# Lagrange, Dirac, and the Imposition of Constraints on Fluid Flow

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Many collaborators beginning with John Greene, then Dieter Pfirsch, and many students over the past 35+ years. Recently, Tassi, Chandre, Vittot, Lingam, D'Avignon, Yoshida, Kawazura, Abdelhamid, Andreussi, and ...

Honoree Francesco PEGORARO!

This talk is an historical tour through Hamiltonian fluid/plasma field theory.

# Fluid and Plasma Theories – Matter Models

Systems that describe the motion of matter as dynamical systems of the form

$$\frac{\partial \Psi}{\partial t} = \mathcal{O}(\Psi)\,, \qquad \mathcal{O} \text{ nonlinear PDEs, intregrodifferential} \ldots$$

Examples:

- kinetic theories
  - Vlasov equation, drift kinetic equations, gyrokinetics, ...
- multifluid fluid theories
  - 2-Fluid coupled to Maxwell's equations, ...
- <u>magnetofluids</u>
  - MHD, HMHD, IMHD, XMHD, etc.
- <u>hybrids</u>

<u>Common Features:</u>

- Nondissipative part is Hamiltonian with corresponding action principle. Dissipation should be real!
- Two descriptions: Lagrangian and Eulerian

# MHD

Momentum:

$$\rho \left( \frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V} \cdot \nabla) \boldsymbol{V} \right) = -\nabla p + \boldsymbol{J} \times \boldsymbol{B} \qquad \leftarrow \qquad \boldsymbol{J} = \nabla \times \boldsymbol{B}$$

Mass:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{V}) = 0$$

Ohm's Law:

$$\boldsymbol{E} + \boldsymbol{V} \times \boldsymbol{B} = \eta \boldsymbol{J} \approx 0 \qquad \leftarrow \qquad \text{ideal MHD}$$

Faraday's Law:

$$\frac{\partial \boldsymbol{B}}{\partial t} = -\nabla \times \boldsymbol{E} = \nabla \times (\boldsymbol{V} \times \boldsymbol{B})$$

Thermo:

$$\frac{\partial s}{\partial t} + \boldsymbol{V} \cdot \nabla s = 0 \qquad \text{or barotropic} \rightarrow \qquad p = p(\rho) = \kappa \rho^{\gamma}$$

# Eulerian Variable Description

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Observables $\{\rho(r,t), \boldsymbol{V}(r,t), \boldsymbol{B}(r,t)\}$ where $r \in D \subset \mathbb{R}^3$ with BC.

# Eulerian Variable Description

Observables $\{\rho(r,t), \boldsymbol{V}(r,t), \boldsymbol{B}(r,t)\}$ constitue a Field Theory.

## Lagrangian Variable Description

Assume a continuum of fluid particles or fluid elements and follow them around.

# Lagrangian Variable Description

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Dynamical canonical variables $\{q(a,t), \pi(a,t)\}$.

# Fluid Kinematics

Lagrange *Mécanique Analytique* (1788) $\rightarrow$ Newcomb (1962) MHD

Lagrangian Variables:

Fluid occupies domain $D \subset \mathbb{R}^3$ e.g. $(x, y, z)$ or $(x, y)$

Fluid particle position $q(a, t)$, $q_t : D \rightarrow D$ bijection, etc.

Particles label: $a$ e.g. $q(a, 0) = a$.

Deformation: $\frac{\partial q^i}{\partial a^j} = q^i_{,j}$

Determinant: $\mathcal{J} = \det(q^i_{,j}) \neq 0 \Rightarrow a = q^{-1}(r, t)$

Identity: $q^i_{,k} a^k_{,j} = \delta^i_j$

Volume: $d^3q = \mathcal{J} d^3a$

Area: $(d^2q)_i = \mathcal{J} a^j_{,i} (d^2a)_j$

Line: $(dq)_i = q^i_{,j} (da)_j$

<u>Eulerian Variables:</u>

Observation point: $r$

Velocity field: $\boldsymbol{V}(r,t) =?$ Probe sees $\dot{q}(a,t)$ for some $a$.

What is $a$ ? $r = q(a,t) \quad \Rightarrow \quad a = q^{-1}(r,t)$

$$\boldsymbol{V}(r,t) = \dot{q} \circ q^{-1} = \dot{q}(a,t)|_{a=q^{-1}(r,t)}$$

# IDEAL MHD

Attributes and the **Lagrange to Euler Map**:

Velocity (vector field):

$$\boldsymbol{V}(r,t) = \dot{q} \circ q^{-1} = \dot{q}(a,t)\big|_{a=q^{-1}(r,t)}$$

Entropy (1-form):

$$s(r,t) = s_0\big|_{a=q^{-1}(r,t)} \ ,$$

Mass (3-form):

$$\rho d^3x = \rho_0 d^3a \quad \Rightarrow \quad \rho(r,t) = \frac{\rho_0}{\mathcal{J}}\bigg|_{a=q^{-1}(r,t)} \ .$$

$B$-Flux (2-form):

$$B \cdot d^2x = B_0 \cdot d^2a \quad \Rightarrow \quad B^i(r,t) = \frac{q^i_{,j} B^j_0}{\mathcal{J}}\bigg|_{a=q^{-1}(r,t)} \ .$$

# Hamiltonian Form

Hamilton's Equations:

$$\dot{p}_i = \{p_i, H\} = -\frac{\partial H}{\partial q^i}\,, \qquad \dot{q}^i = \{q^i, H\} = \frac{\partial H}{\partial p_i}\,,$$

Natural Hamiltonians:

$$H(p,q) = p^2/2 + V(q) = K + V$$

Poisson Bracket:

$$\{F, G\} = \frac{\partial F}{\partial q^i}\frac{\partial G}{\partial p_i} - \frac{\partial G}{\partial q^i}\frac{\partial F}{\partial p_i}$$

Phase Space Coordinates: $z = (q, p)$

$$\dot{z}^i = J_c^{ij}\frac{\partial H}{\partial z^j} = [z^i, H] \qquad (J_c^{ij}) = \begin{pmatrix} 0_N & I_N \\ -I_N & 0_N \end{pmatrix}$$

symplectic 2-form = (cosymplectic form)$^{-1}$: $\omega^c_{ij} J_c^{jk} = \delta_i^k$

# Hamiltonian

Kinetic Energy:

$$K[q] = \frac{1}{2}\int_D d^3a\, \rho_0 |\dot{q}|^2 = \frac{1}{2}\int_D d^3x\, \rho |v|^2$$

Potential Energy:

$$\begin{aligned} V[q] \;&=\; \int_D d^3a \left(\rho_0 \mathcal{U}(\rho_0/\mathcal{J}, s_0) + \frac{1}{2}\frac{|q^i_{,j} B_0^j|^2}{\mathcal{J}^2}\right) \\ &=\; \int_D d^3x \left(\rho U(\rho, s) + \frac{1}{2}|\mathbf{B}|^2\right) \end{aligned}$$

Lagrangian (due to Lagrange for fluid! Newcomb for MHD):

$$L[q] = K - V\,,$$

Legendre $\rightarrow$ Hamiltonian (energy):

$$H[q] = K + V\,,$$

## Hamiltonian Structure – an early field theory

Fréchet Derivative $\rightarrow$ Variational Derivative:

$$\delta F = \frac{d}{d\epsilon} F[q + \epsilon \delta q]\Big|_{\epsilon=0} = DF \cdot \delta q \qquad \rightarrow \qquad \frac{\delta F}{\delta q}$$

Poisson Bracket:

$$\{F, G\} = \int_D d^3a \left( \frac{\delta F}{\delta q^i} \frac{\delta G}{\delta \pi_i} - \frac{\delta G}{\delta q^i} \frac{\delta F}{\delta \pi_i} \right)$$

EOM:

$$\dot{q} = \{q, H\} \qquad \dot{\pi} = \{\pi, H\}$$

# Reduction

Simple particle with canonical coordinates: $(\mathbf{r}, \mathbf{p})$

Equations of motion:

$$\dot{\mathbf{r}} = \frac{\partial H}{\partial \mathbf{p}} \qquad \text{and} \qquad \dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{r}}$$

Angular momentum:

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}$$

Reduction:

$$\{L_x, L_y\} = L_z$$

Casimir:

$$\{|\mathbf{L}|^2, f\} = 0 \qquad \forall\, f$$

If $H(\mathbf{L}) \Rightarrow$ closure, i.e. reduction of system to dimension three!

# Lagrangian to Eulerian Reduction

Lagrange $\mapsto$ Euler:

$$(q, p) \mapsto (\rho, \mathbf{V}, \mathbf{B})$$

Assume functionals $\quad F[q, \pi] = \hat{F}[\rho, s, \boldsymbol{V}, \boldsymbol{B}]$

Chain Rule $\Rightarrow$ noncanonical Eulerian Poisson Bracket:

$$\begin{aligned} \{F, G\}^{MHD} \;=\; & -\int_D d^3x \Big\{ [F_\rho \nabla \cdot G_{\mathbf{V}} + F_{\mathbf{V}} \cdot \nabla G_\rho] \\ & - \left[ \frac{(\nabla \times \mathbf{V})}{\rho} \cdot (F_{\mathbf{V}} \times G_{\mathbf{V}}) \right] \\ & - \left[ \frac{\mathbf{B}}{\rho} \cdot \Big( F_{\mathbf{V}} \times (\nabla \times G_{\mathbf{B}}) - G_{\mathbf{V}} \times (\nabla \times F_{\mathbf{B}}) \Big) \right] \Big\} \end{aligned}$$

where $F_{\boldsymbol{V}} := \delta F / \delta \boldsymbol{V}$ etc. With Hamiltonian

$$H = \int_D d^3x \Big( \rho |\boldsymbol{V}|^2/2 + \rho U(\rho, s) + |\boldsymbol{B}|^2/2 \Big)$$

gives MHD in Eulerian form $\psi_t = \{\psi, H\}$. (pjm & Greene 1980)

# Generalized Hamiltonian Structure

Sophus Lie (1890)

Noncanonical Coordinates:

$$\dot{z}^i = J^{ij}\frac{\partial H}{\partial z^j} = \{z^i, H\}\,, \qquad \{A,B\} = \frac{\partial A}{\partial z^i}J^{ij}(z)\frac{\partial B}{\partial z^j}$$

Poisson Bracket Properties:

antisymmetry $\longrightarrow$ $\{A,B\} = -\{B,A\}\,,$

Jacobi identity $\longrightarrow$ $\{A,\{B,C\}\} + \{B,\{C,A\}\} + \{C,\{A,B\}\} = 0$

G. Darboux: $detJ \neq 0 \Longrightarrow J \rightarrow J_c$ Canonical Coordinates

Sophus Lie: $detJ = 0 \Longrightarrow$ Canonical Coordinates plus Casimirs

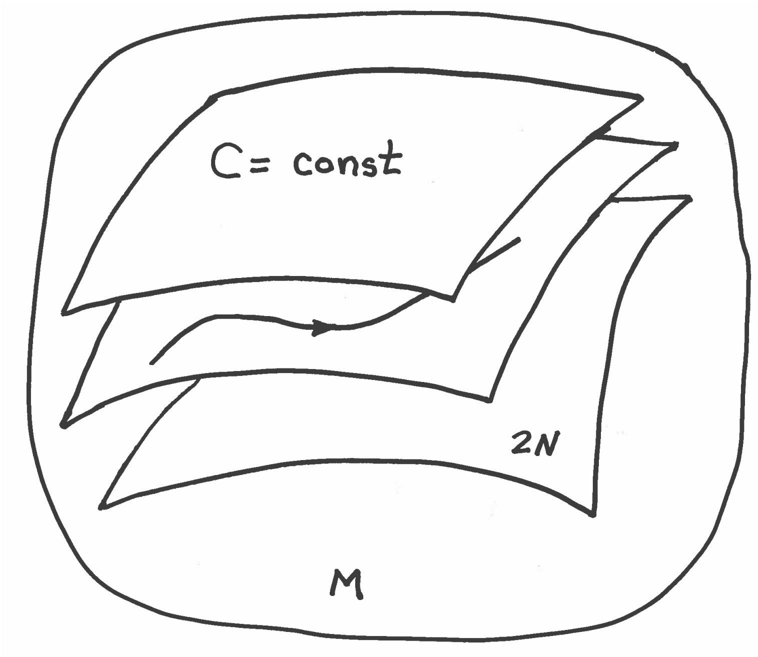
Finite dimensions to infinite dimensions!

# Poisson Manifold $\mathcal{P}$ Cartoon

Degeneracy in $J$ $\Rightarrow$ Casimirs:

$$\{f, C\} = 0 \quad \forall\ f : \mathcal{P} \to \mathbb{R}$$

Lie-Darboux Foliation by Casimir (symplectic) leaves:




Leaves are symplectic rearrangements in infinite dimensions.

# Casimir Invariants

Casimir Invariants $C$:

$$\{F, C\}^{MHD} = 0 \qquad \forall \text{ functionals } F.$$

Casimirs are a consequence of Poisson bracket degeneracy.

Helicities etc. are examples of Casimir invariants. Many new found for more complex models.

## Two Constraints: $\nabla \cdot \mathbf{V} = 0$ vs. $\nabla \cdot \mathbf{B} = 0$

$$\nabla \cdot \frac{\partial \mathbf{B}}{\partial t} = -\nabla \cdot \nabla \times \mathbf{E} = 0 \qquad \Rightarrow \qquad \frac{\partial}{\partial t} \nabla \cdot \mathbf{B} = 0$$ automatic!

$$\nabla \cdot \frac{\partial \mathbf{V}}{\partial t} \neq 0 \qquad \Rightarrow \qquad \frac{\partial}{\partial t} \nabla \cdot \mathbf{V} \neq 0$$ not automatic!

Pressure constraint force must make it so $\Rightarrow$ elliptic equation

$$\nabla^2 p = -\rho_0 \nabla \cdot (\mathbf{V} \cdot \nabla \mathbf{V})$$

This is Eulerian description. What about Lagrangian?

# Lagrange's Multiplier

Extremize function $f(x_1, x_2, \ldots, x_n)$ at fixed $g = g_0$:

$$\delta(f + \lambda g) = 0 \qquad \Rightarrow \qquad n \text{ equations}$$

There are $n + 1$ unknowns $g = g_0 \Rightarrow$ the other.

For the incompressible fluid Lagrange added constraint to Lagrangian

$$\mathcal{L} + \lambda \mathcal{J}$$

Recall

$$\mathcal{J} = \det\left(\partial q(a,t)/\partial a\right) \quad \text{and} \quad \rho = \rho_0/\mathcal{J}$$

So Lagrange's constraint, $\mathcal{J} = 1 \Rightarrow \rho_0$ constant.

What is $\lambda$?

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So, $\mathcal{J} = 1 \Rightarrow \rho_0$ constant.

What is $\lambda$? Pressure!

Lagrange struggled to find it for the general case.

# Dirac

Given two desired constraints, i.e., two phase space functions $D_{1,2}$ and good Poisson bracket $\{\ ,\ \}$:

Dirac bracket:

$$\{f,g\}_D = \frac{1}{\{D_1,D_2\}}\Big(\{D_1,D_2\}\{f,g\} - \{f,D_1\}\{g,D_2\} + \{g,D_1\}\{f,D_2\}\Big)$$

Degeneracy $\Rightarrow$ $D$'s are Casimir Invariants:

$$\{D_{1,2},g\}_D = 0 \quad \forall\ g\colon \mathcal{Z}_p \to \mathbb{R}$$

Dynamics is Hamiltonian and conserves $D_{1,2}$:

$$\dot{D}_{1,2} = \{D_{1,2},H\} \equiv 0$$

for any Hamiltonian, $H$.

# Eulerian Incompressible Fluid

Nguyen and Turski (1999) constraints:

$$\rho(r,t) = \rho_0 \qquad \text{textcolorblueand} \qquad \nabla \cdot \mathbf{V} = \text{constant}$$

Noncanonical Eulerian Poisson Bracket (pjm & Greene 1980):

$$\{F,G\} = \int d^3x \left( G_\rho \nabla \cdot F_{\mathbf{V}} + G_{\mathbf{V}} \cdot \nabla F_\rho + (\nabla \times \mathbf{V}) \cdot (F_{\mathbf{V}} \times G_{\mathbf{V}})/\rho \right)$$

Messy calculation gives bracket that gives known $\nabla^2 p = \ldots$

Chandre, Tassi, pjm, et al. generalized to full MHD $(\rho, \mathbf{V}, \mathbf{B}, s)$ by simply replacing

$$F_{\mathbf{V}} = \frac{\delta F}{\delta V} \quad \text{by} \quad \mathbb{P} F_{\mathbf{V}}$$

where $\mathbb{P}$ is the projector

$$\mathbb{P} := I - \nabla(\nabla^2)^{-1}\nabla \cdot$$

Note $\nabla \cdot (\mathbb{P}\mathbf{V}) \equiv 0$ for any vector field $\mathbf{V}$.

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Except!

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What about Dirac in the Lagrangian variable picture?

What constraints go into the Poisson Bracket?

$$\{F,G\} = \int_D d^3a \left( \frac{\delta F}{\delta q^i}\frac{\delta G}{\delta \pi_i} - \frac{\delta G}{\delta q^i}\frac{\delta F}{\delta \pi_i} \right) .$$

And what Eulerian equations can be obtained?

## **Recent Work** (Andreussi, pjm, and Pegoraro)

Make Dirac bracket with

$$\{F, G\} = \int_D d^3a \left( \frac{\delta F}{\delta q^i} \frac{\delta G}{\delta \pi_i} - \frac{\delta G}{\delta q^i} \frac{\delta F}{\delta \pi_i} \right) ,$$

and <u>convenient</u> compressibility constraints

$$D_1 = \mathcal{J} \qquad \text{and} \qquad D_2 = A^k_\ell \frac{\partial}{\partial a^k} \left( \frac{\pi^\ell}{\mathcal{J}} \right) ,$$

where cofactor matrix

$$A^k_\ell = \frac{1}{2} \epsilon^{kmn} \, \epsilon_{\ell u i} \, \frac{\partial q^u}{\partial a^m} \frac{\partial q^i}{\partial a^n} .$$

Why two? $\nabla \cdot \mathbf{V} =$ constant plus $\rho$ advected but not constant! Lagrange's procedure $\mathcal{J} = 1$ misses this.

# Procedure

- Construct Dirac bracket using $D_{1,2}$
- Eulerianize

Long calculation gives the complete bracket as

$$\begin{aligned}\{f,g\}_D \;=\; & -\int d^3x' \, [\rho \, (\mathbb{P} f_{\mathbf{M}} \cdot \nabla g_\rho - \mathbb{P} g_{\mathbf{M}} \cdot \nabla f_\rho) \\ & +\sigma \, (\mathbb{P} f_{\mathbf{M}} \cdot \nabla g_\sigma - \mathbb{P} g_{\mathbf{M}} \cdot \nabla f_\sigma) \\ & +\mathbf{M} \cdot [(\mathbb{P} f_{\mathbf{M}} \cdot \nabla) \, \mathbb{P} g_{\mathbf{M}} - (\mathbb{P} g_{\mathbf{M}} \cdot \nabla) \, \mathbb{P} f_{\mathbf{M}}] \\ & +\nabla \cdot \mathbf{M} \, (f_{\mathbf{M}} \cdot \mathbb{P} g_{\mathbf{M}} - g_{\mathbf{M}} \cdot \mathbb{P} f_{\mathbf{M}})] \, .\end{aligned}$$

where $\mathbf{M} = \rho \mathbf{V}$ and $\sigma = \rho s$.

This generates dynamics different from usual, but on the constraint surface $\nabla \cdot \mathbf{V} = 0$ it agrees with usual.

It just come out that way! Lagrangian constaints $\neq$ Eulerian.

# Summary

- Surveyed aspects of Lagrangian and Hamiltonian plasma field theory.
- Described Dirac's constraint method, which is of general utility for imposing constraints.
- Redid Lagrange's constraint in Hamiltonian-Dirac framework.
- Did not cover applications: eliminate unreasonable equations, stability with various constraints, physics inspired numerical algorithms, nonlinear dynamical reductions, etc.
- Given reasonable nondissipate dynamics can construct reasonable dissipation, e.g., invariant preserving relaxation to equilbirum.