The second Yamabe invariant with singularities

In this work, Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$ .We suppose that g is a metric with satisfies the assumption (H) i.e : assumption (H) is : g is a metric in the Sobolev space $H_2^P(M, T^*M \otimes T^*M)$ with $p > n$ .There exist a point $p \in M$ and $\delta > 0$ such that g is smooth in the ball $B_p(\delta)$. Then the scalar curvature S_g is in L^p , this condition define the notion of "singularities".i.e:

$(g \in H_2^P(M, T^*M \otimes T^*M) \Leftrightarrow g_{ij} \in H_2^P(M)$ in normal coordonaite and $H_2^P(M) = \{u \in L^p, |\nabla u| \in L^p \text{and } \Delta_g u \in L^p\}$

By Sobolev embedding : $H_k^q(M) \subset C^m$ if $0 > \frac{1}{q} - \frac{k-m}{n}$,we get $H_2^P(M, T^*M \otimes T^*M) \subset C^1(M, T^*M \otimes T^*M)$. Hence the metric with satisfies Assumption (H) is C^1. The Christoffels belong to $H_1^P \subset C^0$, Riemann curvature tensor, Ricci tensor and scalar curvature are in L^p.

Recently F.Madani (The Yamabe problem with singularities) show that there is a metric $\tilde{g} = u^{N-2}g$ conformal to g such that $u \in H_2^P$,$u > 0$ and the scalar curvature $S_{\tilde{g}}$ of $\tilde{g}$ is constant under assumption (H) and (M, g) is not conformal to the sphere S_n with the standard Riemannian structure .The method to solve the Yamabe problem with singularities was the following . Let $u \in H_2^P$,$u > 0$ be a function and $\tilde{g} = u^{N-2}g$, we prefer saying this metric a particular conformal metric to mark the difference betwin the conform class where $N = 2n/(n-4)$.Then, multiplying u by a constant, the following equation is satisfied

$$L_g(u) = S_{\tilde{g}}|u|^{N-2}u \qquad \text{where} \qquad L_g = \Delta_g + \frac{(n-2)}{4(n-1)}S_g$$

Is called the Yamabe operator singular and S_g is the scalar curvature is in L^p. Moreover L_g is weakly conformally invariant. As consequence, solving the Yamabe problem singular is equivalent to find a positive solution $u \in H_2^P$ of $L_g(u) = k|u|^{N-2}u$,where k is a constant.In order to obtain solutions of this equation we define the quantity:

$$\mu = \inf_{u \in H_2^P, u > 0} Y(u) \qquad \text{where} \qquad Y(u) = \frac{\int_M |\nabla u|^2 + \frac{(n-2)}{4(n-1)} S_g u^2 dv_g}{(\int_M |u|^N dv_g)^{2/N}}$$

μ is called the standard Yamabe invariant with singularities (the standard Yamabe invariant singular). Writing the Euler-Lagrange equation associated to Y .we see that there exist a one to one correspondence between critical points of Y and solutions of $L_g(u) = k|u|^{N-2}u$. In particular, if u positive H_2^P function which minimizes Y , then $\tilde{g} = u^{N-2}g$ is the desired metric of constant scalar curvature.

Now we introduce and study two invariants that we call the first Yamabe invariant with singularities and the second Yamabe invariant with singularities respectly (the first Yamabe invariant singular , the second Yamabe invariant singular)., with S_g is in L^p our operator L_g is an elliptic operator on M self- adjoint , has discrete spectrum $spec(L_g) = \{\lambda_{1,g} \ , \lambda_{2,g} \ldots\}$, where the eigenvalue $\lambda_{1,g} \ < \ \lambda_{2,g} \ldots$ appear with their multiplicities. The variational characterization of $\lambda_{1,g}$ is given by

$$\lambda_{1,g} \ = \inf_{u \in H_1^2, u>0} \frac{\int_M |\nabla u|^2 + \frac{(n-2)}{4(n-1)} S_g u^2 dv_g}{(\int_M |u|^2 dv_g)}$$

Let $\langle g \rangle = \{\tilde{g} = u^{N-2} g, u \in C^\infty, u > 0\}$ be a conformal class of g and let $[g]$ be a particular conformal class of g i.e $[g] = \{\tilde{g} = u^{N-2} g \ , \ ,u \in H_2^P$ and $u > 0\}$, Let $k \in \mathbb{N}^*$,we define the k^{th}Yamabe invariant singular μ_k as

$$\mu_k = \inf_{\tilde{g} \in [g]} \lambda_{k,\tilde{g}} Vol(M, \tilde{g})^{2/n}$$

With these notations , μ_1 is the first yamabe invariant singular . In order to find minimizers, we enlarged the particular conformal class to what we call the class generalized metrics conformal to g. A generalized metrics is "metric" of the form $\tilde{g} = u^{N-2} g$, where is no longer necessarily positive and smooth, but $u \in L^N(M), u \geq 0, u \neq 0$. The definitions of $\lambda_{k,\tilde{g}}$ and of $Vol(M, \tilde{g})^{2/n}$ can be extended to generalized metric in our cas.

Firsly we show that :

Theorem 1 *If* $\mu > 0$ *, then for all any* $u \in L_+^N(M)$ *, there exist two functions* v, w *belonging to* H_1^2 *with* $v \geq 0$ *and such that in the sense of distributions.*$L_g(v) = \lambda_{1,\tilde{g}} u^{N-2} v$ *and* $L_g(w) = \lambda_{2,\tilde{g}} u^{N-2} w$ *.Moreover we can normalize* v, w *by*

$$\int_M u^{N-2} w^2 dv_g \ = \ \int_M u^{N-2} v^2 dv_g = 1 \quad \textit{and} \quad \int_M u^{N-2} wv dv_g = 0$$

We study a sequence of metrics $g_m = u_m^{N-2} g$ with $u_m \in H_2^P$, $u_m > 0$ which minimizes the infimum in the definition of μ_1 i.e. a sequence of metrics such that

$$\mu_1 = \lim \lambda_{1,m} (Vol(M, g_m))^{2/n}$$

Then there is not easy to see that $\mu_1 = \mu$ contrary to the standard yamabe invariant, then in this work we must showing if the standard Yamabe invariant singular $\mu \geq 0$, μ_1 it is exactly the standard Yamabe invariant singular , the condition $p > n$ give the fact $\mu_1 \leq \mu$ and for converse contary to the standard second Yamabe invariant , we begin to enlarge the particular conformal class, and we show that μ_1 is attained by generalized metrics $\tilde{g} = u^{N-2} g$ i.e there exist a positive fonction $v \in H_1^2$ such that $L_g(v) = \mu_1 u^{N-2} v$ is satisfait and we proof that $u = v$, we find $\mu_1 \geq \mu$.

Secondly we study the second Yamabe invariant singular μ_2, , in particular we discuss wether μ_2 is attained , we assert that contrary to the Yamabe invariant singular, $\mu_2$cannot be attained by a particular conformal metric .

In particular we show the differences and the dificults betwin the standards first and second Yamabe invariants and the first and second Yamabe invariants singulars and we show how the assumption H give the difference , dificult and the result.

The result we obtain is the following :

Theorem 2 *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$ .We suppose that g is a metric in the Sobolev space $H_2^P(M, T^*M \otimes T^*M)$ with $p > n$ .There exist a point $p \in M$ and $\delta > 0$ such that g is smooth in the ball $B_p(\delta)$,if $\mu \geq 0$ then*

$$\mu_1 = \mu$$

Theorem 3 *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$, we suppose that g is a metric in the Sobolev space $H_2^P(M, T^*M \otimes T^*M)$ with $p > n$ .There exist a point $p \in M$ and $\delta > 0$ such that g is smooth in the ball $B_p(\delta)$ .Assume that μ_2 is attained by a generalized metric $\tilde{g} = u^{N-2}g$, then there exist a nodal solution $w \in c^1$ of equation*

$$L_g(w) = \Delta w + \frac{(n-2)}{4(n-1)} S_g w = \mu_2 u^{N-2} w$$

More over there exist $a, b > 0$ such that

$$u = aw_+ + bw_-$$

With $w_+ = sup(w, 0)$ and $w_- = sup(-w, 0)$.

Theorem 4 *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$, we suppose that g is a metric in the Sobolev space $H_2^P(M, T^*M \otimes T^*M)$ with $p > n$ .There exist a point $p \in M$ and $\delta > 0$ such that g is smooth in the ball $B_p(\delta)$, then μ_2 is attained by a generalized metric in the folowing cases:*

$\mu_2 < \left[(\mu^{n/2} + (K^{-2})^{n/2}\right]^{2/n}$ or $\mu_2 < (K^{-2})$ and

If (M, g) in not locally conformally flat and, $n \geq 11$ and $\mu > 0$, then $\mu_2 < \left[(\mu^{n/2} + (K^{-2})^{n/2}\right]^{2/n}$.,

If (M, g) in not locally conformally flat and, $\mu = 0$ and $n \geq 9$, then $\mu_2 < (K^{-2})$.

Reference

[1]Benalili M and Boughazi H :The second Yamabe invariant with singularities ,les annales de mathematiques Blaise Pascal ,volume 19 ,n=1(2012),p.147-176.