Asymptotic behaviour of stationary solutions to the Vlasov–Poisson system in a polygon

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Consider the stationary solution to the Vlasov–Poisson system in a domain $\Omega \subset \mathbf{R}^d$, corresponding to the Maxwell–Boltzmann distribution: $f(t,\boldsymbol{x},\boldsymbol{v}) \propto \exp -(\frac{1}{2}|\boldsymbol{v}|^2 + \phi(\boldsymbol{x}) - \phi_{\rm e}(x))$. This distribution function models the equilibrium of a population of charged particles under the effect of its self-consistent potential ϕ and of an external (confining) potential $\phi_{\rm e}$ created by "fixed" charges and/or an applied voltage. Integrating in the kinetic variable $\boldsymbol{v}$, we find that the former is solution to the nonlinear *Boltzmann–Poisson problem*:

$$-\Delta\phi = \kappa\, \mathrm{e}^{\phi_{\rm e}-\phi} := \rho \text{ in } \Omega, \quad \int_\Omega \rho\,\mathrm{d}x = M, \quad \phi = 0 \text{ on } \Gamma_1 \cup \Gamma_2, \quad \partial_\nu \phi = 0 \text{ on } \Gamma_3. \tag{1}$$

where the *mass* $M > 0$ of the distribution is given and $\kappa > 0$ is unknown; while the latter is solution to the linear problem:

$$-\Delta\phi_{\rm e} = \rho_{\rm e} \text{ in } \Omega, \quad \phi_{\rm e} = 0 \text{ on } \Gamma_1, \quad \phi_{\rm e} = \phi_{\rm in} \text{ on } \Gamma_2, \quad \partial_\nu \phi_{\rm e} = 0 \text{ on } \Gamma_3. \tag{2}$$

Above, we have set: $\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, with Γ_2 and Γ_3 possibly empty.

When Ω is non-convex and non-smooth, the potentials $\phi,\ \phi_{\rm e}$ are *singular*: and their gradients (which make up the electrostatic field) are unbounded near re-entrant edges or vertices (the well-known *point effect*). In the case of a polygon in $\mathbf{R}^2$, the singular part of the solution is described by a finite number of *singularity coefficients* $\lambda,\ \lambda_{\rm e}$: in local polar coordinates $\phi(r,\theta) = \lambda\, r^\alpha \sin(\alpha\,\theta) + \phi_{\rm reg}(r,\theta)$, with $\phi_{\rm reg} \in H^2(\Omega)$.

We are interested in the asymptotic behaviour of the coefficients $\kappa,\ \lambda$ under various limits, such as $M \to +\infty$ and $\phi_{\rm in} \to +\infty$. The usual boundary layer techniques *do not work*, as the limiting problems seem to be ill-posed. We introduce a new approach [1], based on the theory of *large* solutions to the *boundary blow-up problem*

$$\Delta u = p(x)\, \mathrm{e}^u \text{ in } \Omega, \quad u \to +\infty \text{ near } \Gamma_B, \quad u = g_D \text{ on } \Gamma_D, \quad \partial_\nu u = g_N \text{ on } \Gamma_N, \tag{3}$$

where Γ_D and Γ_N can be empty, but: $\Gamma_D \neq \emptyset \implies \Gamma_N \neq \emptyset$ and $\overline{\Gamma}_B \cap \overline{\Gamma}_D = \emptyset$. The problem (3) has been studied by many authors in the case $\Gamma_B = \partial\Omega$; see [2] for a review. Under "reasonable" conditions, we show that it has a unique solution when Ω is a polygon. Using large solutions as auxiliary variables, we determine the asymptotic behaviour of κ and λ.

References

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