Introduction to Pressure-Anisotropy-Driven Instabilities.

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Reading: MNRAS 405, 291 (2010) and refs therein

§1. Kinetic Description of a Magnetised Plasma.

In order to start having an adequate theoretical understanding of the primary topic of these lectures, it is necessary first to gain a good mastery of the general kinetic framework in which dynamics of magnetised plasma is studied. I will spend quite a large fraction of time alloted for these lectures on this extended introduction because it involves material that is not generally part of standard plasma courses — although it provides a natural connection between fluid descriptions (MHD, etc.) and the fully kinetic ones.

Our goal is to develop a theoretical framework for plasmas that are strongly magnetised and weakly collisional in the sense that

$$\underbrace{\nu_{ie}, \nu_{ii}, \nu_{ei}, \nu_{ee}}_{\text{collision frequencies}} \ll \underbrace{\Omega_i, \Omega_e}_{\text{Larmor frequencies}}$$

or $$\underbrace{\rho_i, \rho_e}_{\text{Larmor radii}} \ll \underbrace{\lambda_{mfp}}_{\text{mean free path}}$$

We will also mostly consider low-frequency dynamics,

$$\omega \ll \Omega_i, \Omega_e$$

and (less generally) long wavelengths: $k\rho_i, k\rho_e \ll 1$.

1.1 Kinetic Equation

Let us start from "the beginning":
the Vlasov - Landau - Maxwell system of equations:
distribution function $f_s(t,\vec{r},\vec{v})$ of species $s\,(=e,i)$ satisfies

$$\overbrace{\underbrace{\frac{\partial f_s}{\partial t} + \vec{v}\cdot\nabla f_s}_{\text{streaming}} + \underbrace{\frac{e_s}{m_s}\left(\vec{E}+\frac{\vec{v}\times\vec{B}}{c}\right)}_{\text{Lorentz force}}\cdot\frac{\partial f_s}{\partial \vec{v}}}^{\text{Vlasov}} = \overbrace{\underbrace{\left(\frac{\partial f_s}{\partial t}\right)_c}_{\text{collisions}}}^{\text{Landau}} \qquad (1)$$

Maxwell

$$\cancel{\nabla\cdot\vec{E}} = 4\pi\sum_s e_s n_s, \quad n_s = \int d^3\vec{v}\, f_s$$
Poisson

$$\nabla\cdot\vec{B} = 0$$

$$\vec{j} = \sum_s e_s n_s \vec{u}_s, \quad \vec{u}_s = \frac{1}{n_s}\int d^3\vec{v}\,\vec{v}\, f_s$$

$$\frac{\partial\vec{B}}{\partial t} = -c\nabla\times\vec{E}$$
Faraday

$$\nabla\times\vec{B} = \frac{4\pi}{c}\vec{j} + \cancel{\frac{1}{c}\frac{\partial\vec{E}}{\partial t}}$$
Ampère

displacement current neglected when non-relat.

$\sim k^2\lambda_{De}^2 \ll 1$ neglected for wavelength $\gg \lambda_{De}$ — quasineutrality

$\frac{\omega^2/k^2}{c^2} \ll 1$

because

$$\frac{\frac{1}{c}\omega E}{kB} \sim \frac{\frac{1}{c^2}\omega^2\frac{1}{k}B}{kB} \sim \frac{\omega^2}{k^2c^2}$$

because

$$\frac{kE}{4\pi e\,\delta n} \sim \frac{k^2\varphi}{4\pi e\,\delta n} \sim \frac{k^2 T}{4\pi e^2 n} \sim \frac{k^2 m_e v_{the}^2}{4\pi e^2 n} \sim \frac{k^2 v_{the}^2}{\omega_{pe}^2}$$

Our intuition likes imagining plasma as a fluid, with some density $n_s = \int d^3\vec{v}\, f_s$

and velocity $\vec{u}_s = \frac{1}{n_s}\int d^3\vec{v}\,\vec{v}\, f_s$

(and perhaps pressure, temperature or some generalization thereof). This is rooted in the fact that gases we are used to (e.g. our atmosphere) are very collisional, so ~~[illegible]~~ the $\left(\frac{\partial f}{\partial t}\right)_c$ term dominates ($\nu \gg \omega$) and so to lowest order the distribution function is a local Maxwellian:

$$F_s = \frac{n_s}{(\pi v_{ths}^2)^{3/2}} e^{-\frac{(\vec{v}-\vec{u}_s)}{v_{ths}^2}} \quad , v_{ths} = \sqrt{\frac{2T_s}{m_s}}$$

and then all we need to do is derive equations for n_s, $\vec{u}_s$, T_s and also sometimes for perturbations of the particle distribution function around F_s (to calculate transport coefficients: viscosity, thermal diffusivity etc.)

Here we will be concerned with a situation in which collisions are not quite so dominant ($\nu \sim \omega$ or $\ll \omega$). How do we generalise this fluid approach then?

Let's first make a minor preliminary step, namely, change variables

$$\vec{v} \longrightarrow \vec{w} = \vec{v} - \vec{u}_s(t, \vec{r}) \quad \text{peculiar velocity}$$

where $\vec{u}_s = \frac{1}{n_s} \int d^3\vec{v}\, \vec{v} f_s$ the <u>exact</u> mean flow velocity.

So our particle kinetics will always be relative to the mean flow of the plasma.

$$\left(\frac{\partial}{\partial t}\right)_{\vec{v}} = \left(\frac{\partial}{\partial t}\right)_{\vec{w}} + \left(\frac{\partial \vec{w}}{\partial t}\right)_{\vec{v}} \cdot \frac{\partial}{\partial \vec{w}} = \frac{\partial}{\partial t} - \frac{\partial \vec{u}_s}{\partial t} \cdot \frac{\partial}{\partial \vec{w}}$$

$$(\nabla)_{\vec{v}} = (\nabla)_{\vec{w}} + (\nabla \vec{w})_{\vec{v}} \cdot \frac{\partial}{\partial \vec{w}} = \nabla - (\nabla \vec{u}_s) \cdot \frac{\partial}{\partial \vec{w}}$$

so $\vec{v} \cdot \nabla \to \vec{u}_s \cdot \nabla - (\vec{u}_s \cdot \nabla \vec{u}_s) \cdot \frac{\partial}{\partial \vec{w}} + \vec{w} \cdot \nabla - (\vec{w} \cdot \nabla \vec{u}_s) \cdot \frac{\partial}{\partial \vec{w}}$

Let $\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{u}_s \cdot \nabla$ convective derivative for species s

Then the kinetic equation becomes

$$\boxed{\underset{1}{\frac{df_s}{dt}} + \underset{2}{\vec{w}\cdot\nabla f_s} + \Big(\underset{3}{\frac{e_s}{m_s}\frac{\vec{w}\times\vec{B}}{c}} + \underset{4}{\vec{a}_s} - \underset{5}{\vec{w}\cdot\nabla\vec{u}_s}\Big)\cdot\frac{\partial f_s}{\partial\vec{w}} = \underset{6}{\left(\frac{\partial f_s}{\partial t}\right)_c}} \qquad (2)$$

where $\vec{a}_s = \frac{e_s}{m_s}\left(\frac{\vec{u}_s\times\vec{B}}{c} + \vec{E}\right) - \frac{d\vec{u}_s}{dt}$ acceleration (independent of $\vec{w}$!)

To this we must now attach Maxwell's equations:

$$\sum_s e_s n_s = 0 \quad \text{quasineutrality} \qquad (3)$$

$$\frac{\partial\vec{B}}{\partial t} = -c\,\nabla\times\vec{E} \quad \text{Faraday} \qquad (4)$$

$$\nabla\times\vec{B} = \frac{4\pi}{c}\vec{J} = \frac{4\pi}{c}\sum_s e_s n_s \vec{u}_s \quad \text{Ampère} \qquad (5)$$

and the constraint $\int d^3\vec{w}\,\vec{w}\,f_s = 0$, which can be thought of as implicitly determining $\vec{u}_s$.

We are now going to spin out a fluid-like description for our plasma with these equations as the starting point.

1.2 Moment Equations.

Take moments of (2): NB use $\int d^3\vec{w}\,\vec{w}\, f_s = 0$!

$$\int d^3\vec{w}\,(2): \quad \frac{d n_s}{dt} + (\nabla\cdot\vec{u}_s)\, n_s = 0$$

from (1)   from (5) after integration by parts

NB: $\int d^3\vec{w}\left(\frac{\partial f_s}{\partial t}\right)_c = 0$ conservation of particles

This is the continuity equation:

$$\boxed{\frac{\partial n_s}{\partial t} + \nabla\cdot(\vec{u}_s n_s) = 0} \qquad (6)$$

$$\int d^3\vec{w}\, m_s\vec{w}\,(2): \quad \underbrace{\nabla\cdot\int d^3\vec{w}\, m_s\vec{w}\vec{w} f_s}_{\equiv\ \hat{P}_s} - \vec{a}_s n_s m_s = \vec{R}_s$$

particle momentum (relative to mean flow)

from (2)   from (4)   from (6)

$\hat{P}_s$ pressure tensor

$\vec{R}_s$ interspecies friction force

$$\vec{R}_s \doteq \int d^3\vec{w}\, m_s\vec{w}\left(\frac{\partial f_s}{\partial t}\right)_c$$

This is the momentum equation:

$$\boxed{m_s n_s \frac{d\vec{u}_s}{dt} = -\nabla\cdot\hat{P}_s + e_s n_s\left(\vec{E} + \frac{\vec{u}_s\times\vec{B}}{c}\right) + \vec{R}_s} \qquad (7)$$

We are interested in mass flow (momentum) ~~[illegible]~~, so we add $(7)_i + (7)_e$ and use

$$m_e\vec{u}_e + m_i\vec{u}_i \approx m_i\vec{u}_i \equiv m_i\vec{u} \quad (m_i \gg m_e)$$

$$\hat{P} \doteq \sum_s \hat{P}_s\,, \qquad \sum_s \vec{R}_s = 0 \ \text{(momentum conservation by elastic collision)}$$

$$\sum_s e_s n_s = 0 \ \text{(quasineutrality)}$$

and $$\sum_s e_s n_s \vec{u}_s = \vec{J} = \frac{c}{4\pi}\nabla\times\vec{B} \ \text{(Ampère)}$$

This gives

$$\boxed{m_i n_i \frac{d\vec{u}}{dt} = -\nabla\cdot\hat{P} + \frac{(\nabla\times\vec{B})\times\vec{B}}{4\pi}} \qquad (8)$$

Lorentz force

where $\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{u}\cdot\nabla\vec{u}$

The rhs can be written in a nice divergence form:

$$\boxed{m_i n_i \frac{d\vec{u}}{dt} = -\nabla\cdot\left(\hat{P} + \underbrace{\hat{\mathbb{1}}\frac{B^2}{8\pi} - \frac{\vec{B}\vec{B}}{4\pi}}_{\text{Maxwell stress}}\right)} \qquad (9)$$

all the kinetic physics is in this tensor!

So, we have equations for n_i and $\vec{u}$, but still need to calculate $\vec{B}$ and $\hat{P}$.
We'll deal with $\vec{B}$ first, which is easier, and then discuss $\hat{P}$ at great length as this is where all the interesting (for the purposes of these lectures) physics is contained.

1.3. Magnetic Field.

Faraday: $$\boxed{\frac{\partial \vec{B}}{\partial t} = - c \nabla \times \vec{E}} \qquad (4)$$

So we need to calculate $\vec{E}$. Note we do this not from Poisson eqn (where $\nabla \cdot \vec{E}$ is small as $k^2 \lambda_{De}^2$ & so $\vec{E}$ is not explicitly present), but from the electron momentum equation, $(7)_e$, which is also known as the "generalised Ohm's law":

$$\vec{E} + \frac{\vec{u}_e \times \vec{B}}{c} = \frac{\vec{R}_e}{e n_e} - \frac{\nabla \cdot \hat{P}_e}{e n_e} - \cancel{\frac{m_e}{e} \frac{d\vec{u}_e}{dt}}$$

e. field in the frame of the el. fluid

friction $\Downarrow$ resistivity $= \vec{j}/\sigma$

"electron thermal force" ("thermoelectric term")

electron inertia small because $m_e \ll m_i$ $\quad \frac{m_e \omega u_e / e}{u_e B / c} \sim \frac{\omega}{\Omega_e} \ll 1$

Since $e n_e (\vec{u}_i - \vec{u}_e) = \vec{j}$ this is $\frac{\vec{u}_i \times \vec{B}}{c} - \frac{\vec{j} \times \vec{B}}{c e n_e}$ ← "Hall term"

because $\vec{R}_e = - \nu_{ei} m_e n_e (\vec{u}_e - \vec{u}_i) = \frac{\nu_{ei} m_e}{e} \vec{j}$

So $\frac{1}{\sigma} = \frac{\nu_{ei} m_e}{e^2 n_e}$

So, $$\boxed{\vec{E} = - \frac{\vec{u} \times \vec{B}}{c} + \vec{j}/\sigma - \frac{\vec{j} \times \vec{B}}{c e n_e} - \frac{\nabla \cdot \hat{P}_e}{e n_e}} \qquad (10)$$

Since $\vec{j} = \frac{c}{4\pi} \nabla \times \vec{B}$, everything here is expressed in terms of $\vec{u}$, $\vec{B}$ or $n_e \left(= \frac{e_i}{e} n_i \equiv Z n_i\right)$ except $\hat{P}_e$, which we have not yet discussed.

Substitute (10) into (4):

Hall thermoelectric

$$\underbrace{\frac{\partial \vec{B}}{\partial t} = \nabla\times(\vec{u}\times\vec{B}) + \eta\Delta\vec{B}}_{\text{usual MHD "induction equation"}} + c\nabla\times\left[\frac{(\nabla\times\vec{B})\times\vec{B}}{4\pi e n_e} + \frac{\nabla\cdot\hat{P}_e}{e n_e}\right]$$

$$\eta = \frac{c^2}{4\pi\sigma} = \frac{c^2\nu_{ei} m_e}{4\pi e^2 n_e} \qquad (11)$$

$= \nu_{ei} d_e^2$ mag. diffusivity (small when coll's are small)

It is useful to know that the Hall and thermoelectric terms are only important at short spatial scales.

$$\frac{\eta k^2 B}{\omega B} \sim \frac{\nu_{ei}}{\omega} k^2 d_e^2 \ll 1$$

Hall:

$$\frac{c k^2 B^2/4\pi e n_e}{\omega B} \sim \frac{c^2 m_i}{4\pi e^2 n_i}\frac{k^2}{\omega}\frac{eB}{m_i c} \sim k^2 d_i^2 \frac{\Omega_i}{\omega}$$

If $\omega \sim k v_A \sim \frac{kB}{\sqrt{4\pi n_i m_i}}$, then $\frac{\omega}{\Omega_i} \sim \frac{kB\, m_i c}{\sqrt{4\pi m_i n_i}\, eB} \sim k d_i$,

so Hall term $\sim k d_i \ll 1$ as long as we stay above the ion inertial scale.

Thermoelectric:

$$\frac{c k^2 v_{the}^2 m_e \cancel{n_e}/e\cancel{n_e}}{\omega B} \sim \frac{k^2 v_{thi}^2}{\omega}\underbrace{\frac{m_i c}{eB}}_{\sim\Omega_i^{-1}} \sim k^2 \rho_i^2 \frac{\Omega_i}{\omega} \sim$$

$\sim k\rho_i \frac{\rho_i}{d_i} \sim k\rho_i\sqrt{\beta_i}$ where $\beta_i = \frac{n_i T_i}{B^2/8\pi}$, $T_i = \frac{m_i v_{thi}^2}{2}$

$\sim \frac{k d_i}{\sqrt{\beta_i}}$ small as long as we stay above the ion inertial and ion Larmor scales.

(for now, let $\beta_i \sim 1$).

1.4 Gyrotropic Plasmas.

Now let us tackle the pressure tensor

$$\hat{P}_s = \int d^3\vec{w}\, m_s \vec{w}\vec{w}\, f_s$$

In general, in order to know what it is, we still need to solve the kinetic equation (2), so despite all the work we have done so far, no real simplificati[on] has yet been achieved.

Note that term ③ in eq. (2) can be written so:

$$\frac{e_s}{m_s}\frac{\vec{w}\times\vec{B}}{c}\cdot\frac{\partial f_s}{\partial \vec{w}} = -\Omega_s\left(\frac{\partial f_s}{\partial \vartheta}\right)_{w_\perp, w_\parallel}$$

where $\Omega_s = \frac{e_s B}{m_s c}$ is the Larmor frequency and ϑ is the gyroangle – angle at which the particle orbits the magnetic field.

Ex. Prove this! (just let $w_x = w_\perp \cos\vartheta$, $w_y = w_\perp \sin\vartheta$ etc.
– cylindrical coordinates locally in velocity space)
↳ wrt $\vec{B}$

So, eq. (2) can be written so:

$$\underset{③}{\Omega_s\left(\frac{\partial f_s}{\partial \vartheta}\right)_{w_\perp, w_\parallel}} = \underset{①}{\frac{d f_s}{dt}} + \underset{②}{\vec{w}\cdot\nabla f_s} + \underset{④}{(\vec{a}_s} - \underset{⑤}{\vec{w}\cdot\nabla\vec{u}_s)}\cdot\frac{\partial f_s}{\partial \vec{w}} - \underset{⑥}{\left(\frac{\partial f_s}{\partial t}\right)_c} \quad (12)$$

Let us now consider a situation when the lhs is $\gg$ the rhs. Then, to lowest order,

$$\frac{\partial f_s}{\partial \vartheta} = 0 \quad \text{and so} \quad f_s = f_s(t, \vec{r}, w_\perp, w_\parallel)$$

– The distribution function is gyrotropic, i.e., independent of the gyroangle.

To lowest order in what?

$$\frac{①}{③} \sim \frac{\omega}{\Omega_s} \ll 1 \quad \text{low frequency}$$

$$\sim \frac{k u_s}{\Omega_s} \sim k\rho_s \frac{u_s}{v_{ths}} \ll 1 \quad \text{long wavelength}$$

$$\frac{②}{③} \sim \frac{k v_{ths}}{\Omega_s} \sim k\rho_s \ll 1 \quad \text{long wavelength}$$

from gen'd Ohm's law
use $\nabla\cdot\hat{P}_e / e n_e$ (biggest term)

$$\frac{④}{③} \sim \frac{a_s}{\Omega_s v_{ths}} \sim \frac{\vec{E} + \frac{\vec{u}_s \times \vec{B}}{c}}{\frac{m_s}{e_s}\Omega_s v_{ths}} \sim k\rho_s \ll 1$$

or smaller

$$\frac{⑤}{③} \sim \frac{k v_{ths} u_s}{\Omega_s v_{ths}} \sim k\rho_s \, Ma \ll 1$$

$$\frac{⑥}{③} \sim \frac{\nu_s}{\Omega_s} \ll 1 \quad \text{weakly collisional (= magnetised)}$$

[otherwise coll's will dominate and f_s will become isotropic]

Thus, we are safe in this approximation if

$$\omega \ll \Omega_i \quad \text{and} \quad k\rho_i \ll 1 \quad \text{and} \quad \nu_i \ll \Omega_i$$

magnetised weakly collisional plasma

For such a plasma, the pressure tensor is greatly simplified:

$$\hat{P}_s = \int d^3\vec{w}\, \underbrace{m_s\langle \vec{w}\vec{w}\rangle_\vartheta}_{\frac{w_\perp^2}{2}(\mathbb{1}-\hat{b}\hat{b}) + w_\parallel^2 \hat{b}\hat{b} \quad \text{where } \hat{b} = \frac{\vec{B}}{B}} f_s(t,\vec{r},w_\perp,w_\parallel) =$$

$$= (\mathbb{1}-\hat{b}\hat{b}) \underbrace{\int d^3\vec{w}\, m_s \frac{w_\perp^2}{2} f_s}_{\equiv p_{\perp s}} + \hat{b}\hat{b} \underbrace{\int d^3\vec{w}\, m_s w_\parallel^2 f_s}_{\equiv p_{\parallel s}}$$

$$= \begin{pmatrix} p_{\perp s} & & 0 \\ & p_{\perp s} & \\ 0 & & p_{\parallel s} \end{pmatrix}$$

so we simply have two scalar pressures, perp. and parallel to the local direction of $\vec{B}$.

Denoting $p_\perp = \sum_s p_{\perp s}$

and $p_\parallel = \sum_s p_{\parallel s}$, we get

$$\nabla\cdot\hat{P} = \nabla\cdot\left[(\mathbb{1}-\hat{b}\hat{b})p_\perp + \hat{b}\hat{b}p_\parallel\right] = \nabla p_\perp - \nabla\cdot\left[\hat{b}\hat{b}(p_\perp - p_\parallel)\right]$$

and so the momentum equation is

$$\boxed{m_i n_i \frac{d\vec{u}}{dt} = -\nabla\Big(p_\perp + \frac{B^2}{8\pi}\Big) + \nabla\cdot\Big[\hat{b}\hat{b}\Big(p_\perp - p_\parallel + \frac{B^2}{4\pi}\Big)\Big]} \qquad (13)$$

usual scalar pressure, incl. magnetic

pressure anisotropy stress

Maxwell stress

1.5 Origin of Pressure Anisotropy.

Let us momentarily interrupt the formal flow and ask where pressure anisotropies might come from and how large they are likely to be. This discussion is qualitative and we will subsequently rederive everything more rigorously.

If the magnetic field in a plasma changes sufficiently slowly ($\omega \ll \Omega_i$) and particles rarely collide ($\nu_{ii} \ll \Omega_i$), then each particle has an adiabatic invariant (called <u>first adiabatic invariant</u>) $\mu = \frac{m w_\perp^2}{2B}$ (we'll prove that $\mu = \text{const}$ directly from eq. (12) later.)

Physically, this can be thought of as the magnetic moment of a current loop formed by a gyroorbit or angular momentum of the gyrating particle ($m w_\perp r = m w_\perp \cdot w_\perp / \Omega_i = m w_\perp^2 / (eB/mc) \propto m w_\perp^2 / B$).

Now the sum of all these μ's is

$$\int d^3\vec{w}\, \mu f = \frac{p_\perp}{B}$$

(in fact $\underset{\#\text{particles}}{N} \frac{p_\perp}{\underset{\text{density}}{n} B}$, but let $n = \text{const}$ for now)

Let us express this expectation that μ is conserved:

$$\frac{1}{p_\perp}\frac{dp_\perp}{dt} \sim \frac{1}{B}\frac{dB}{dt} - \nu \frac{p_\perp - p_\parallel}{p_\perp} \qquad (14)$$

non-rigorous at this stage, derivation later.

conservation

collisional tendency to isotropize pressure.

Thus, we expect that if ambient magnetic field changes in a plasma, this should cause $p_\perp$ to change, so as to preserve μ — a process possibly attenuated by collisions if they are large enough to compete.

If they indeed are large enough, we can make a simple estimate of the ~~relative~~ pressure anisotropy:

Example: solar wind is expanding, B is dropping, expect ~1 negative pressure anisotropy at 1AU

$$\Delta \equiv \frac{p_\perp - p_\parallel}{p_\perp} \sim \frac{1}{\nu}\frac{1}{B}\frac{dB}{dt} \sim \frac{\text{rate of change of } B}{\text{coll. rate}} \quad ^{*)}$$

Let us recall the induction equation [eq. (11), dropping all the small-scale terms]:

$$\frac{\partial \vec{B}}{\partial t} = \nabla\times(\vec{u}\times\vec{B}) = -\vec{u}\cdot\nabla\vec{B} + \vec{B}\cdot\nabla\vec{u} \quad (\nabla\cdot\vec{u} = 0$$

for simplicity, incompressible)

$$\vec{B}\cdot \Big| \quad \frac{d\vec{B}}{dt} = \vec{B}\cdot\nabla\vec{u}$$

$$\frac{1}{B}\frac{dB}{dt} = \hat{b}\cdot(\nabla\vec{u})\cdot\hat{b}$$

← rate of strain

Thus, $$\frac{p_\perp - p_\parallel}{p_\perp} \sim \frac{\hat{b}\cdot(\nabla\vec{u})\cdot\hat{b}}{\nu}$$

or $$p_\perp - p_\parallel \sim \left(\frac{p_\perp}{\nu}\right)\hat{b}\hat{b}:\nabla\vec{u}$$

$$\sim \frac{m_i n_i v_{thi}^2}{\nu_{ii}} \sim \text{collisional viscosity}$$

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*) Note that this means that the electron pressure anisotropy is usually smaller than the ion one:

$$\Delta_e/\Delta_i \sim \frac{\nu_i}{\nu_e} \sim \sqrt{\frac{m_e}{m_i}} \ll 1 \quad \text{(as indeed is the case in SW)}$$

In eq. (13) we have therefore

$$\nabla\cdot\left[\hat{b}\hat{b}(p_\perp - p_\parallel)\right] \sim \nabla\cdot\left[\frac{p_\perp}{\nu}\hat{b}\hat{b}\hat{b}\hat{b}:\nabla\vec{u}\right]$$

— precisely the familiar (to some, at least!) parallel (Braginskii) viscosity term [so, in the coll. limit, equations can be closed]

Thus, pressure anisotropy = parallel viscosity, although, as we are going to see shortly, while its effect on large scales is dissipative, at small scales it will be <u>wildly destabilising</u>.

Intuitively this is because $p_\perp \neq p_\parallel$ is a non-equilibrium situation and so is a source of free energy; the pressure-anisotropic system will want to relax towards isotropy. It can do so via collisions, of course, but it can (and will) also be impatient with their sluggishness and find ways of exciting instabilities, which will then push it towards equilibrium — a common phenomenon.

From eq. (13), we can also estimate under what conditions $p_\perp - p_\parallel$ is likely to prove an important effect: clearly we must compare it with $B^2/4\pi$

So $\dfrac{p_\perp - p_\parallel}{p} \ll \dfrac{B^2}{4\pi p} \sim \dfrac{2}{\beta}$ pressure anisotropy irrelevant

$\dfrac{p_\perp - p_\parallel}{p} > \dfrac{2}{\beta}$ pressure anisotropy potentially important.

1.6 Kinetic MHD.

Let us now bring our quest for a fluid system of equations to a kind of completion by working out the evolution equations for $p_\perp$ and $p_\parallel$ (one of which will be a somewhat corrected form of eq. (14)).

Let us go back to eq. (12). We agreed that to lowest order, $\frac{\partial f_{0s}}{\partial \vartheta} = 0$, so $f_s = f_{0s} + \delta f_s$ and

$$\Omega_s \frac{\partial \delta f_s}{\partial \vartheta} = \frac{d f_{0s}}{dt} + \vec{w}\cdot\nabla f_{0s} + \left(\vec{a}_s - \vec{w}\cdot\nabla\vec{u}_s\right)\cdot\frac{\partial f_{0s}}{\partial \vec{w}} - \left(\frac{\partial f_{0s}}{\partial t}\right)_c \qquad (15)$$

We can annihilate the lhs by averaging this equation over gyroangles. Since $f_{0s} = f_{0s}(t, \vec{r}, w_\perp, w_\parallel)$, the gyroaverage of the rhs will give us a closed equation for the distribution function.

It turns out that mathematically the least cumbersome calculation can be done if we use $(w, w_\parallel)$ instead of $(w_\perp, w_\parallel)$ as variables.

↳ $w = w_\perp^2 + w_\parallel^2$ [they also have the advantage that for isotropic distribution, $\partial f/\partial w_\parallel = 0$, so $f = f(t, \vec{r}, w)$]

NB: $\int d^3\vec{w} = 2\pi \int_0^\infty dw\, w \int_{-w}^{w} dw_\parallel$

The time and spatial derivatives in the rhs of (15) must be carefully transformed because our change of of variables mixes phase space:

$$(t, \vec{r}, \vec{w}) \rightarrow (t, \vec{r}, w, w_\parallel, \vartheta), \text{ where } w_\parallel = \vec{w}\cdot\hat{b}(t, \vec{r})$$

and $f_{0s} = f_{0s}(t, \vec{r}, w, w_\parallel)$ [drop subscripts in what follows]

Then
$$\left(\frac{df}{dt}\right)_{\vec w} = \left(\frac{df}{dt}\right)_{w,w_\parallel} + \left(\frac{dw_\parallel}{dt}\right)_{\vec w}\left(\frac{\partial f}{\partial w_\parallel}\right)_w$$
$$= \frac{df}{dt} + \frac{d\hat b}{dt}\cdot\vec w\,\frac{\partial f}{\partial w_\parallel}$$
$$(\nabla f)_{\vec w} = (\nabla f)_{w,w_\parallel} + (\nabla w_\parallel)_{\vec w}\left(\frac{\partial f}{\partial w_\parallel}\right)_w$$
$$= \nabla f + (\nabla\hat b)\cdot\vec w\,\frac{\partial f}{\partial w_\parallel}$$

Also
$$\frac{\partial f}{\partial\vec w} = \frac{\vec w}{w}\frac{\partial f}{\partial w} + \hat b\frac{\partial f}{\partial w_\parallel}\ .$$

So, the gyroaveraged eqn (15) is
$$\frac{df}{dt} + \underbrace{\frac{d\hat b}{dt}\cdot\underbrace{\langle\vec w\rangle}_{=w_\parallel\hat b}}_{=0}\frac{\partial f}{\partial w_\parallel} + \underbrace{\langle\vec w\rangle}_{=w_\parallel\hat b}\cdot\nabla f + \underbrace{\langle\vec w\vec w\rangle}_{=\frac{w_\perp^2}{2}(\mathbb{1}-\cancel{\hat b\hat b})+w_\parallel^2\cancel{\hat b\hat b}}:(\nabla\hat b)\frac{\partial f}{\partial w_\parallel} +$$

because $\hat b\cdot\frac{d\hat b}{dt} = \frac{1}{2}\frac{d\hat b^2}{dt} = 0$

NB: $(\nabla\hat b)\cdot\hat b = \frac{1}{2}\nabla\hat b^2 = 0$

$$+\vec a\cdot\left(\underbrace{\frac{\langle\vec w\rangle}{w}}_{=\frac{w_\parallel}{w}\hat b}\frac{\partial f}{\partial w} + \hat b\frac{\partial f}{\partial w_\parallel}\right) - (\nabla\vec u):\underbrace{\frac{\langle\vec w\vec w\rangle}{w}}_{=\frac{w_\perp^2(\mathbb{1}-\hat b\hat b)}{2w}+\frac{w_\parallel^2}{w}\hat b\hat b}\frac{\partial f}{\partial w} - \underbrace{\langle\vec w\rangle}_{=w_\parallel\hat b}\cdot(\nabla\vec u)\cdot\hat b\frac{\partial f}{\partial w_\parallel}$$
$$= \left\langle\left(\frac{\partial f}{\partial t}\right)_c\right\rangle$$

$\nabla\cdot\hat b = -\frac{\hat b\cdot\nabla B}{B}$

$$\boxed{\begin{aligned}&\frac{df}{dt} + w_\parallel\hat b\cdot\nabla f + \underbrace{\frac{w_\perp^2}{2}(\nabla\cdot\hat b)\frac{\partial f}{\partial w_\parallel}}_{\text{mirror force}} + \vec a\cdot\hat b\left(\frac{w_\parallel}{w}\frac{\partial f}{\partial w} + \frac{\partial f}{\partial w_\parallel}\right)\\ &\quad - (\nabla\cdot\vec u)\frac{w_\perp^2}{2w}\frac{\partial f}{\partial w} + (\hat b\hat b:\nabla\vec u)\left[\left(\frac{w_\perp^2}{2} - w_\parallel^2\right)\frac{1}{w}\frac{\partial f}{\partial w} - w_\parallel\frac{\partial f}{\partial w_\parallel}\right]\\ &\quad = \left\langle\left(\frac{\partial f}{\partial t}\right)_c\right\rangle\end{aligned}}\qquad(16)$$

This equation, coupled with the definitions of $p_\perp$ & $p_\parallel$, the induction equation (11) (without the small-scale terms) & the momentum equation (13) constitute a closed system, known as "Kinetic MHD".

Note that the continuity eqn (6) is redundant (formally) as it can be obtained from (16) by integrating over velocities. The same is true about the parallel projection of the momentum equation, $(13)\cdot\hat{b}$, because $\vec{a}$ contains the $\frac{d\vec{u}}{dt}$ term.

In standard treatments (e.g. Kulsrud's review in the Handbook of Plasma Physics - 1983 or his earlier, more detailed, Varenna lecture notes), eq. (16) is written in $(w_\perp, w_\parallel)$ [or sometimes $(w_\perp, v_\parallel)$] variables — or $(\mu, w_\parallel)$ where $\mu = \frac{m w_\perp^2}{2B}$.

In this last form, μ conservation becomes manifest in the sense that the kinetic equation for $f(t, \vec{r}, \mu, w_\parallel)$ contains no μ derivatives:

$$\frac{df}{dt} + w_\parallel \hat{b}\cdot\nabla f - \Big(\hat{b}\cdot\frac{d\vec{u}}{dt} + w_\parallel \hat{b}\hat{b}:\nabla\vec{u} + \underbrace{\frac{\mu}{m}\hat{b}\cdot\nabla B}_{\text{mirror force}} - \frac{e}{m}E_\parallel\Big)\frac{\partial f}{\partial w_\parallel} = \left(\frac{\partial f}{\partial t}\right)_c \qquad (17)$$

(Ex. Derive this by changing variables from eq. (16))

This is nice and compact, but mixing B into velocity variables introduces some unwelcome complications into practical calculations, so I recommend eq. (16) over this.

The final piece we need to take care of is the parallel electric field contained in

$$\vec{a}_s \cdot \hat{b} = \frac{e_s}{m_s}\vec{E}\cdot\hat{b} - \hat{b}\cdot\frac{d\vec{u}_s}{dt}$$

[NB btw that all $\vec{u}$'s in (16) are $\vec{u}_s$'s, so $\vec{u}_i = \vec{u}$ set by (13) and $\vec{u}_e = \vec{u} - \frac{1}{en_e}\vec{j}$, $\vec{j} = \frac{c}{4\pi}\nabla\times\vec{B}$ ← but this term is small compared to $\vec{u}$ if $kd_i \ll 1$]

For this we use the generalised Ohm's law (10):

$$\vec{E}\cdot\hat{b} = \underbrace{\frac{1}{\sigma}\vec{j}\cdot\hat{b}}_{} - \frac{1}{en_e}\underbrace{(\nabla\cdot\hat{P}_e)\cdot\hat{b}}_{}$$

small:

$$\frac{\frac{1}{\sigma}j_\parallel}{\frac{1}{en_e}\nabla_\parallel p_{\parallel e}} \sim \frac{\nu_{ei}\cancel{m_e}\, c\cancel{k}B\,\cancel{e}\cancel{n_e}}{e^{\cancel{2}}n_e 4\pi\cancel{k}\,\cancel{m_e}v_{the}^2\cancel{n_e}}$$

$$\sim \frac{\nu_{ei}cB}{4\pi e n_e v_{the}^2} \sim$$

$$\sim \frac{\nu_{ei}c^2 m_e}{4\pi e^2 n_e}\frac{eB}{m_e c}\frac{1}{v_{the}^2}$$

$$\sim \frac{\nu_{ei}}{\Omega_e}\frac{d_e^2}{\rho_e^2} \ll 1$$

$$\left\{\nabla\cdot\left[p_{\perp e}(\mathbb{1}-\hat{b}\hat{b}) + p_{\parallel e}\hat{b}\hat{b}\right]\right\}\cdot\hat{b} =$$

$$= \hat{b}\cdot\nabla p_{\perp e} - \left\{\nabla\cdot\left[\hat{b}\hat{b}(p_{\perp e}-p_{\parallel e})\right]\right\}\cdot\hat{b}$$

$$= \cancel{\hat{b}\cdot\nabla p_{\perp e}} - (\nabla\cdot\hat{b})(p_{\perp e}-p_{\parallel e}) - \hat{b}\cdot\nabla(\cancel{p_{\perp e}} - p_{\parallel e}) =$$

$$= \hat{b}\cdot\nabla p_{\parallel e} - (p_{\perp e}-p_{\parallel e})\underbrace{(\nabla\cdot\hat{b})}_{= -\frac{\hat{b}\cdot\nabla B}{B}\text{ from }\nabla\cdot\vec{B}=0}$$

the usual thermoelectric term.

small (or can be)

$$\frac{p_{\perp e}-p_{\parallel e}}{p_{\parallel e}} \sim \frac{\omega}{\nu_{ei}} \ll 1$$

if electrons are collisional (assuming $\omega \sim \nu_{ii} \sim \sqrt{\frac{m_e}{m_i}}\nu_{ei}$)

Thus, usually, the dominant term is the thermoelectric one

$$\boxed{\vec{E}\cdot\hat{b} = -\frac{\hat{b}\cdot\nabla p_{\parallel e}}{en_e}} \quad (18)$$

$+ (p_{\perp e}-p_{\parallel e})\frac{\nabla\cdot\hat{b}}{en_e}$ if we wish to keep the el. anisotropy

NB: The approximations involving the smallness of the m_e/m_i mass ratio are not essential here. Formally speaking, we can keep all those small terms and do the mass-ratio expansion later on, as a subsidiary one.

Let us summarise the equations:

$$\frac{\partial n_i}{\partial t} + \nabla\cdot(n_i\vec{u}) = 0 \qquad [\text{superfluous}] \qquad (6)$$

$$m_i n_i \frac{d\vec{u}}{dt} = -\nabla\left(p_\perp + \frac{B^2}{8\pi}\right) + \nabla\cdot\left[\hat{b}\hat{b}\left(p_\perp - p_\parallel + \frac{B^2}{4\pi}\right)\right] \qquad (13)$$

$$\frac{\partial \vec{B}}{\partial t} = \nabla\times(\vec{u}\times\vec{B}) \qquad (11)$$

$$p_\perp = \sum_s p_{\perp s}, \qquad p_{\perp s} = \int d^3\vec{w}\, m_s \frac{w_\perp^2}{2} f_s$$

$$p_\parallel = \sum_s p_{\parallel s}, \qquad p_{\parallel s} = \int d^3\vec{w}\, m_s w_\parallel f_s$$

$$\frac{df_s}{dt} + w_\parallel \hat{b}\cdot\nabla f_s + \frac{w_\perp^2}{2}(\nabla\cdot\hat{b})\frac{\partial f}{\partial w_\parallel} + \left(\frac{e_s}{m_s}E_\parallel - \hat{b}\cdot\frac{d\vec{u}}{dt}\right)\left(\frac{w_\parallel}{w}\frac{\partial f}{\partial w} + \frac{\partial f}{\partial w_\parallel}\right)$$
$$- (\nabla\cdot\vec{u})\frac{w_\perp^2}{2w}\frac{\partial f}{\partial w} + (\hat{b}\hat{b}:\nabla\vec{u})\left[\left(\frac{w_\perp^2}{2} - w_\parallel^2\right)\frac{1}{w}\frac{\partial f}{\partial w} - w_\parallel \frac{\partial f}{\partial w_\parallel}\right]$$
$$= \left\langle \left(\frac{\partial f}{\partial t}\right)_c \right\rangle, \quad \text{where } w_\perp^2 = w^2 - w_\parallel^2. \qquad (16)$$

and $$E_\parallel = -\frac{\hat{b}\cdot\nabla p_{\parallel e}}{e n_e} + (p_{\perp e} - p_{\parallel e})\frac{\nabla\cdot\hat{b}}{e n_e} \qquad (18)$$

So these are the equations of KMHD.
When we come to the treatment of pressure-anisotropy-driven instabilities, we will conclude that these equations are in fact ill-posed in the sense that, unless the instabilities are stabilised, these equations will give rise to perturbations that grow the faster the greater is k — and require inclusion of non-gyrotropic FLR terms to be regularised at small scales.
(I'll derive some of those terms later on)

1.7 CGL Equations

Chew-Goldberger-Low.

I promised evolution equations for $p_\perp$ and $p_\parallel$, but so far have only delivered a kinetic equation for the gyrotropic distribution function from which pressures can be calculated.

I am now going to make a further step by taking the $m\frac{w_\perp^2}{2}$ and $m w_\parallel^2$ moments of eq. (16) and hence deriving the desired eqns for $p_\perp$ and $p_\parallel$. So, $\int d^3\vec w\, m\frac{w_\perp^2}{2}(16)$:

$$\frac{dp_\perp}{dt} + \hat b\cdot\nabla \underbrace{\int d^3\vec w\, m\frac{w_\perp^2}{2} w_\parallel f}_{=\,q_\perp} + (\nabla\cdot\hat b)\underbrace{\int d^3\vec w\, m\frac{w_\perp^4}{4}\frac{\partial f}{\partial w_\parallel}}_{=\,-2q_\perp} +$$

$q_\perp$ ← parallel flux of "perp energy"

$$\int d^3\vec w\, m\frac{(w^2-w_\parallel^2)^2}{4}\frac{\partial f}{\partial w_\parallel} \overset{\text{by parts}}{=} \int d^3\vec w\, m (w^2-w_\parallel^2) w_\parallel f = 2q_\perp$$

$$+\left(\frac{e_s}{m_s}E_\parallel - \hat b\cdot\frac{d\vec u}{dt}\right)\underbrace{\int d^3\vec w\, m\frac{w^2-w_\parallel^2}{2}\left(\frac{w_\parallel}{w}\frac{\partial f}{\partial w}+\frac{\partial f}{\partial w_\parallel}\right)}_{=\,0}$$

$$\int d^3\vec w\, m(w^2-w_\parallel^2)\left(w_\parallel\frac{\partial f}{\partial w^2}+\frac{1}{2}\frac{\partial f}{\partial w_\parallel}\right) \overset{\text{by parts}}{=} -\int d^3\vec w\, m\left(w_\parallel f - w_\parallel f\right) = 0$$

$$-(\nabla\cdot\vec u)\underbrace{\int d^3\vec w\, m\frac{(w^2-w_\parallel^2)^2}{2}\frac{\partial f}{\partial w^2}}_{=\,-\int d^3\vec w\, m(w^2-w_\parallel^2)f \,=\, -2p_\perp} +$$

$$+(\hat b\hat b:\nabla\vec u)\underbrace{\int d^3\vec w\, m\,\frac{w^2-w_\parallel^2}{2}\left[(w^2-3w_\parallel^2)\frac{\partial f}{\partial w^2} - w_\parallel\frac{\partial f}{\partial w_\parallel}\right]}_{=\,-p_\perp} =$$

$$-\int d^3\vec w\, m\left(\cancel{\frac{w^2-3w_\parallel^2}{2}} + \frac{w^2-w_\parallel^2}{2} - \cancel{\frac{w^2-3w_\parallel^2}{2}}\right)f = -p_\perp$$

$$= \int d^3\vec w\, m\frac{w_\perp^2}{2}\left\langle\left(\frac{\partial f}{\partial t}\right)_c\right\rangle \Bigg\} = -\nu(p_\perp - p_\parallel)$$

← this can be calculated using a model Coll. operator

see Note on p.22

e.g. Lorentz

$$\nu\frac{\partial}{\partial\xi}\frac{1-\xi^2}{2}\frac{\partial f}{\partial\xi}$$

where $\xi = w_\parallel/w$

Assemble:

$$\frac{dp_\perp}{dt} = -\hat{b}\cdot\nabla q_\perp - 2q_\perp\nabla\cdot\hat{b} - 2p_\perp\nabla\cdot\vec{u} + p_\perp\hat{b}\hat{b}:\nabla\vec{u} - \nu(p_\perp - p_\parallel)$$

$$= -\nabla\cdot\overbrace{(\hat{b}q_\perp)}^{\equiv\vec{q}_\perp} - q_\perp\nabla\cdot\hat{b} - \nu(p_\perp - p_\parallel) + p_\perp\underbrace{(\hat{b}\hat{b}:\nabla\vec{u} - \nabla\cdot\vec{u})}_{=\frac{1}{B}\frac{dB}{dt}} - p_\perp\underbrace{\nabla\cdot\vec{u}}_{=-\frac{1}{n}\frac{dn}{dt}} \qquad (19)$$

$$\boxed{p_\perp\frac{d}{dt}\ln\frac{p_\perp}{nB} = -\nabla\cdot\vec{q}_\perp - q_\perp\nabla\cdot\hat{b} - \nu(p_\perp - p_\parallel)} \qquad (20)$$

This is the more rigorous version of eq. (14), incorporating now the effect of compressibility and heat fluxes. It is worth noting the size of the heat flux terms:

$$\frac{q_\perp}{u p_\perp} \sim \frac{\cancel{m} v_{th}^3\,\delta f/f}{u\,\cancel{m} v_{th}^2} \sim \frac{v_{th}}{u}\frac{\delta f}{f}$$

asymmetric part of the distribution function, usually small

This is small if flows are sonic $u \sim v_{th}$ ("Braginskii ordering")
and order unity if $\frac{u}{v_{th}} \sim \frac{\delta f}{f} \ll 1$
("drift ordering", in the context of pressure anisotropies these are called "Mikhailovskii terms")

NB: obviously, for electrons, heat fluxes are always important!

So pressure anisotropies are caused both by changes in n and B ($\Leftarrow$ caused by flows) and by heat fluxes.

Let us complete the derivation and get $p_\parallel$:

$\int d^3\vec{w}\, m w_\parallel^2$ (16):

$$\frac{dp_\parallel}{dt} + \hat{b}\cdot\nabla \underbrace{\int d^3\vec{w}\, m w_\parallel^3 f}_{\substack{= \\ q_\parallel \leftarrow \text{parallel flux of "par. energy"}}} + (\nabla\cdot\hat{b}) \underbrace{\int d^3\vec{w}\, m w_\parallel^2 \frac{w^2-w_\parallel^2}{2}\frac{\partial f}{\partial w_\parallel}}_{=}$$

$$-\int d^3\vec{w}\, m\,(w_\parallel w^2 - 2w_\parallel^3) f = -\int d^3\vec{w}\, m\,(w_\perp^2 w_\parallel - w_\parallel^3) f = -2q_\perp + q_\parallel$$

$$+\left(\frac{e_s}{m_s}E_\parallel - \hat{b}\cdot\frac{d\vec{u}}{dt}\right)\underbrace{\int d^3\vec{w}\, m w_\parallel^2 \left(2w_\parallel \frac{\partial f}{\partial w^2} + \frac{\partial f}{\partial w_\parallel}\right)}_{=\,0}$$

$$-(\nabla\cdot\vec{u})\underbrace{\int d^3\vec{w}\, m w_\parallel^2 (w^2 - w_\parallel^2)\frac{\partial f}{\partial w^2}}_{=} +$$

$$-\int d^3\vec{w}\, m w_\parallel^2 f = -p_\parallel$$

$$+(\hat{b}\hat{b}:\nabla\vec{u})\underbrace{\int d^3\vec{w}\, m w_\parallel^2 \left[(w^2 - 3w_\parallel^2)\frac{\partial f}{\partial w^2} - w_\parallel \frac{\partial f}{\partial w_\parallel}\right]}_{=} =$$

$$-\int d^3\vec{w}\, m\,(w_\parallel^2 - 3w_\parallel^2) f = 2p_\parallel$$

$$= \left.\int d^3\vec{w}\, m w_\parallel^2 \left\langle \left(\frac{\partial f}{\partial t}\right)_c \right\rangle \right\} = -2\nu(p_\parallel - p_\perp)$$

see Note

again from model, but can also be inferred from the requirement the energy $2p_\perp + p_\parallel$ is conserved by collisions

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Note: Lorentz operator, the simplest model for collisions:

$$\left\langle \left(\frac{\partial f}{\partial t}\right)_c \right\rangle = \nu \frac{\partial}{\partial \xi}\frac{1-\xi^2}{2}\frac{\partial f}{\partial \xi} = \nu \frac{\partial}{\partial w_\parallel}\frac{(w^2 - w_\parallel^2)}{2}\frac{\partial f}{\partial w_\parallel}$$

So
$$\int d^3\vec{w}\, m w_\parallel^2\, \nu \frac{\partial}{\partial w_\parallel}\frac{(w^2-w_\parallel^2)}{2}\frac{\partial f}{\partial w_\parallel} = -\nu \int d^3\vec{w}\, m w_\parallel (w^2 - w_\parallel^2)\frac{\partial f}{\partial w_\parallel} =$$
$$= \nu \int d^3\vec{w}\, m\,(w^2 - 3w_\parallel^2) f = 2\nu(p_\perp - p_\parallel)$$
$$\int d^3\vec{w}\, m \frac{w^2-w_\parallel^2}{2}\nu\frac{\partial}{\partial w_\parallel}\frac{w^2-w_\parallel^2}{2}\frac{\partial f}{\partial w_\parallel} = +\nu\int d^3\vec{w}\, m w_\parallel \frac{w^2-w_\parallel^2}{2}\frac{\partial f}{\partial w_\parallel} = -\nu(p_\perp - p_\parallel)$$

Assemble:

$$\frac{dp_\parallel}{dt} = -\hat{b}\cdot\nabla q_\parallel - (\nabla\cdot\hat{b})(q_\parallel - 2q_\perp) - p_\parallel(\nabla\cdot\vec{u}) - 2p_\parallel \hat{b}\hat{b}:\nabla\vec{u} - 2\nu(p_\parallel - p_\perp)$$

$$= -\nabla\cdot(\underbrace{\hat{b}q_\parallel}_{\equiv\vec{q}_\parallel}) + 2q_\perp\nabla\cdot\hat{b} - 2\nu(p_\parallel - p_\perp)$$

$$- 2p_\parallel \underbrace{(\hat{b}\hat{b}:\nabla\vec{u} - \nabla\cdot\vec{u})}_{= \frac{1}{B}\frac{dB}{dt}} - 3p_\parallel \underbrace{\nabla\cdot\vec{u}}_{= -\frac{1}{n}\frac{dn}{dt}} \qquad (21)$$

$$\boxed{p_\parallel \frac{d}{dt}\ln\frac{p_\parallel B^2}{n^3} = -\nabla\cdot\vec{q}_\parallel + 2q_\perp\nabla\cdot\hat{b} - 2\nu(p_\parallel - p_\perp)} \qquad (22)$$

this is to do with the so-called "second adiabatic invariant" or bounce invariant

Equations (20), (22) are known as CGL equations — often in the version without the heat fluxes, in which case they are referred to as "double-adiabatic" equations (but that approximation tends to be very wrong because heat fluxes are not small, especially for electrons). If we must keep the heat fluxes, then obviously we still need the kinetic equation (16) to calculate them — this is the usual story with kinetic theory, moment equations do not close.

You might wonder why we bothered to derive these equations then. Well,

1) we learned some physics: what sets $p_\perp$ and $p_\parallel$
2) we have identified the key quantities that we might want to invent closures for: $p_\perp, p_\parallel, q_\parallel, q_\perp$.

Both pressure anisotropies and heat fluxes cause instabilities and one might take the view that instead of solving the kinetic equ (which is ill posed anyway) we should set them to marginal values!

1.8 Pressure Anisotropy.

Since this is going to be the key quantity, let's work out what it is. Using eqs (~~19~~) and (21), we get

$$\frac{d}{dt}(p_\perp - p_\parallel) = p_\perp\left(\frac{1}{B}\frac{dB}{dt} + \frac{1}{n}\frac{dn}{dt}\right) - \nabla\cdot\vec{q}_\perp - q_\perp \nabla\cdot\hat{b} - \nu(p_\perp - p_\parallel)$$
$$-\left\{p_\parallel\left(-2\frac{1}{B}\frac{dB}{dt} + 3\frac{1}{n}\frac{dn}{dt}\right) - \nabla\cdot\vec{q}_\parallel + 2q_\perp\nabla\cdot\hat{b} - 2\nu(p_\parallel - p_\perp)\right\}$$

$$= (p_\perp + 2p_\parallel)\frac{1}{B}\frac{dB}{dt} + (p_\perp - 3p_\parallel)\frac{1}{n}\frac{dn}{dt} - 3q_\perp\nabla\cdot\hat{b} - \nabla\cdot(\vec{q}_\perp - \vec{q}_\parallel) - 3\nu(p_\perp - p_\parallel) \quad (23)$$

If we choose to express things in terms of total pressure $p = \frac{2}{3}p_\perp + \frac{1}{3}p_\parallel$, then

$$p_\perp = p + \frac{1}{3}(p_\perp - p_\parallel)$$
$$p_\parallel = p - \frac{2}{3}(p_\perp - p_\parallel)$$

Things simplify a bit when collisions are dominant, pressure anisotropy small and can be found by setting the rhs of (23) to 0:

$$\boxed{\frac{p_\perp - p_\parallel}{p} \approx \frac{1}{\nu}\left[\frac{1}{B}\frac{dB}{dt} - \frac{2}{3}\frac{1}{n}\frac{dn}{dt} - \frac{\nabla\cdot(\vec{q}_\perp - \vec{q}_\parallel) + 3q_\perp\nabla\cdot\hat{b}}{3p}\right]} \quad (24)$$

Note. Under the same assumption of high collisionality,

$$q_\perp = \frac{1}{3}q_\parallel = -\frac{1}{2}n\frac{v_{th}^2}{\nu}\hat{b}\cdot\nabla T \quad \text{where } T = \frac{p}{n}$$

this prefactor depends on the coll. operator used.

This follows from expanding around a Maxwellian equilibrium. [Note this again: when $\nu \gg \omega$, equations are closed: Braginskii theory]

1.9 Heating

Finally, it is sometimes useful – and revealing – to know the equation for total energy

$$\int d^3\vec{w}\,\frac{mw^2}{2} f = p_\perp + \tfrac{1}{2}p_\parallel = \tfrac{3}{2}p = \tfrac{3}{2}\underbrace{nT}_{\text{by definition}}$$

Again using (19) and (21),

$$\underbrace{\frac{3}{2}\frac{d}{dt}nT}_{=\ \frac{3}{2}n\frac{dT}{dt} + \underbrace{\frac{3}{2}p}_{=\ (\cancel{p_\perp} + \frac{1}{2}p_\parallel)}\frac{1}{n}\frac{dn}{dt}} = -\nabla\cdot\underbrace{\left(\vec{q}_\perp + \tfrac{1}{2}\vec{q}_\parallel\right)}_{\equiv\ \vec{q}\ \text{heat flux}} + \underbrace{(p_\perp - p_\parallel)\frac{1}{B}\frac{dB}{dt}}_{\text{viscous heating}} + \underbrace{\left(\cancel{p_\perp} + \tfrac{3}{2}p_\parallel\right)\frac{1}{n}\frac{dn}{dt}}_{\text{compressional heating}} \qquad (25)$$

$$\boxed{\frac{3}{2}n\frac{dT}{dt} = -\nabla\cdot\vec{q} + (p_\perp - p_\parallel)\frac{1}{B}\frac{dB}{dt} + p_\parallel\frac{1}{n}\frac{dn}{dt}} \qquad (26)$$

Note that eqs. (23) and (26) can be used instead of (20), (22)

In a simple incompressible (collisional, heatfluxless) situation, the viscous heating term is explicitly positive:

$$(p_\perp - p_\parallel)\frac{1}{B}\frac{dB}{dt} = \frac{p}{\nu}\left(\frac{1}{B}\frac{dB}{dt}\right)^2 = \frac{p}{\nu}\left(\hat{b}\hat{b}:\nabla\vec{u}\right)^2$$

Ex. Work out the total energy conservation law for

$$\mathcal{E} = \underbrace{\frac{m_i n_i u^2}{2}}_{\text{kinetic}} + \underbrace{\frac{B^2}{8\pi}}_{\text{magnetic}} + \underbrace{\sum_s \frac{3}{2}n_s T_s}_{\text{thermal}} \qquad (27)$$