5.1. The Rindler metric

In this section we will study the Minkowski space from the point of view of an observer in accelerated motion. The plan is the following:

1. find the universe line of an uniformly accelerated observer;
2. determine the time and space coordinates of the other events (if possible) using a syncronization procedure based on the exchange of light signals;
3. write the metric as a function of these coordinates.

5.1.1. Constant acceleration

In special relativity an uniform acceleration is defined by the requirement that the acceleration in the reference frame of the accelerated object is constant. If we consider the quadrivector

$$\frac{du^\nu}{d\tau}$$

we have in the rest frame

$$\frac{du^\nu}{d\tau} \overset{*}{=} \begin{pmatrix} 0 \\ \vec{a} \end{pmatrix}$$

where we choose the x axis in the acceleration direction. With a Lorentz transformation we can rewrite this in the inertial frame

$$\frac{du^\nu}{d\tau} = \frac{d}{d\tau}\begin{pmatrix} \gamma \\ \gamma\vec{v} \end{pmatrix} = \begin{pmatrix} \gamma\vec{v}\cdot\vec{a} \\ \vec{a} + (\gamma - 1)\,\frac{\vec{v}\cdot\vec{a}}{v^2}\vec{v} \end{pmatrix}$$

If we suppose that initially the object has zero velocity, we see that $\vec{v}$ will be always in the same direction of $\vec{a}$. This means that we can restrict our equations to the acceleration direction only, obtaining

$$\frac{d}{d\tau}\begin{pmatrix} \gamma \\ \gamma v \end{pmatrix} = \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}\begin{pmatrix} \gamma \\ \gamma v \end{pmatrix}$$

which can be solved as

$$\begin{pmatrix} \gamma \\ \gamma v \end{pmatrix} = \exp\left[\begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}\tau\right]\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cosh a\tau \\ \sinh a\tau \end{pmatrix}$$

With another integration we get the universe line of the object

$$\begin{aligned} \gamma &= \frac{dt}{d\tau} = \cosh a\tau \\ \gamma v &= \frac{dx}{d\tau} = \sinh a\tau \end{aligned}$$

which gives

$$\begin{aligned} x_o^t &= \frac{1}{a}\sinh a\tau \\ x_o^x &= \frac{1}{a}(\cosh a\tau - 1) + x_0 \end{aligned}$$

For future convenience we choose x_0 in such a way that asymptotically $x = t$, which means that $x_0 = a^{-1}$ and

$$x_o^t = \frac{1}{a}\sinh a\tau$$
$$x_o^x = \frac{1}{a}\cosh a\tau$$

5.1.2. Observer coordinates

The accelerated observer will consider an event to happen at his time τ, if a light signal emitted at $\tau - \Delta$ and reflected at the event will be received at the symmetric time $\tau + \Delta$. This means that the τ coordinate of an event will be given by

$$\tau\left(x^{\mu}\right) = \frac{\tau_+ + \tau_-}{2}$$

where τ_+ and τ_- are the two solutions of

$$\left(x_o^t\left(\tau\right) - x^t\right)^2 - \left(x_o^x\left(\tau\right) - x^x\right)^2 = \left(x^y\right)^2 + \left(x^z\right)^2$$

The same observer will assign a distance δ to the event given by

$$\delta\left(x^{\mu}\right) = \frac{\tau_+ - \tau_-}{2}$$

Explicitly, it must be

$$\left(\frac{1}{a}\sinh a\tau_+ - x^t\right)^2 - \left(\frac{1}{a}\cosh a\tau_+ - x^x\right)^2 = \left(x^y\right)^2 + \left(x^z\right)^2$$
$$\left(\frac{1}{a}\sinh a\tau_- - x^t\right)^2 - \left(\frac{1}{a}\cosh a\tau_- - x^x\right)^2 = \left(x^y\right)^2 + \left(x^z\right)^2$$

Expanding we get

$$\begin{pmatrix}\cosh a\tau_+ & -\sinh a\tau_+\\ \cosh a\tau_- & -\sinh a\tau_-\end{pmatrix}\begin{pmatrix}x^x\\ x^t\end{pmatrix} = \frac{a}{2}\begin{pmatrix}1\\1\end{pmatrix}\left[\frac{1}{a^2} - \left(x^t\right)^2 + \left(x^x\right)^2 + \left(x^y\right)^2 + \left(x^z\right)^2\right]$$

or

$$\begin{pmatrix}\cosh a\left(\tau+\delta\right) & -\sinh a\left(\tau+\delta\right)\\ \cosh a\left(\tau-\delta\right) & -\sinh a\left(\tau-\delta\right)\end{pmatrix}\begin{pmatrix}x^x\\ x^t\end{pmatrix} = \frac{a}{2}\begin{pmatrix}1\\1\end{pmatrix}\left[\frac{1}{a^2} - \left(x^t\right)^2 + \left(x^x\right)^2 + \left(x^y\right)^2 + \left(x^z\right)^2\right]$$

It must be

$$x^x\left[\cosh a\left(\tau+\delta\right) - \cosh a\left(\tau-\delta\right)\right] = x^t\left[\sinh a\left(\tau+\delta\right) - \sinh a\left(\tau-\delta\right)\right]$$

or

$$x^x\sinh a\tau = x^t\cosh a\tau$$

which means that

$$\tanh a\tau = \frac{x^t}{x^x}$$

In other words, we will assign the same time coordinate to all the events in the hyperplane

$$x^t = x^x \tanh a\tau$$

Substituting we obtain the distance from the event

$$\cosh a\delta \quad = \tfrac{1}{2a} \quad \frac{1}{2\sqrt{(x^x)^2 - (x^t)^2}} \left\{ \frac{1}{a^2} - \left[\left(x^t\right)^2 - (x^x)^2 - (x^y)^2 - (x^z)^2 \right] \right\}$$

A set of observers:

$$x^t_{\vec{a}} = \frac{1}{a} \sinh a\tau$$
$$x^i_{\vec{a}} = \frac{a^i}{a^2} \cosh a\tau$$