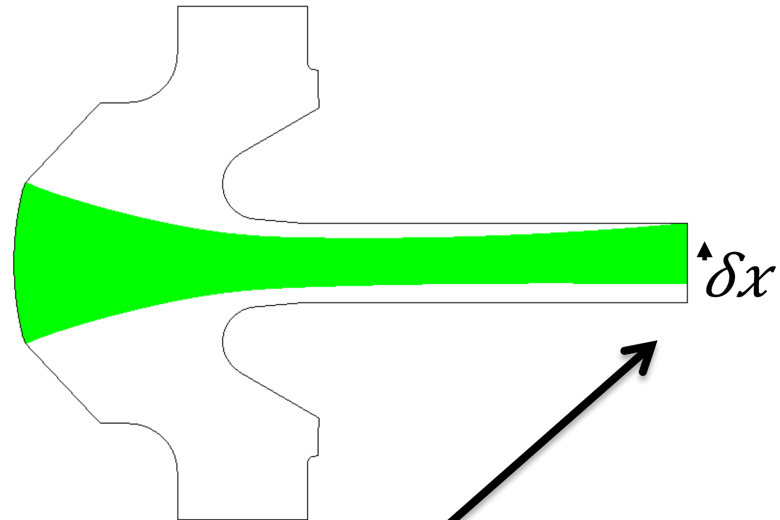
Task 1-4: Theoretical Study of Statistical Variations

Successful Test of Adjoint Method !

Vector plot of the 'sensitivity' or Green's function



'Direct' MICHELLE Simulation with Perturbed Anode Voltages

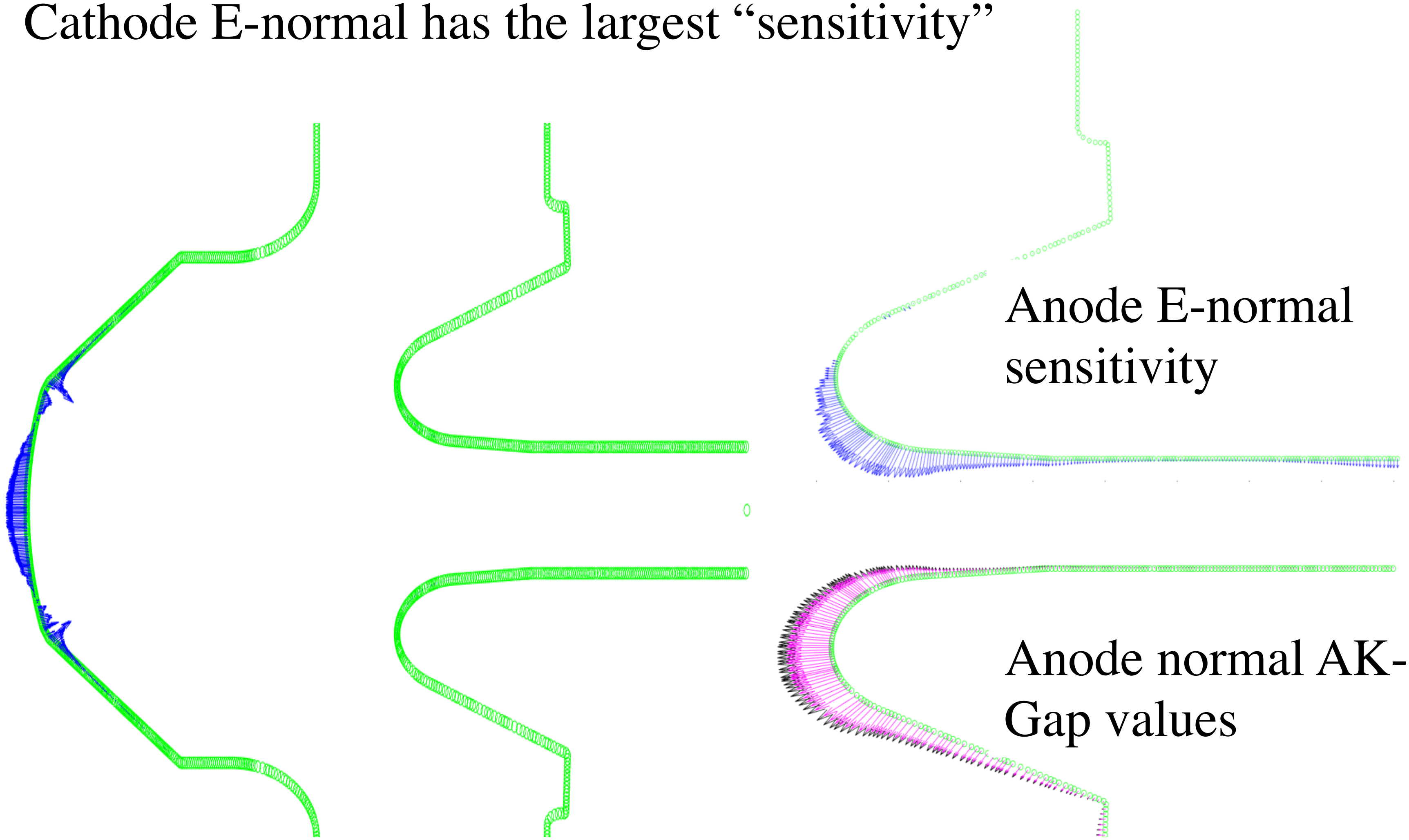


$$\delta x = -\frac{q\varepsilon_0}{\lambda I}\int_S da\,\mathbf{n}\cdot\delta\Phi\nabla\delta\hat{\Phi}$$

Predicted displacement / Calculated displacement = 0.9969

RMS radius sensitivity

Cathode E-normal has the largest "sensitivity"



$$\lambda I R_{RMS} \delta R_{RMS} = \lambda \sum_j I_j \left(\mathbf{x}_j \cdot \delta \mathbf{x}_j \right)\Big|_{T_j} = -q \varepsilon_0 \int_S da\, \delta \boldsymbol{\Phi} \left(\mathbf{n} \cdot \nabla \delta \hat{\boldsymbol{\Phi}} \right)$$

Conclusion: Next Steps

Add Magnetic field

sensitivity function

$$\sum_j I_j \left(\delta\hat{\mathbf{p}}_j \cdot \delta\mathbf{x}_j - \delta\mathbf{p}_j \cdot \delta\hat{\mathbf{x}}_j\right)\Big|_{T_j} = -q\varepsilon_0 \int_S da\,\delta\Phi_A\, n\cdot\nabla\delta\hat{\Phi}_S - \mu_0 \int d^3x\,\delta\mathbf{j}_m \cdot q\delta\hat{\mathbf{A}}_s$$

Change in magnetization current

Add time dependence

Implement in an optimization routine

Thank You

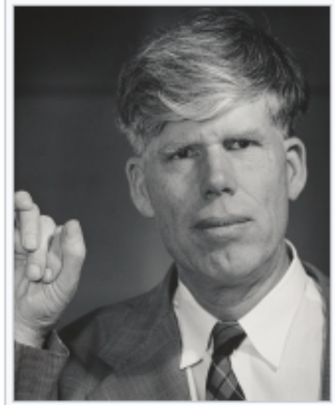
Happy Birthday Francesco

Signal to Noise Ratio in a Gyroklystron

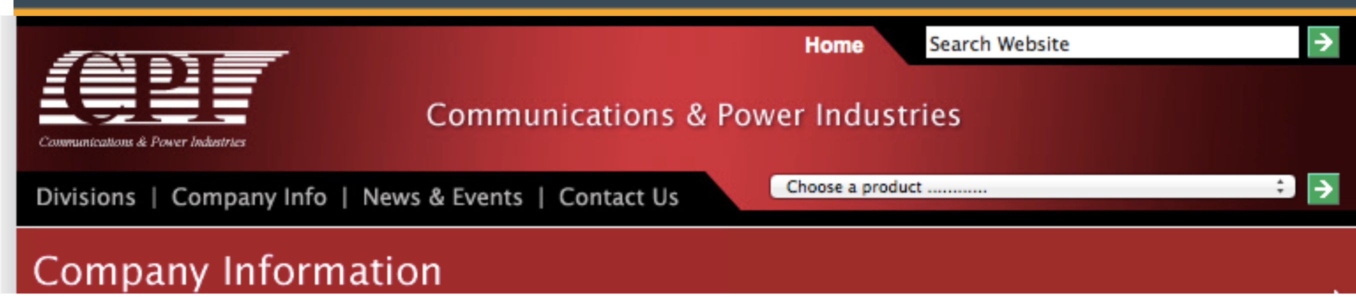
Klystron: invented in 1937 by the Varian brothers. One of the first Palo Alto High Tech. firms.

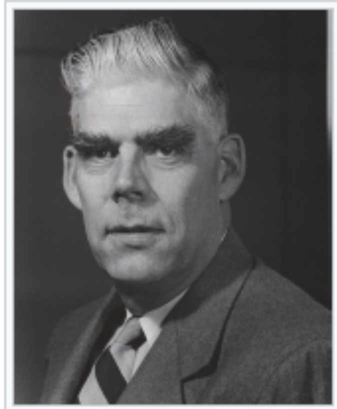
High Power Source of Microwaves

Radar, Particle Accelerators, (LHC 16 x 300 kW), etc



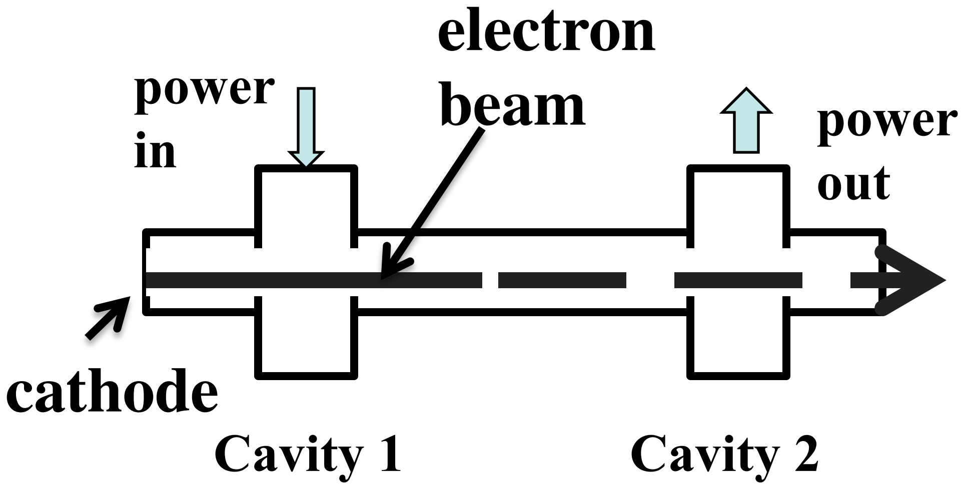
Russell Varian (1898–1959). Photograph by Ansel Adams.



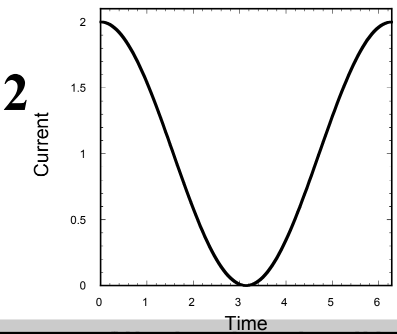


Sigurd Varian (1901–1961) Photograph by Ansel Adams.

Velocity Modulation   Ballistic Bunching



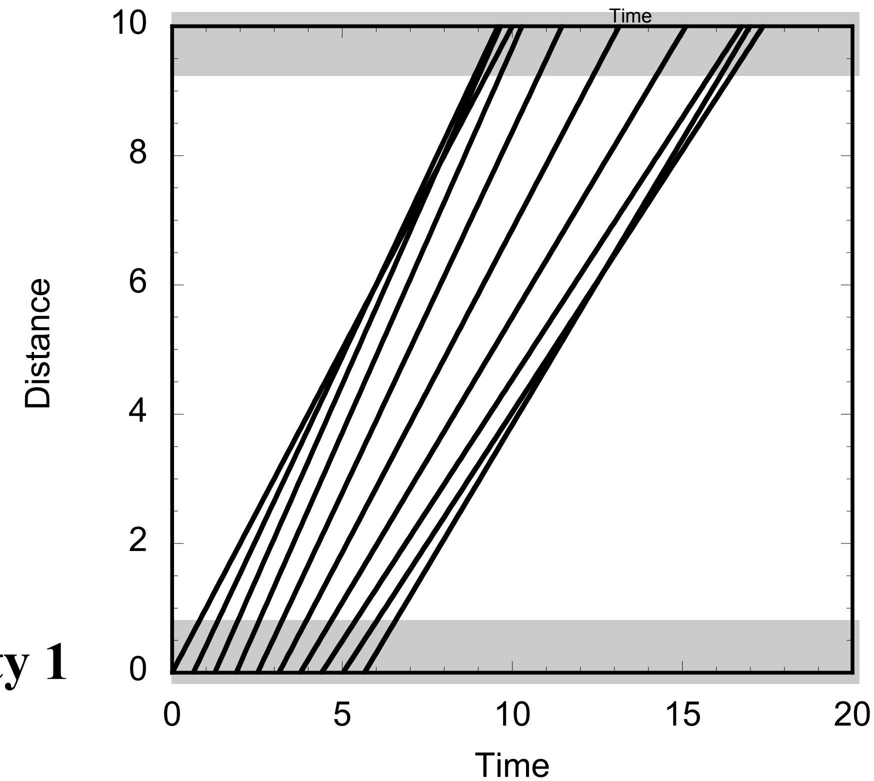
Cavity 2
I(t)



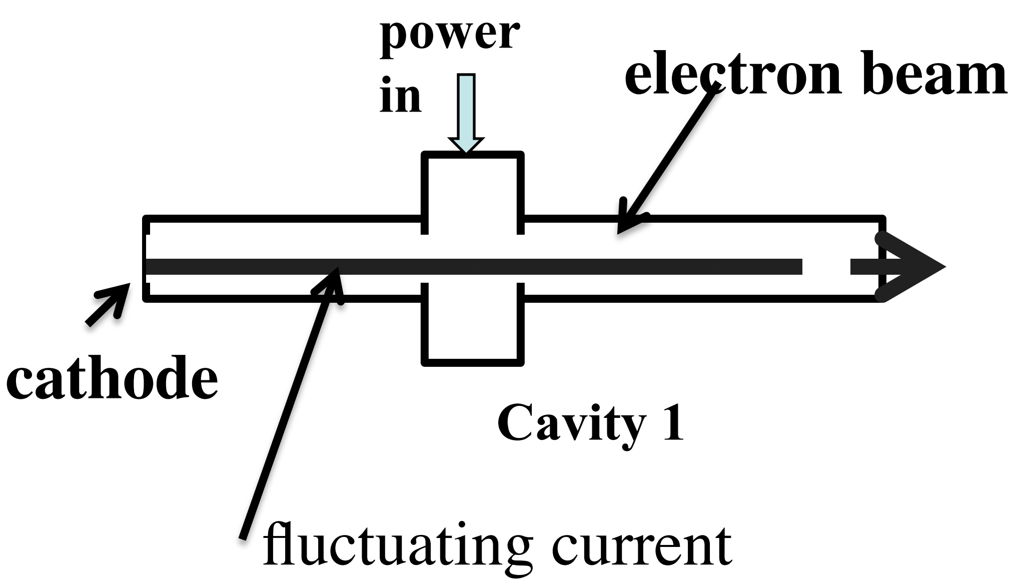
Field in cavity 1 gives small time dependent velocity modulation

Fast electrons catch up to slow electrons giving large current modulation.

Cavity 1



Shot Noise in Input Cavity



Signal to noise ratio determined by ratio of injected signal power in cavity to fluctuating beam power due to discrete electronic charge.

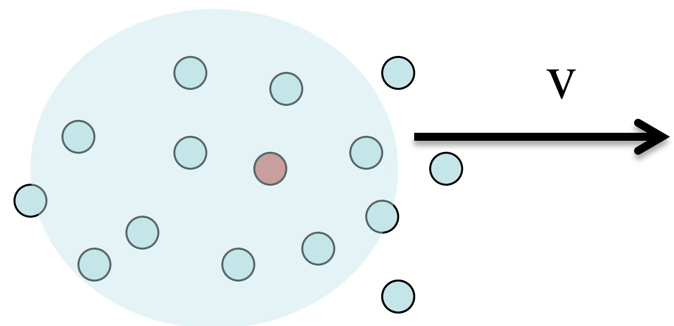
If arrival times are independent and identically distributed, fluctuations are a white noise process.

$$\left\langle I^2(t)\right\rangle = \int \frac{d\omega}{2\pi} e\left\langle I\right\rangle$$

But, this is wrong: electrons become correlated on transit from cathode to cavity.

Shielding Cloud

Direct calculation



For an ensemble (N>>1) of initial conditions at that cathode of test electrons, calculate the shielding cloud and total current fluctuation that excites the relevant mode in the cavity.

Adjoint approach:

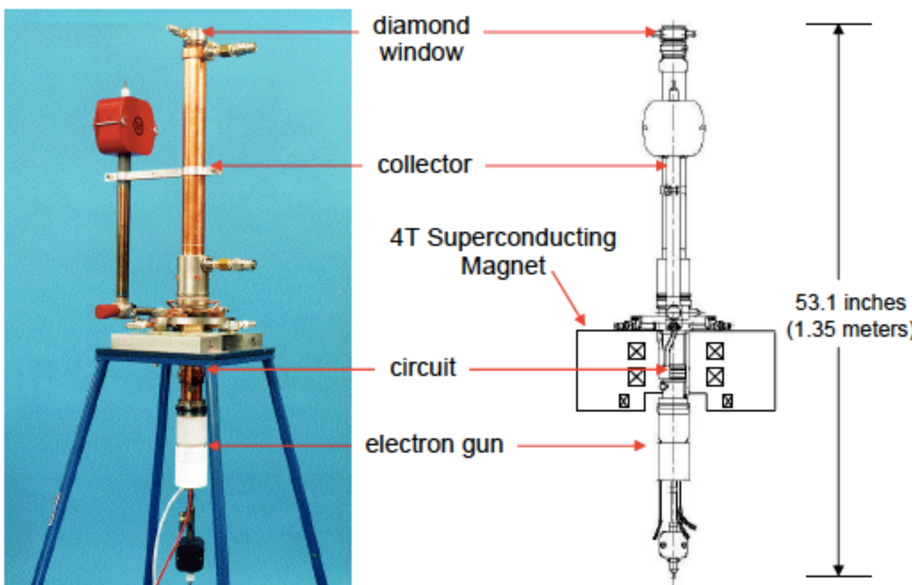
For a given cavity mode profile, integrate the kinetic equation (once) backward in time to find the sensitivity function, average over initial ensemble.

Gyroklystron

VGB-8194 Gyrotron Amplifier

CPI Communications & Power Industries microwave power products division

CPI gyrotron amplifiers are the only commercially available W-band amplifiers with high peak and average output powers. The VGB-8194, a five-cavity amplifier, can be operated at peak output powers up to 100 kW and average output powers up to 10 kW. The full-width-half-maximum bandwidth is 700 MHz and the saturated gain is 35 dB. A CVD diamond window, developed for use on high-power gyrotron oscillators, has been adapted for the amplifier. The VGB-8194 is available with a refrigerator-cooled 4 Telsa superconducting magnet which does not require liquid cryogens.



Features

- High Power
- Broad Bandwidth
- High Gain
- Axial Output
- CVD Diamond Output Window
- Cryogen-Free Superconducting Magnet

Typical Operating Parameters

Peak Output Power	100 kW
Avg. Output Power	10 kW
Center Frequency	93.5-94.5 GHz
Bandwidth (-3 dB)	700 MHz
Gain	35 dB
Beam Voltage	65 kV
Mod-Anode Voltage	17 kV
Beam Current	6 A
Output Mode	TE_{01}

Strong applied magnetic field.

Electrons gyrate as they pass through the device.

Operating frequency

$$\omega \simeq \frac{eB}{m\gamma}$$

Relativistic factor

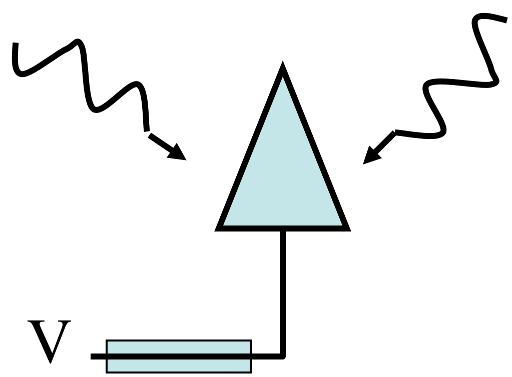
Shielding cloud is unstable!
Must control growth.

TMA, W Manheimer, and A Fliflet, PoP (2001)

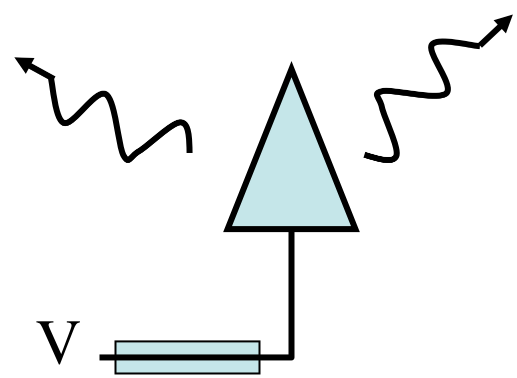
EM Reciprocity

Example:

- Antenna sending and receiving radiation patterns are equal due to time reversal symmetry of ME.
- Direct calculation of receiving pattern requires many simulations
- Instead, calculate sending pattern and invoke reciprocity

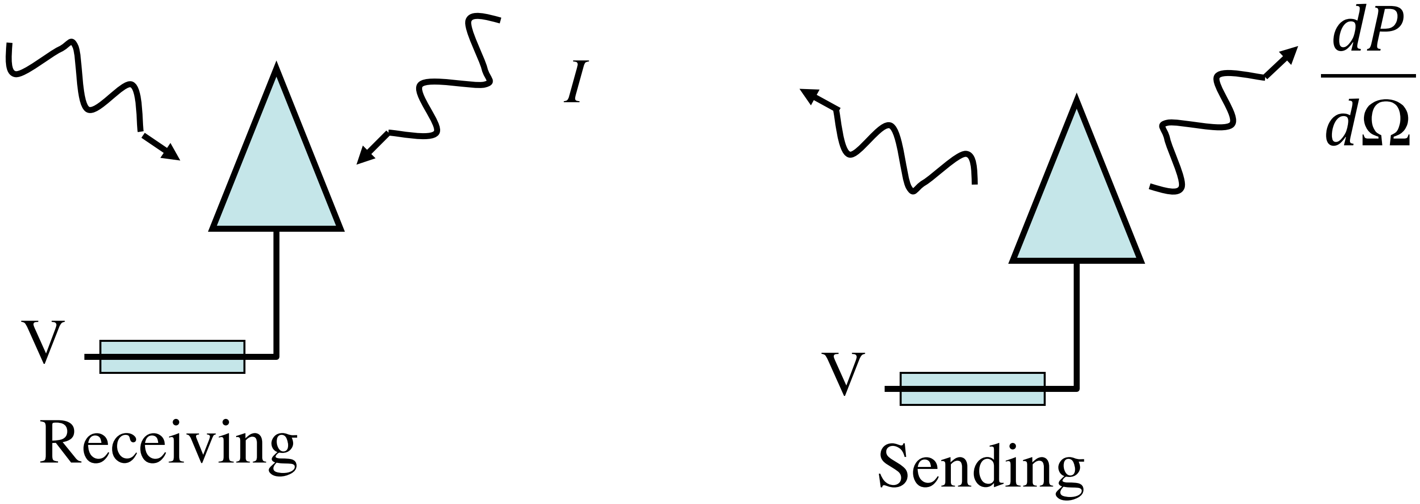


Receiving



Sending

Effective Area – Antenna Gain



Power received → $P_R = A_e(\Omega) I$ ← Incident intensity

Effective area → $A_e(\Omega) = \dfrac{\lambda^2 G(\Omega)}{4\pi}$ ← gain →

$$G(\Omega) = \frac{dP}{d\Omega} / P_T$$

← Power per unit solid angle

$$P_T = \int \frac{dP}{d\Omega} d\Omega$$

Adjoint Problems

- Calculation of RF current drive in magnetic confinement plasma configurations, TMA and KR Chu, PoF 25, (1982)
- Calculation of RF induced transport in magnetic confinement plasmas, TMA and K. Yoshioka, PoF 29, (1986), Nucl. Fusion, 26 (1986).
- Spontaneous poloidal flow spin-up, Bull Am Phys Soc. (1994).
- Shot noise on gyrotron beams, TMA, W Manheimer, and A Fliflet, PoP (2001).
- Beam optics sensitivity function TMA, D. Chernin, J. Petillo, TBP

CODING ERROR !!!

1985 Volvo 240 DL

RF Induced Transport

TMA and K. Yoshioka, PoF 29, (1986)

Perturbed neoclassical DF

$$v_{\parallel}\mathbf{b}\cdot\nabla f+\frac{\partial}{\partial\mathbf{v}}\cdot\Gamma=C(f)$$

Response to a radial gradient

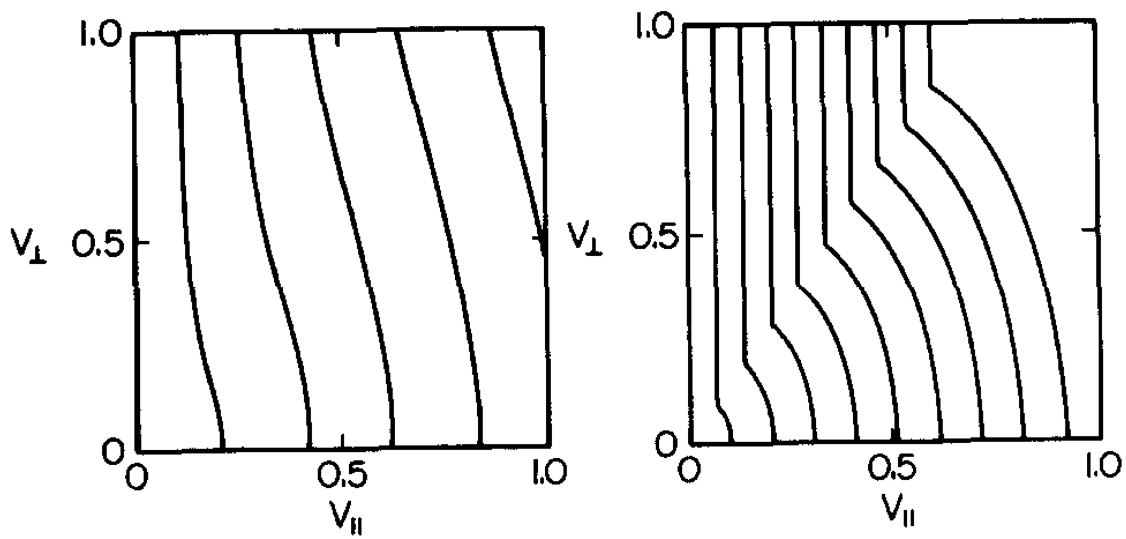
$$v_{\parallel}\mathbf{b}\cdot\nabla g+v_{d}\cdot\nabla f_{M}=C(g)$$

Fluctuation induced radial flux

$$\left\langle\int d^3v\, f\, v_{d}\cdot\nabla\psi\right\rangle=\left\langle\int d^3v\,\Gamma\cdot\frac{\partial}{\partial\mathbf{v}}\, g\big/ f_{M}\right\rangle$$

Radial Flux driven by RF

$$\left\langle \int d^3v \; f \, v_d \cdot \nabla \psi \right\rangle = \left\langle \int d^3v \; \Gamma \cdot \frac{\partial}{\partial \mathbf{v}} \, g \Big/ f_M \right\rangle$$



G-Band (220 GHz) Folded Waveguide Amplifier Structure
Joye et al., IEEE Trans ED (2014) 60W, 15 GHz BW