Variational Integrators for the Vlasov-Poisson System

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Variational integrators [1] provide a systematic way to derive geometric numerical methods. The framework is that of Lagrangian field theory, for which, usually, equations of motion are derived by Hamilton's action principle and then discretised. In the application of the variational integrator theory, the order of these steps is reversed. At first, the Lagrangian and the accompanying variational principle are discretised, then discrete equations of motion are obtained directly by applying the discrete variational principle to the discrete Lagrangian. The advantage of this approach is that the resulting discretisation automatically preserves a discrete multisymplectic form (and therefore has good long time energy behaviour) as well as momenta associated to symmetries of the system. A discrete version of Noether's theorem yields the discrete expressions of these momenta.

An inevitable prerequisite for the derivation of variational integrators is the existence of a variational formulation for the considered dynamical system. Even though this is the case for a large class of systems, there are many interesting examples which do not belong to this class, e.g., equations of advection-diffusion type like they are often found in fluid dynamics or plasma physics. We propose the application of the variational integrator method to so called adjoint Lagrangians, which formally allow us to embed any dynamical system into a larger Lagrangian system by doubling the number of variables. Thereby we are able to extend the applicability of the variational integrator method significantly. An extended Noether theorem [2, 3, 4] allows us to determine conservation laws for such systems, both on the continuous and the discrete level.

We present the fundamental theory of variational integrators for field theories in a language adapted to problems from plasma physics, trying to make the geometric framework more transparent and therefore accessible to non specialists. We explain the detailed derivation of variational integrators and conservation laws for the advection equation as a prototypical example that shares many similarities with the Vlasov-Poisson system but is more instructive due to its reduced complexity. Finally, we present several variational integrators for the Vlasov-Poisson system which preserve the total particle number, momentum and energy as well as norms of the distribution function exactly (up to machine precision). These properties will be demonstrated by typical test cases from plasma and astro physics.

References

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