ÉCOLE POLYTECHNIQUE
UNIVERSITÉ PARIS-SACLAY

CONCOURS D'ADMISSION 2016 – FILIÈRE UNIVERSITAIRE INTERNATIONALE
SESSION AUTOMNE 2015

PHYSICS

(Duration : 2 hours)

The two problems are independent. If you are not able to solve a question, we advise you to assume the result of that question and to move to the next one. The different questions are, to a large extent, independent of one another.
The figures and captions are part of the text : they should be read carefully.

The use of electronic calculators is forbidden.

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1 Surface Tension and Capillarity

Surface tension appears at the interface between a liquid phase (such as water) and a gas phase (such as air) due to the molecular attraction amongst the molecules of the liquid. The net effect is that molecules close to the surface of a liquid are subject to an inward force similar to the tension in a stretched membrane. Because of surface tension, the interface between the liquid and the gas has a tendency to minimize its area. Hence, in order to increase the area A of the interface by an amount dA, one has to perform a work $\delta W = \sigma dA$. The physical quantity σ is the surface tension, with units J.m^{-2}.

For water, we shall take $\sigma \simeq 72.5 \times 10^{-3}$ J.m^{-2} at temperature $T = 20°$ C.

1. We consider a spherical droplet of liquid of radius R at equilibrium with the surrounding air at atmospheric pressure P_0. We call $P_{\rm in}$ the pressure inside the droplet. Show that

$$P_{\rm in} - P_0 = \frac{2\sigma}{R}$$

This formula is know as Laplace's law.

Hint : Suppose that the radius of the droplet increases by an amount dR. Calculate the work due to pressure forces and identify it to the increase of surface energy.

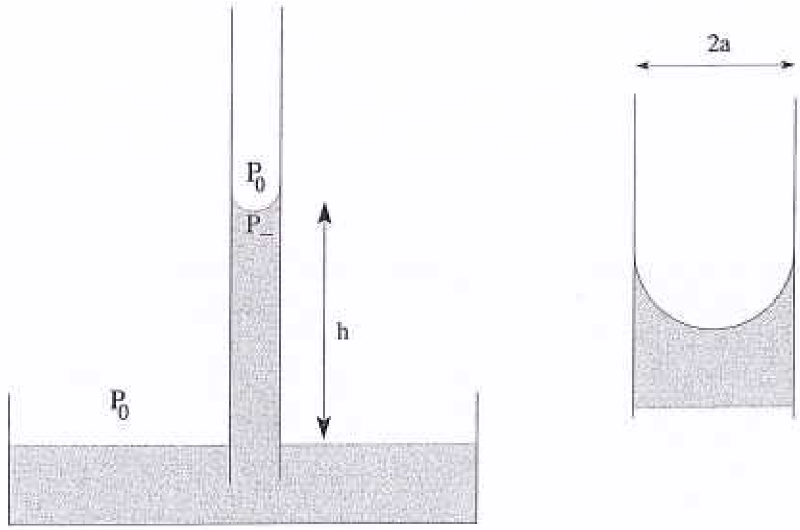

FIGURE 1 – A capillary tube of small diameter $2a$ and open at its two ends is immersed in a basin of water. Some liquid rises in the tube thanks to surface tension forces. The external pressure is the atmospheric pressure P_0. The pressure just below the interface is denoted by P_-. A blow-up of the picture is drawn on the right : the shape of the interface is taken to be a hemisphere.

For a droplet of water, calculate the required value of R to have an internal pressure that exceeds the atmospheric pressure by 1% ? ($P_0 \simeq 10^5 \text{N/m}^2$).

2. We study the rise of water due to capillarity in a thin cylindrical tube made of clean glass of radius a immersed in a basin full of water. We suppose that the interface has the shape of a full hemisphere (see Figure 1). Using Laplace's law, show that the height h of water in the tube is given by

$$h = \frac{2\sigma}{\rho g a}$$

where $\rho = 1$ kg/l is the density of water and $g = 9.8$ m/s^2 is the standard acceleration due to gravity.

Calculate the value of h for $a = 0.5$ mm.

3. Consider the more general case where the interface makes an angle ψ with the wall of the tube (see Figure 2). The value of the contact angle ψ is determined by the properties of the liquid, the gas and the material of the tube. Show that the height h is now given by

$$h = \frac{2\sigma}{\rho g a} \cos\psi$$

Hint : show that the radius of curvature of the interface is given by $R = a/\cos\psi$.

What happens when $\psi > \frac{\pi}{2}$? Do you know a liquid for which this is the case ?

4. We now study some thermodynamical properties of surface phenomena. It can be shown that the surface tension is a function of temperature $\sigma(T)$. The amount of work required to

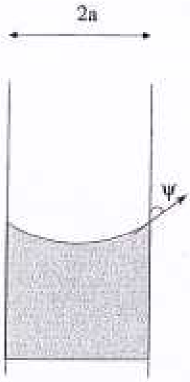

FIGURE 2 – More general interface : the boundary angle between the capillary tube and the surface of the water is given by ψ. Note that the picture drawn in Figure 1 corresponds to $\psi = 0$.

increase the interface by dA is

$$\delta W = \sigma(T) dA$$

(i) From the first and the second principles of Thermodynamics, show that the elementary amount of heat transferred to the interface when its temperature is increased by dT and its surface by dA is given by

$$\delta Q = C_A dT - T \frac{d\sigma(T)}{dT} dA$$

where C_A is the specific heat per unit area.

Hint : Write $\delta Q = C_A dT + l dA$ and give the expressions for the variations of the internal energy U and of the entropy S (discard the contribution of pressure forces). Use the fact that dU and dS are exact differentials to prove $l = -T \frac{d\sigma(T)}{dT}$.

The surface tension of water changes linearly with temperature from 75×10^{-3}N/m at 5°C to 70×10^{-3}N/m at 35°C. Calculate the heat entering a water surface when its area is increased by unity under isothermal conditions at 27°C.

(ii) Recalling that U and S are extensive with respect to A and writing $C_A = Ac(T)$, prove that

$$c(T) = -T \frac{d^2\sigma(T)}{dT^2}$$

(iii) Deduce the following formulas for the thermodynamic potentials :

$$\begin{aligned} U &= U_0 + A(\sigma - T \frac{d\sigma(T)}{dT}) \\ S &= S_0 - A \frac{d\sigma(T)}{dT} \\ F &= U - TS = F_0 + A\sigma(T) \end{aligned}$$

Using extensivity once again, show that $U_0 = S_0 = F_0 = 0$.

(iv) An atomizer produces water droplets of diameter $2R = 10^{-5}$cm. A cloud of droplets at 35°C coalesces to form a single drop of mass 1g. Estimate the temperature of the drop if there is no heat exchange with the surroundings. What is the increase of entropy in the process ?

Hint : Calculate the variation of the total surface. Because there is no heat exchange and no external work performed, the total energy is conserved : the loss of surface energy of the cloud contributes to the internal energy of the single drop. We recall that the bulk heat capacity of water is $C_v \simeq 4.18$ J K^{-1} g^{-1}.

2 Adiabatic invariants in classical mechanics

In this problem we study the motion of a mechanical particle in one-dimension, governed by the following equation

$$m\frac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2} = -\frac{\partial \mathcal{U}(x)}{\partial x},$$

where $\mathcal{U}(x)$ represents an external potential. The motion of the particle is determined by the initial conditions at time $t = 0$:

$$x(0) = x_0 \quad \text{and} \quad \dot{x}(0) = v_0 .$$

2.1 Phase Portraits

The phase space of the system is a two dimensional space with coordinates $(x, p = m\dot{x})$. A phase portrait is a geometric representation of a trajectory of the system starting from arbitrary initial conditions (x_0, mv_0) in the phase plane.

1. For a free particle, show that the phase portrait is a set of horizontal lines.

2. Study the case of a particle confined in a box of size L in the region $-L/2 \leq x \leq L/2$. The particle bounces elastically on the walls of the box. Draw the phase portrait for a given value of the velocity v of the particle. What is the value of the surface enclosed by the trajectory in the phase space ?

3. When $\mathcal{U}(x) = \frac{1}{2}m\omega^2 x^2$, we have a harmonic oscillator : show that the trajectories in the phase space are ellipses. Calculate the values of the axis of the ellipse as a function of A, the amplitude of the oscillator. Prove that the total energy E satisfies

$$E = \frac{1}{2}m\omega^2 A^2$$

and that the area under the trajectory of total energy E is given by E/ν, where ν is the frequency of the oscillator.

Plot qualitatively the trajectories when a damping force $-\gamma\dot{x}$ is added.

2.2 Particle in a box

We consider a particle of mass m and speed v that bounces elastically against the walls of a box of size L ; the walls are located at $x = \pm L/2$. The dynamics is confined in one dimension.

We suppose that the box is enlarged very slowly from L to $2L$ by moving the walls apart : the right hand wall and the left hand wall moving at speeds $\dot{L}/2$ and $-\dot{L}/2$, respectively.

1. Find the variation Δv of the speed of the particle after each collision and calculate the time τ between two successive collisions. What is the variation ΔL of the length L during the same time interval τ ? Deduce that Δv and ΔL satisfy

$$\frac{\Delta v}{v} + \frac{\Delta L}{L} = 0$$

Conclude that the value of vL remains constant. This quantity vL is called an *adiabatic invariant.* What is its dimension ? What does it represent in the phase space ? What is the value of v at the end of the process ?

2. From this adiabatic invariant, obtain the equation $PV^{5/3}$ = Constant for the adiabatic curves in a mono-atomic perfect gas.

Hint : Relate v to temperature using equipartition of energy and L to the total volume V. What happens for a diatomic gas ?

3. Can you relate this adiabatic invariant to the quantization of the energy levels of a particle in a box ?

2.3 Harmonic oscillator

We consider a pendulum, i.e. a mass m attached to a thin wire of length l and oscillating with a small amplitude $\theta(t)$. The motion is supposed to be harmonic.

1. Show that the motion is given by

$$\theta = \theta_{\max} \cos \omega t \quad \text{and} \quad \dot{\theta} = -\omega\theta_{\max} \sin \omega t$$

where $\omega = \sqrt{g/l}$ and $\theta_{\max}$ is the maximal amplitude.

Calculate the total energy E as a function of $m, g, l, \theta_{\max}$.

2. Show that the force T due to the tension of the thread is given by

$$T = mg(1 - \frac{\theta^2}{2}) + ml\dot{\theta}^2$$

Calculate the average value $\langle T \rangle$ of the tension during one period of oscillation.

We now suppose that the length l of the wire changes with time by a very slow process such that $\dot{l} \ll l\omega$. To give a precise picture, we suppose that an operator (for example, a person) pulls very slowly the wire and shortens it (see Figure 3). For the process to be very slow, the force applied by the operator exceeds the tension T of the wire only by an infinitesimal (and negligible) amount.

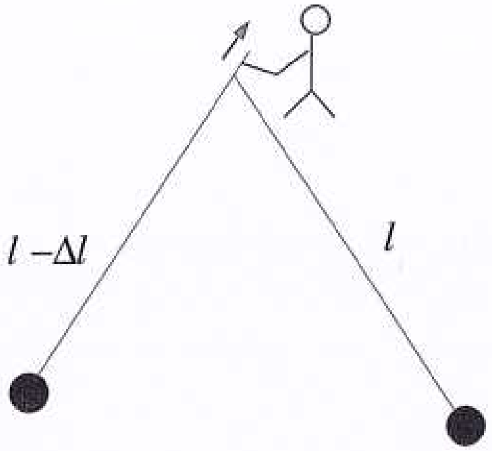

FIGURE 3 – An operator is pulling gently the wire of the pendulum. As a result, the length of the wire is becoming smaller by a very slow process.

3. Calculate the average work done by the operator, that is the work done by the average tension $\langle T \rangle$ when the length varies by Δl. Why are we allowed to replace T by its average ?

4. Show that the variation of the energy of the pendulum when l varies by Δl is given by a sum of two contributions : a term giving the variation of potential energy and a term related to the variation of the amplitude. Show that the second term satisfies

$$\frac{\Delta E}{E} = -\frac{1}{2}\frac{\Delta l}{l}\,.$$

5. Deduce from the previous question that the quantity E/ν, where ν is the frequency of the oscillator, is invariant throughout the process. This is an adiabatic invariant.

6. Suppose that the length l is reduced to $l/2$ by a very slow process. What is the maximal amplitude of the oscillations at the end of the process ?

7. Give an interpretation of the adiabatic invariant E/ν in the phase space (see question 3 in section **3.1**. What are the dimensions of E/ν ? Do you know a fundamental physical constant having the same dimensions ? Can you relate E/ν and the quantization of the energy levels of a harmonic oscillator ?

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I. Molecular sizes

The aim of this problem is to find the size of the molecules of a liquid from the latent heat of vaporization (L) and the surface tension (γ) of the pure liquid.

Let M be the molar mass of the liquid and ε the interaction energy of one molecule with all the other molecules in the liquid. The molecules are assumed to be spherical with a diameter d. Their interaction is long ranged with a power-law decay :

$$U(r) = \frac{A}{r^n}$$

whenever $r > d$.

1) For which values of n is the energy ε well-defined? (hint: express the interaction energy of one molecule with all the other molecules contained in a macroscopic volume).

In the following, the molecules are assumed to be at the nodes of a simple cubic lattice, with a mesh size equal to d.

2) Express the mass density ρ of the liquid as a function of the molar mass M, the molecule diameter d and Avogadro constant N_A.

3) Let L be the latent heat of vaporization of the liquid (equal to the amount of heat required to transform 1 kg of liquid into vapor). Express L as a function of ε, M and N_A.

4) The surface tension γ of the liquid is defined as follows : when the open surface of the liquid (the surface separating the liquid and the air) is increased by an amount of dA, then the energy of the liquid increases accordingly by the amount :

$$dU = \gamma dA$$

5) Express γ as a function of ε and d.

6) Infer from this result that :

$$d = \frac{2\gamma}{\rho L}$$

7) Experimental data are given in the following table :

	ρ g/cm^3	γ 10^{-3}N/m	L kJ/kg
water	1	73	2265
mercury	13,5	450	295
octane	0,703	22	298
glycerin	1,26	63	974

Find the diameters of the corresponding molecules. Conclude.

II. Gas leakage modeling

Notations and reminders :

(i) Boltzmann constant : k_B

(ii) gas particle density: n (number of molecules per unit volume)

(iii) gas molecular mass : m (mass of one molecule of the gas)

(iv) $\gamma = \frac{C_p}{C_v} = 1.4$ for a diatomic gas, with C_v (resp. C_p) the mass heat capacity at constant volume (resp. constant pressure).

(v) Sound speed in a perfect gas : $c = \sqrt{\gamma \frac{k_B T}{m}}$

(vi) vitesse moyenne des molécules dans un gaz parfait : $\bar{v} = \sqrt{\frac{8k_B T}{\pi m}}$

A vessel, surrounded by vacuum, contains a perfect gas at pressure P and temperature T. At time t=0, a hole of diameter D is bored through the vessel wall.

1) The mean free path of the molecules in the gas is denoted λ. What is the physical meaning of the mean free path ? The molecules of the gas are assumed to be hard

spheres of diameter d and the particle volume density is n. Then the mean free path can be written as :

$$\lambda \simeq \frac{1}{nd^2}$$

Express n as a function of P and T.

In the following, the thickness L of the vessel wall is assumed to be equal to λ.

2) Let us first assume that D < λ (« Knudsen regime»). Estimate the mass flow rate Q_m of the leaking gas. The gas pressure and temperature will be considered to be invariant in the vessel. It may further be assumed that one sixth of all the molecules in the vicinity of the hole have the same velocity of norm $\bar{v}$ pointing outwards in the normal direction to the wall. Justify this assumption.
How does the mass flow depend on D ?

From now on there is also a perfect gas outside the vessel. Both gases inside and outside the vessel are made of the same molecules, and are at the same temperature T ; the gas inside is at pressure P + ΔP and the gas outside at pressure P.

3) What is the new leak mass flow Q'_m?

4) Let us then assume that λ < D (« Poiseuille regime »). The gas can now be considered as a viscous fluid. If D is smaller than a critical value D_{max}, which will be specified in the next question, the gas flow is laminar through the pore (cylindric hole of diameter D and length L). In this case, the resistance to flow R_H, defined as :

$$R_H = \frac{\Delta P}{Q_v}$$

where ΔP is the excess of pressure inside the vessel (with respect to outside) and Q_v is the gas volume flow rate, can be written as :

$$R_H = \frac{128\eta L}{\pi D^4}$$

and the dynamic viscosity η of a perfect gas is :

$$\eta = \frac{1}{3} nm\bar{v}\lambda$$

Express the mass flow rate of the leaking gas as a function of ΔP, $\bar{v}$, n, d, L and D. How does it vary with D ?

5) It is reminded that the flow is laminar as long as Reynolds number

$$Re = \frac{< v > D}{\eta}$$

is smaller than 2000. Here $< v >$ is the velocity of the gas flow averaged over the pore cross-section. Compute $< v >$ and infer D_{max} from it.

6) Let us finally assume that D > D_{max} (« Bernoulli regime »). It is reminded that the first law of thermodynamics for open systems can be written :

$$h + \frac{1}{2}v^2 = const$$

where h is the enthalpy per unit mass of gas and v the velocity of the gas flow, assumed to be uniform over the pore cross-section. Moreover $h = C_pT$ up to an arbitrary additive constant.
The flow is fast enough to be adiabatic and slow enough to be quasistatic. What is the physical meaning of these assumptions ?
Then according to Laplace law : $P^{1-\gamma}T^{\gamma} = const$. Prove that the velocity v of the gas flow satisfies the following equation:

$$1 + \frac{\gamma - 1}{2}\frac{v^2}{c^2} = \left(\frac{P + \Delta P}{P}\right)^{\frac{\gamma-1}{\gamma}}$$

Infer from it the mass flow rate of the leaking gas. How does it depend on D ? For which values of ΔP does the sudden opening of a hole in the vessel wall result in a bang?

7) Plot the shape of the functional dependence of the mass flow rate Q_m on the hole diameter D in the most general case. Specify the values of D that correspond to the transition from Knudsen to Poiseuille regime on the one hand, and Poiseuille to Bernoulli regime on the other.

III. A charged particle moving in magnetic and electric fields

A point particle of mass m and charge q moves in a region where both a uniform electrostatic field $\vec{E}$ and a uniform magnetostatic field $\vec{B}$ are present. $\vec{E}$ and $\vec{B}$ are assumed to be orthogonal (see Figure). All relativistic effects are neglected in this problem.

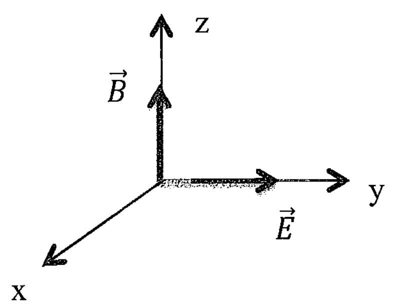

Figure : relative disposition of the $\vec{E}$ et $\vec{B}$ fields

1) Write the equation of motion of the particle.

2) Solve this equation for the particle velocity in the most general case. It is assumed that the initial velocity (velocity at time t=0), named $\vec{V}$, is normal to $\vec{B}$.

3) Express the time-average velocity $\vec{U}$ as a function of $\vec{E}$ et $\vec{B}$. Show that this result has an obvious interpretation in the reference frame moving at $\vec{U}$ relative to the original frame.

4) Give a physical description of the particle motion and plot the various trajectories depending on the ratio $\frac{VB}{E}$ (reminder : V is the initial velocity of the particle). What kind of trajectory is obtained for $V = 0$?

5) What is the condition on $\vec{E}$ and $\vec{B}$ that allows us to neglect relativistic effects ?

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PHYSIQUE

(Durée : 2 heures)

Les trois problèmes sont indépendants. Si vous n'arrivez pas à traiter une question, nous vous recommandons d'accepter le résultat de cette question et de passer à la question suivante. Les différentes questions sont largement indépendantes.

L'usage de la calculette est interdit.

I. Taille des molécules

On cherche à évaluer la taille d'une molécule à partir de la chaleur latente de vaporisation (L) et de la tension superficielle (γ) du liquide pur.

On note M la masse molaire du liquide pur et ε l'énergie d'interaction d'une molécule avec toutes les autres molécules du liquide. On suppose que les molécules sont des sphères de diamètre d, qui interagissent par un potentiel à longue portée, de telle sorte que l'interaction entre deux molécules séparées d'une distance r>d s'écrit :

$$U(r) = \frac{A}{r^n}$$

1) A quelle condition sur n l'énergie ε est-elle définie ? (indication : exprimer l'énergie d'interaction d'une molécule avec toutes les autres molécules du liquide occupant un volume macroscopique).

   Dans la suite du problème on supposera pour simplifier que les molécules sont disposées aux nœuds d'un réseau cubique simple, dont la maille élémentaire est un cube de côté d.

2) Exprimer la masse volumique du liquide en fonction de la masse molaire M, du diamètre d et du nombre d'Avogadro N_A.

3) On note L la chaleur latente massique du liquide (variation d'enthalpie lors de la transformation d'un kg de liquide en vapeur). Exprimer L en fonction de ε, M et N_A.

4) La tension superficielle γ est définie de la façon suivante : si la surface libre du liquide (interface liquide-air) varie d'une quantité dA, alors la variation correspondante de l'énergie interne du liquide s'écrit :

$$dU = \gamma dA$$

5) Exprimer γ en fonction de ε et d.

6) En déduire que :

$$d = \frac{2\gamma}{\rho L}$$

7) On donne le tableau suivant :

	ρ g/cm³	γ 10^{-3}N/m	L kJ/kg
eau	1	73	2265
mercure	13,5	450	295
octane	0,703	22	298
glycérol	1,26	63	974

Donner les valeurs correspondantes des diamètres des molécules. Conclure.

II. Modélisation d'une fuite de gaz

Notations et rappels :

(i) constante de Boltzmann : k_B

(ii) densité particulaire du gaz : n (nombre de molécules par unité de volume)

(iii) masse moléculaire du gaz : m (masse d'une molécule du gaz)

(iv) $\gamma = \frac{C_p}{C_v} = 1.4$ pour un gaz diatomique avec C_v (resp. C_p) la capacité thermique massique à volume constant (resp. pression constante)

(v) vitesse du son dans un gaz parfait : $c = \sqrt{\gamma \frac{k_B T}{m}}$

(vi) vitesse moyenne des molécules dans un gaz parfait : $\bar{v} = \sqrt{\frac{8 k_B T}{\pi m}}$

Une enceinte, placée dans le vide, contient initialement un gaz parfait à la pression P et la température T. A l'instant t=0, on perce un trou de diamètre D dans la paroi de l'enceinte.

1) On note λ le libre parcours moyen des molécules du gaz. Que représente le libre parcours moyen ? On rappelle que, pour un gaz de densité particulaire n, dont les

molécules peuvent être assimilées à des sphères dures de diamètre d, le libre parcours moyen s'écrit :

$$\lambda \simeq \frac{1}{nd^2}$$

Exprimer n en fonction de P et T.

Dans toute la suite on admettra que l'épaisseur L de la paroi de l'enceinte est égale à λ.

2) On suppose d'abord que D < λ (« régime de Knudsen »). Estimer le débit massique Q_m de la fuite de gaz. On négligera la variation de pression et de température du gaz dans l'enceinte due à la fuite. On supposera de plus pour simplifier qu'un sixième des molécules du gaz présentes près de l'orifice ont une vitesse dirigée vers l'extérieur selon la normale à l'orifice et dont la norme est égale à la vitesse moyenne $\bar{v}$ des molécules.
Comment varie le débit massique avec D ?

On se place désormais dans le cas plus général où le gaz parfait est également présent à l'extérieur de l'enceinte : les gaz à l'intérieur et à l'extérieur de l'enceinte sont à la même température T ; le gaz à l'intérieur est à la pression P + ΔP et celui à l'extérieur à la pression P.

3) Que devient le débit de fuite ?

4) On suppose maintenant que λ < D (« régime de Poiseuille »). Le gaz peut alors être considéré comme un fluide visqueux et on rappelle que la viscosité dynamique d'un gaz parfait est donnée par l'équation :

$$\eta = \frac{1}{3} n m \bar{v} \lambda$$

Si D est inférieur à une valeur seuil D_{max}, dont la valeur sera précisée à la question suivante, l'écoulement est de type laminaire à travers le pore de diamètre D et de longueur L. Dans ce cas, la résistance hydraulique R_H, définie par

$$R_H = \frac{\Delta P}{Q_v}$$

où ΔP est la différence de pression entre l'intérieur et l'extérieur de l'enceinte et Q_v le débit volumique de gaz, s'écrit :

$$R_H = \frac{128 \eta L}{\pi D^4}$$

Calculer le débit massique de la fuite en fonction de ΔP, $\bar{v}$, n, d, L et D. Comment varie-t-il maintenant avec D ?

5) On rappelle que l'écoulement est laminaire tant que le nombre de Reynolds

$$Re = \frac{< v > D}{\eta}$$

est inférieur à 2000. $< v >$ désigne ici la vitesse d'écoulement du gaz moyennée sur une section du pore. Calculer $< v >$ et en déduire la valeur de D_{max}.

6) On suppose enfin que $D > D_{max}$ (régime de Bernoulli). On rappelle que le premier principe généralisé (pour un sysème ouvert) s'écrit alors :

$$h + \frac{1}{2}v^2 = cte$$

où h est l'enthalpie massique du gaz et v la vitesse du jet de gaz, supposée constante dans toute la section du pore. On rappelle également que $h = C_p T + cte$. On supposera que l'écoulement est assez rapide pour être considéré adiabatique et assez lent pour être considéré quasistatique. Justifier ces hypothèses.
On a alors l'équation de Laplace : $P^{1-\gamma}T^{\gamma} = cte$. Montrer que la vitesse v du jet satisfait l'équation :

$$1 + \frac{\gamma - 1}{2}\frac{v^2}{c^2} = \left(\frac{P + \Delta P}{P}\right)^{\frac{\gamma-1}{\gamma}}$$

En déduire le débit massique de la fuite. Comment varie-t-il avec D ? Pour quelles valeurs de la différence de pression ΔP l'ouverture brutale du trou se traduit-elle par une détonation?

7) Tracer l'allure du débit massique Q_m en fonction du diamètre D du trou dans le cas général. On précisera les valeurs du diamètre D correspondant aux transitions entre les régimes de Knudsen et de Poiseuille, d'une part, et de Poiseuille et de Bernoulli, d'autre part.

III. Mouvement d'une particule chargée

Une particule ponctuelle de masse m et de charge q est soumise à un champ électrique $\vec{E}$ et un champ magnétique $\vec{B}$, tous deux constants et uniformes. On suppose de plus que $\vec{E}$ et $\vec{B}$ sont orthogonaux (voir Figure). Dans ce problème, on néglige tous les effets relativistes.

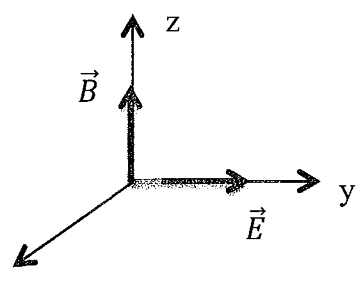

Figure : disposition relative des champs $\vec{E}$ et $\vec{B}$

1) Ecrire l'équation du mouvement de la particule.

2) Résoudre cette équation pour la vitesse dans le cas le plus général. On supposera pour simplifier que la vitesse initiale (vitesse au temps t=0), notée $\vec{V}$, est orthogonale à $\vec{B}$.

3) Exprimer la vitesse moyenne $\vec{U}$ en fonction des vecteurs $\vec{E}$ et $\vec{B}$. Retrouver ce résultat en effectuant un changement de référentiel (se placer dans le référentiel en translation à la vitesse moyenne $\vec{U}$).

4) Décrire le mouvement de la particule et tracer les différentes allures possibles de la trajectoire en fonction du rapport $\frac{VB}{E}$ où V est la vitesse initiale de la particule. Quelle est la nature de la trajectoire quand $V = 0$?

5) A quelle condition sur $\vec{E}$ et $\vec{B}$ pouvait-on négliger les effets relativistes ?

CONCOURS D'ADMISSION 2016 – FILIÈRE UNIVERSITAIRE INTERNATIONALE

SESSION AUTOMNE 2015

MATHEMATICS

(Duration : 2 hours)

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This exam is composed of 5 exercises. One can solve them in *any order*. Generally speaking, the first questions are simpler and the final questions are more difficult. One should therefore *not spend too much time* on these final questions before having solved the other exercices.

In solving a given exercise, one is allowed to use the results of the preceding questions *(including those one could not prove)*. All statements must be *clearly* and *completely* justified.

I.

Let P be a polynomial in $\mathbb{R}[X]$ satisfying $(*)$: $P(P(X)) = P(X)^2$.

1. Are there constant polynomials solutions to $(*)$?

From now on, we assume that P is a non-constant polynomial solution to $(*)$.

2. Determine the degree of P.

3 What is the value of the coefficient of X^2 in P ?

4. Find all the polynomials P solutions to $(*)$?

5. By an analogous reasoning, find the solutions $P \in \mathbb{C}[X]$ of the equation :

$$P(P''(X)) = (P(X+1) - P(X) - P'(X))^3 .$$

II.

Let n be a positive integer, E be an n-dimensional $\mathbb{C}$-vector space and $(e_1, \ldots, e_n)$ be a basis of E. Let u be the endomorphism of E defined by the relations :

$$\text{for all } i \in \{1, \ldots, n\} : u(e_i) = e_i + f$$

where $f = \sum_{k=1}^{n} e_k$.

1. What is the matrix of u in the basis $(e_1, \ldots, e_n)$? We call it U in the following questions.

2. Is the matrix U invertible ?

3. Let J be the matrix which has ones in all positions ($J_{i,j} = 1$ for all $1 \leq i,j \leq n$). Compute J^2.

4. Find the eigenvalues of U. For each of them, give a basis of the associated eigenspace.

5. Compute, for each non-negative integer m, the m-th power of U as a function of J.

III.

Let M_0 be the 3×3 square matrix :

$$M_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} .$$

1. Compute ${M_0}^3$ in terms of M_0.

Let now M be a 3×3 square matrix with real coefficients, $M \in \mathcal{M}(\mathbb{R}^3)$, such that

$$M^3 = -M \ .$$

We assume $M \neq 0$.

2. Is the matrix M diagonalizable as a real matrix? What about diagonalizability as a complex matrix ?

From now on, we are only interested in the reduction of M as a real matrix.

3. Show that

$$\mathbb{R}^3 = \text{Ker } M \oplus \text{Ker } (M^2 + I) \ ,$$

where I stands for the identity matrix in dimension 3.

4. Prove that the dimension of Ker $(M^2 + I)$ is even. Then deduce dim Ker $M = 1$.

5. Show that there exists a vector x in $\mathbb{R}^3$ such that the family $\{x, Mx\}$ is free.

6. Prove that M is similar to M_0.

IV.

In this exercise, we are interested in the solutions to the equation (E) : $\tan x = x$.

1. Show that for each integer n, (E) has exactly one solution in the interval $(n\pi - \pi/2, n\pi + \pi/2)$. We denote it by x_n.

2. Prove that, when n tends to infinity, $x_n - n\pi$ tends towards $\pi/2$.

3. Compute the limit of the quantity $v_n = n(x_n - n\pi - \pi/2)$ when n tends to infinity.

V.

We define the functions

$$I(x) = \int_0^{+\infty} \frac{e^{-t}}{\sqrt{t}} \cos tx \, dt \qquad \text{and} \qquad J(x) = \int_0^{+\infty} \frac{e^{-t}}{\sqrt{t}} \sin tx \, dt .$$

1. Explain why I and J are well defined on $\mathbb{R}$.

2. Show that I and J are differentiable and compute their derivatives I' and J'.

3. Integrating by parts, prove that

$$I'(x) = -\frac{1}{2}J(x) - xJ'(x) .$$

In the same way, find a relation between J', I and I'.

4. Deduce that I and J both satisfy a differential equation of the type

$$2(1 + x^2)y' + xy = f$$

where f is a certain function (not necessarily the same for I and J).

5. Solve the preceding system and determine I and J in terms of x uniquely.

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CONCOURS D'ADMISSION 2016 – FILIÈRE UNIVERSITAIRE INTERNATIONALE
SESSION DE PRINTEMPS

MATHEMATICS

(Duration : 2 hours)

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This exam is composed of four exercises. One can solve them *in any order*. It is not needed to solve all the exercises to obtain the best possible mark.

Generally speaking, the difficulty of the questions is an increasing function of their number so that *the final questions are more difficult*. It is therefore not recommended to spend too much time on the final questions of an exercise before having solved the first questions of the others.

In solving a given exercise, one is allowed to use the results of the preceding questions *(including those one could not prove)*.

All statements must be *clearly* and *completely* justified.

I. Matrices with a robust diagonal

A matrix $M \in \mathcal{M}_n(\mathbb{C})$ is said to have a robust diagonal if the elements on its diagonal coincide with its eigenvalues (with the same multiplicities). We denote by $\mathcal{D}_n$ the subset of $\mathcal{M}_n(\mathbb{C})$ composed of all the matrices with a robust diagonal.

1) Identify $\mathcal{D}_1$.

2) Show that every matrix in $\mathcal{M}_n(\mathbb{C})$ is similar to a matrix with a robust diagonal.

3) Identify $\mathcal{D}_2$. Is this set open ? closed ? path-connected ? convex ?

4) For a given integer n, is the set of all matrices with a robust diagonal $\mathcal{D}_n$ a vector space ?

5) Determine, among the real symmetric matrices, those belonging to $\mathcal{D}_n$. (Hint : one may introduce the quantity $\mathrm{tr}({}^tMM)$).

II. Study of a sequence

Let c be a non-negative real number and $(u_n)_{n\geq 1}$ be the sequence defined by $u_1 = 1$ and the induction relation

$$u_{n+1} = \sqrt{u_n + cn}.$$

1) Find a real number σ such that for any integer $n \geq 1$, one has $u_n \leq \sigma\sqrt{n}$.

2) Find an equivalent of u_n, when n tends to infinity, of the form $u_n \sim \alpha n^\beta$ (where α and β are real numbers).

3) Compute the limit of $u_n - \alpha n^\beta$ as n tends to infinity.

III. Homogeneous polynomials

For a given positive integer n, we denote by $\mathcal{H}_n \subset \mathbb{R}[X, Y]$ the set of homogeneous polynomials with real coefficients of degree n in two variables (X and Y). We recall that these are the polynomials P having the property that $P(\lambda X, \lambda Y) = \lambda^n P(X, Y)$ for any real number λ.

We denote by $\mathcal{D}_n$ the subset of $\mathcal{H}_n$ composed of the polynomials divisible by $X^2 + Y^2$ and we denote by $\mathcal{L}_n$ the subset of $\mathcal{H}_n$ composed of those polynomials P satisfying

$$\frac{\partial^2 P}{\partial X^2} + \frac{\partial^2 P}{\partial Y^2} = 0.$$

1) Prove that, for any integer n, $\mathcal{H}_n$, $\mathcal{D}_n$ and $\mathcal{L}_n$ are vector spaces.

2) Prove that any polynomial in $\mathcal{H}_n$ can be written as

$$\sum_{k=0}^{n} c_k X^k Y^{n-k}$$

where the c_k's are real numbers. What is the dimension of $\mathcal{H}_n$?

3) What are the relations that the coefficients of P must satisfy if $P \in \mathcal{L}_n$?

4) Using question 3), give a basis of $\mathcal{L}_n$.

5) Prove that $\mathcal{D}_n \cap \mathcal{L}_n = \{0\}$.

6) Prove that $\dim \mathcal{D}_n = \dim \mathcal{H}_{n-2}$.

7) Prove that $\mathcal{D}_n \oplus \mathcal{L}_n = \mathcal{H}_n$.

IV. Study of a power series

1) For which integers n does the equality $\sin(n\pi\sqrt{5}) = 0$ hold?

Let R be the radius of convergence of the power series $\displaystyle\sum_{n\geq 1} \frac{z^n}{\sin(n\pi\sqrt{5})}$.

2) Prove that $R \leq 1$.

3) Prove that, for all $t \in [0, \pi/2]$, one has $\sin t \geq t - t^3/6$.

4) Prove that for any integers p and q, non simultaneously equal to zero, one has

$$|p\sqrt{5} - q| \geq \frac{1}{p\sqrt{5} + q}.$$

5) Using questions 3) and 4), find a real number $c > 0$ such that, for all integers $n \geq 1$, one has

$$|\sin(n\pi\sqrt{5})| \geq \frac{c}{n}.$$

6) Compute R.

CONCOURS D'ADMISSION 2016 – FILIÈRE UNIVERSITAIRE INTERNATIONALE
SESSION DE PRINTEMPS

MATHÉMATIQUES

(Durée : 2 heures)

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Ce sujet est composé de quatre exercices. On peut les traiter dans *n'importe quel ordre*. Il n'est pas nécessaire de résoudre tous les exercices pour obtenir la note maximale.

D'une façon générale, la difficulté des questions est une fonction croissante de leur numéro de sorte que *les questions finales sont les plus dures*. On ne doit donc pas consacrer trop de temps aux dernières questions d'un exercice avant d'avoir résolu les premières questions des autres.

Dans la résolution d'un exercice donné, on peut toujours utiliser le résultat des questions précédentes *(même si on ne les a pas résolues)*.

Toute affirmation doit être *clairement* et *complètement* justifiée.

I. Matrices à diagonale solide

On dit qu'une matrice $M \in \mathcal{M}_n(\mathbb{C})$ est à diagonale solide si les éléments de sa diagonale sont ses valeurs propres (avec mêmes multiplicités). On note $\mathcal{D}_n$ le sous-ensemble de $\mathcal{M}_n(\mathbb{C})$ composé de toutes les matrices à diagonale solide.

1) Identifier $\mathcal{D}_1$.

2) Montrer que toute matrice de $\mathcal{M}_n(\mathbb{C})$ est semblable à une matrice à diagonale solide.

3) Identifier $\mathcal{D}_2$. Cet ensemble est-il ouvert ? fermé ? connexe par arcs ? convexe ?

4) Pour n un entier donné, l'ensemble de toutes les matrices à diagonale solide $\mathcal{D}_n$ est-il un espace vectoriel ?

5) Déterminer les matrices symétriques réelles M appartenant à $\mathcal{D}_n$. (Indication : on pourra introduire la quantité $\mathrm{tr}({}^tMM)$).

II. Étude d'une suite

Soit c un nombre réel positif et $(u_n)_{n\geq 1}$ la suite définie par $u_1 = 1$ et la relation de récurrence

$$u_{n+1} = \sqrt{u_n + cn}.$$

1) Trouver un réel σ tel que pour tout entier $n \geq 1$, on ait $u_n \leq \sigma\sqrt{n}$.

2) Trouver un équivalent de u_n, lorsque n tend vers l'infini, de la forme $u_n \sim \alpha n^\beta$ (où α et β sont des nombres réels).

3) Calculer la limite de $u_n - \alpha n^\beta$ lorsque n tend vers l'infini.

III. Polynômes homogènes

Pour n un entier strictement positif quelconque, on note $\mathcal{H}_n \subset \mathbb{R}[X,Y]$ l'ensemble des polynômes homogènes à coefficients réels de degré n en deux variables (X et Y) : on rappelle qu'il s'agit des polynômes P ayant la propriété que $P(\lambda X, \lambda Y) = \lambda^n P(X,Y)$ quel que soit le réel λ.

On note $\mathcal{D}_n$ le sous-ensemble de $\mathcal{H}_n$ composé des polynômes divisibles par X^2+Y^2 et par $\mathcal{L}_n$ celui des polynômes P de $\mathcal{H}_n$ tels que

$$\frac{\partial^2 P}{\partial X^2} + \frac{\partial^2 P}{\partial Y^2} = 0.$$

1) Démontrer que, quel que soit l'entier n, $\mathcal{H}_n$, $\mathcal{D}_n$ et $\mathcal{L}_n$ sont des espaces vectoriels.

2) Démontrer que tout polynôme de $\mathcal{H}_n$ peut s'écrire sous la forme

$$\sum_{k=0}^{n} c_k X^k Y^{n-k}$$

où les c_k sont des réels ? Quelle est la dimension de $\mathcal{H}_n$?

3) Quelles relations doivent vérifier les coefficients de P si $P \in \mathcal{L}_n$?

4) Déduire de la question 3) une base de $\mathcal{L}_n$.

5) Démontrer que $\mathcal{D}_n \cap \mathcal{L}_n = \{0\}$.

6) Démontrer que $\dim \mathcal{D}_n = \dim \mathcal{H}_{n-2}$.

7) Démontrer que $\mathcal{D}_n \oplus \mathcal{L}_n = \mathcal{H}_n$.

IV. Étude d'une série entière

1) Pour quels entiers n a-t-on $\sin(n\pi\sqrt{5}) = 0$?

Soit R le rayon de convergence de la série entière $\displaystyle\sum_{n\geq 1} \frac{z^n}{\sin(n\pi\sqrt{5})}$.

2) Démontrer que $R \leq 1$.

3) Démontrer que, pour tout $t \in [0, \pi/2]$, on a : $\sin t \geq t - t^3/6$.

4) Démontrer que quels que soient les entiers p et q, non nuls simultanément, on a

$$|p\sqrt{5} - q| \geq \frac{1}{p\sqrt{5} + q}.$$

5) En utilisant les questions 3) et 4), trouver un réel $c > 0$ tel que, pour tout entier $n \geq 1$, on ait

$$|\sin(n\pi\sqrt{5})| \geq \frac{c}{n}.$$

6) Que vaut R?