Supplement 4-8

Maxwell’s Equations

Perhaps you’ve noticed that my books describe themselves using the word “honest” in the sales blurb on the back cover. What they mean by this is that they promise never to hide the whole truth from you, for example by throwing an equation at you without explaining where it came from. Like all the books in the series, Electricity and Magnetism is straight-shootin’ and plain-talkin’, so it should be clear to you that the treatment of E&M so far, whatever its good points, has often been nonmathematical, intuitive in style, and not very rigorous. For example, the magnetostatics equations for the magnetic field of a solenoid or a long, straight wire came with warning labels clearly stating that they came from calculations using fancy math (vector calculus) that was not covered in the main body of the book.

This supplement deals with that fancy math. By the end, you’ll have the same kind of deeper understanding of electromagnetism that Maxwell had on the now-legendary starry night when he told his wife what starlight was. To understand this material, you’ll need to know calculus and to have learned about the dot product (book 2, section 3.7), and the vector cross product and the right-hand rule (supplement 2-7). You do not need to know vector calculus already — you’ll learn a little basic vector calculus here, and more if you go on to supplement 4-9. Throughout this chapter, we use the notation “∂ ” for a partial derivative; for instance, if a function f depends on two variables, x, and y, then $\partial f / \partial x$ means the derivative of f with respect to x, with y being held constant.

8.1 Maxwell's Equations

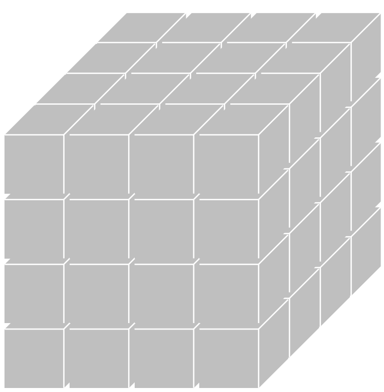
Maxwell's approach

The philosophy of this book is always intuition first, rigor later. In keeping with that approach, we first state Maxwell's equations in an intuitive form, and only later develop the mathematical bag of tricks needed to manipulate them. Maxwell set out to make a unified mathematical framework for all the complicated phenomena that had been observed about electricity and magnetism. A verbal summary of these phenomena is as follows:

- The sources and sinks of the ***E*** field are positive and negative charges.
- The ***B*** field has no sources or sinks. (There are no magnetic charges.)
- A "whirlpool" ***E*** field is induced around a changing ***B*** field.
- A "whirlpool" ***B*** field is induced around a changing ***E*** field or a current.

Note the symmetry of these relationships, which is violated only by the fact that nature does not seem to provide us with any magnetic charges, or with "magnetic currents" made by moving magnetic charges.

The problem with turning these statements into equations is that they refer to field patterns that stretch out into space around the thing that is causing them. In theory, the effects reach infinitely far away, and overlap with all the other patterns formed by all the other things going on in the universe. But since electromagnetic effects only propagate at the speed of light, we cannot relate the distant fields to their causes at the same instant in time. By the time space aliens learn about bebop from our radio signals, the radio antennas that made the signals will have long since ceased to exist. Thus an equation like Coulomb's law, $E=kq/r^2$, cannot be part of the fundamental description of electromagnetism because it refers to effects that occur at a distance without referring to time in any way; Coulomb's law can only apply to static situations.



(a) A Maxwellian lattice of cubes.

Maxwell's approach to this difficulty can be described by imagining that we divide space up into a tiny cubes. Each cube is so small that we need not worry about the time taken for electromagnetic effects to propagate across it. Maxwell's accomplishment was

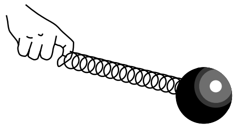
(1) to describe how the field pattern in each tiny cube relates to the cube's contents (charges, currents, or other time-varying fields); and

(2) to figure out how to assemble all the information about the tiny cubes in order to find out what is happening on larger scales.

Of course, space doesn't come equipped with tiny cubes any more than it comes equipped with coordinate axes, street signs, or public drinking fountains. Just as we found in book 1 of this series that Newtonian physics gave the same predictions regardless of what coordinate system we chose, we will find here that it really doesn't matter how we set up our cubes. Specifically, we will end up letting the size of the cubes approach zero using limit techniques reminiscent of ordinary calculus, and it also turns out that it doesn't matter at all how the cubes are oriented (see homework).

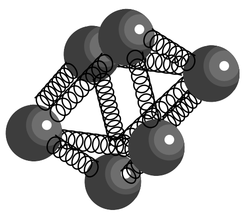
Tiny meters

What measurable electromagnetic properties does a tiny cube of space have? The ***E*** and ***B*** vectors certainly qualify, and these can be defined, respectively, in terms of the force on a test charge and the torque on a magnetic test dipole. For instance, we could take a test charge tethered to a tiny spring and build a tiny "***E***-meter" — tiny enough to fit in a Maxwellian cube — in which the stretching of the spring would serve as a measure of electric fields strength.



(b) The ***E***-meter tells us the direction of the electric field and also its strength, as measured by the length of the stretched spring.An infinitesimally small hand is required.

But notice that in our four verbal statements about what kinds of fields result from certain conditions, none simply refer directly to the fields at one point. They all refer to field patterns, such as whirlpool patterns or the diverging and converging patterns surrounding a source or a sink. We can measure the "divergingness" or "curliness" of the electric field using devices like the div-***E***-meter and curl-***E***-meter shown in figures (c) and (d). All of this has been stated in terms of the electric field, but it is equally applicable mathematically to the magnetic field, even though we would not be able to physically construct the corresponding meters for lack of magnetic charges.

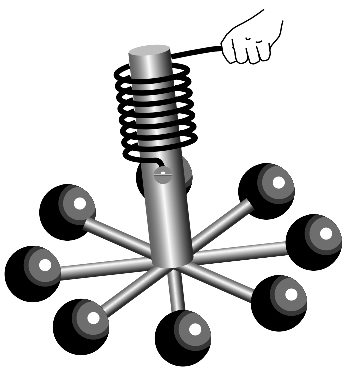


(c) The div-***E***-meter consists of a set of positive charges connected by springs. (The particular geometry of the framework turns out to be unimportant.) If the volume of the device is greater than its equilibium value, the electric field must be diverging from a source somewhere inside. Conversely, a compression of the framework indicates a convergence of the field (negative divergence, i.e. a sink).

In general, what we have constructed are two mathematical operations, call them *div* and *curl*. They measure something about a function ***F***(***x***) that uses a vector as its input and gives a vector as its output. For example, the magnetic field function, ***B***(***r***), takes a position vector as its input and gives as its output the magnetic field vector that exists at that location in space. Such a vector-to-vector function is called a vector field.

The div takes a vector field and gives back a scalar function *g*(***x***)=div ***F***(***x***), whose value is the result of putting the div-meter there. The divergence of the electric field, div ***E***, in a certain cube relates to the charges that are present in that cube.

The curl, on the other hand, needs to be defined as a vector; the curl has a definite direction, since we operate the curl-meter by playing with it until we find how to orient its axis in the direction that produces a maximum reading. We then define the curl vector to lie along the axis; of the two opposite directions that lie along the axis, we choose the one that obeys the right-hand rule (supplement 2-7). The curl takes a vector field and gives back a new vector field, ***G***(***x***)=curl ***F***(***x***).



(d) The curl-E-meter has a set of positive charges arranged around the circumference of a circle. Any torque acting on the charges causes it to twist against the resistance of the coil spring. To operate the curl-***E***-meter properly, we must try it out in every possible orientation, and find the orientation in which it gives the maximum reading.

All of this is perhaps easier to imagine using more prosaic examples than the electromagnetic ones. In a flowing fluid, the velocity of flow at a particular point is a vector, so the velocity is a vector field, ***v***(***r***). You produce a negative divergence (convergence) in this field at the bottom end of the straw when you sip a soda. If the fluid's velocity field had a nonvanishing curl, you could use it to run a paddlewheel. Temperature, on the other hand, is a scalar function *T*(***r***) — it depends on a vector, but its output is a scalar. We cannot define the div or curl of the temperature, since it is a scalar field, not a vector field.

Note that the div and curl are like the derivative operator in two ways. First, they take a function and give you a new function. Second, they produce a zero function when used on a constant function. To see the latter property, imagine what the div-meter would do in a constant field: it would neither swell nor shrink, but simply accelerate in the direction of the field. Likewise the curl-meter will not respond to a constant field: a counterclockwise torque on one side will be exactly balanced by a clockwise torque on the other.

We therefore speak of the div and curl as "derivative operations" or "derivative operators." In fact, these two operators turn out to be the *only* interesting derivative operators that act on a vector field in three dimensions, in the following sense:

(1) One could easily take any derivative operator and change its definition in a trivial way by multiplying its definition by a constant. We do not consider this to be a really new derivative operator. (And in fact we have not even bothered to state how our meters are calibrated in any absolute sense.)

(2) Minor changes in the geometry of the meters, e.g. changing the 8-charge curl-meter to a 9-charge one, don't matter in the limit of very small meters. Such changes are equivalent to a change in the calibration, e.g. the 9-charge curl meter will simply gives readings that are higher by a factor of 9/8.

(3) If one comes up with any change in the design of the meters that is great enough to do more than simply give rescaled readings, then it can be proven that the new operation will not be physically useful, because it will not be rotationally invariant. As an example, we could define an operation that takes a field $\boldsymbol{F}(\boldsymbol{r})$ and produces a scalar field $g(\boldsymbol{r})$ defined by $g=\mathrm{d}F_x/\mathrm{d}x$. Unlike the div operator, this operator is useless in physics because it violates the symmetry of space: it treats the x axis differently than any other axis.

Maxwell's equations

We need one more preliminary step before we can cast our verbal description of electromagnetism into Maxwell's mathematical form. We want the size of the tiny cubes to be irrelevant, so it would not make sense to state the div $\boldsymbol{E}$ equation in terms of the charge contained within the cube. Instead we need to use the *density of charge*, ρ, which has units of coulombs per cubic meter. Likewise instead of current we need *current density*, $\boldsymbol{J}$, a vector whose direction is the direction of the local current flow, and whose magnitude is the number of amperes per square meter flowing through a surface oriented perpendicular to the flow.

Later on we will discuss the standard calibration of the div and curl operations. Taking these for granted, we can now state Maxwell's equations:

$$\mathrm{div}\,\boldsymbol{E} = \frac{1}{\varepsilon_o}\rho$$

$$\mathrm{div}\,\boldsymbol{B} = 0$$

$$\mathrm{curl}\,\boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

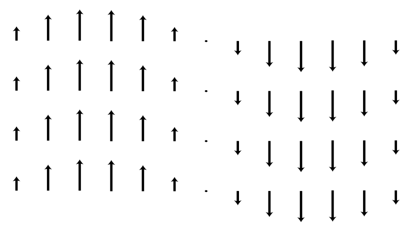
$$\mathrm{curl}\,\boldsymbol{B} = \mu_o\varepsilon_o\frac{\partial \boldsymbol{E}}{\partial t} + \mu_o\boldsymbol{J}$$

where we save writing by using the symbol $\varepsilon_o=1/(4\pi k)$ rather than expressing everything in terms of the Coulomb constant k.

Not worrying about the numerical constants in front, the correspondence between the equations and the previous verbal description should be fairly easy to make out. The negative sign in the third equation is the only major feature has not been mentioned before; this represents the observed fact that the induced $\boldsymbol{E}$ field curls left-handedly around the $\Delta\boldsymbol{B}$ vector.

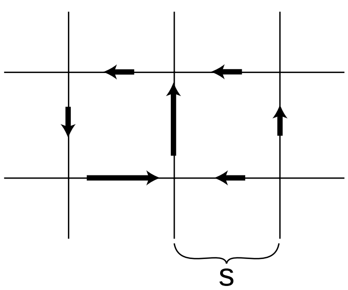
Self-Check

Describe the curl and divergence of the sine wave shown in the figure.



8.2 The Curl, Formally

The curl of a two-dimensional field



In this section we mathematize the definition of the curl, which will allow us for example to find the current density if we already know the magnetic field. We start by restricting ourselves to the case of a two-dimensional field, so that the cubes become squares. The figure shows an example. The field exists in all the infinitely many points throughout the diagram, but to be suggestive I have shown field vectors at the midpoints of the squares' sides. All the vectors happen to lie along the grid lines, although this will certainly not happen in every case. The long vectors have magnitudes of 2, the short ones 1.

It is intuitively clear that a curl-meter the same size as a square will spin counter-clockwise if inserted in the left-hand square, and with a little thought you can convince yourself that the curl is clockwise in the square on the right. We represent these curls using the right-hand rule. The field has a curl coming out of the page ($+z$ direction) in the left-hand square and going into the page in the right square. In general, we can predict how the curl-meter will behave simply by moving around the circle in a counter-clockwise direction and adding up the contributions from the field vectors, counting them as positive if they are in the counterclockwise direction and negative if they are clockwise. The torque on the curl-meter will also be proportional to the size of the squares, since torque depends on leverage. Motivated by these ideas, we define the *circulation* about a square as the kind of sum described above, with each term in the sum being multiplied by the length of the side:

circulation of the field around the left-hand square

$= +2s + s + s + 2s$

[starting on the right and going counterclockwise]

$= +6s$

circulation of the field around the right-hand square

$= +s + s - 2s - s$

The divergence is zero everywhere; there is no place you could put the div-meter and cause it to feel itself pulled outward or inward. The curl-meter would feel a clockwise torque if you placed in anywhere in the central region of the figure, so the curl is into the page (right-hand rule) in that region. In the left and right edge regions, the curl is out of the page.

$= - s$

Note that if the fields had had components perpendicular to the squares' edges (i.e. inward or outward), they would have been irrelevant, since they could not have made any torque on a curl-meter inserted in the square. In general, we define the circulation around a square in terms of a vector dot product,

$$\text{circulation} = \sum \boldsymbol{F}_i \cdot \boldsymbol{s}_i \quad ,$$

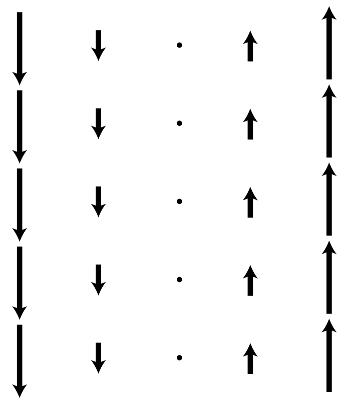
where each vector $\boldsymbol{s}_i$ has magnitude s and points along the side in the counterclockwise direction. Note that we have not been very explicit in saying at what point along each side the field's value should be taken; we can use the center points, which means throwing away information about the field at lots of other points such as the corners, but intuitively this won't matter much, since we intend to make the squares very small, so the field doesn't "have much room" to do anything funny between one field-sampling point and the next. Stated in this way, the definition can be applied to other polygons besides squares. In general, it can be applied to any closed curve (curve that has no loose ends) by approximating the curve with line segments from an infinitesimally small grid, but we will not need this idea until supplement 4-9.

The curl tells us what happens to the curl-meter reading when the meter gets extremely small. In fact, the torque on the curl-meter will be exactly *zero* in the limit of an infinitely small curl-meter, which is not very helpful. It approaches zero for two reasons. First, the torques on the curl-meter get smaller as the amount of leverage gets smaller. Second, the only way to get a nonvanishing torque is if the field is not the same on both sides of the meter, but if the field varies smoothly, then there cannot be that much of a change across an infinitesimally small distance. Together, these effects cause the torque on the curl-meter to shrink proportionately to s^2, which is the area of one square. We therefore define the curl's z-component as

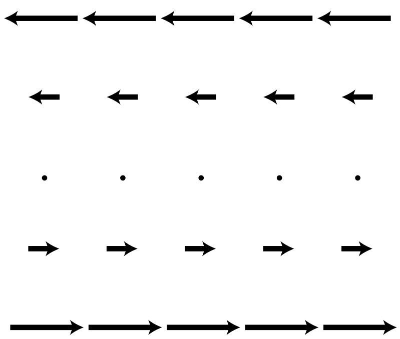
$$(\text{curl } \boldsymbol{F})_z = \lim_{s \to 0} \frac{\text{circulation of the field } \boldsymbol{F} \text{ around a square}}{\text{area of the square}} \quad .$$

The limit will typically be finite, since both the numerator and the denominator shrink like s^2. The curl of a two-dimensional field will never have components within the plane of the field (x or y components), since the field cannot exert a torque on the curl-meter if the curl-meter's axis is in the plane.

As an example, let's try to come up with the simplest possible example of a field that has a nonzero curl. A constant field won't work, because the circulation contributed by each side of the square will cancel with the opposite side. Suppose, then, that we don't want the left and right sides' contribution to the circulation to cancel out. This means that the y-component of the field must depend on x. The simplest such field is given by $x\hat{y}$.



The field $x\hat{\boldsymbol{y}}$.



The field $-y\hat{\boldsymbol{x}}$.

Example

Problem: Compute the curl of the field $\boldsymbol{F}=x\hat{\boldsymbol{y}}$ shown in the figure at the point (0,0).

Solution: The circulation around a square of side s centered on the origin can be approximated by evaluating the field at the midpoints of its sides,

$x = s/2$,	$y = 0$	$\boldsymbol{F} = (s/2)\hat{\boldsymbol{y}}$	$\boldsymbol{F}\cdot\boldsymbol{s}_1 = s^2/2$
$x = 0$,	$y = s/2$	$\boldsymbol{F} = 0$	$\boldsymbol{F}\cdot\boldsymbol{s}_2 = 0$
$x = -s/2$,	$y = 0$	$\boldsymbol{F} = -(s/2)\hat{\boldsymbol{y}}$	$\boldsymbol{F}\cdot\boldsymbol{s}_3 = s^2/2$
$x = 0$,	$y = -s/2$	$\boldsymbol{F} = 0$	$\boldsymbol{F}\cdot\boldsymbol{s}_4 = 0$,

which gives a circulation of s^2, and a curl of $\hat{\boldsymbol{z}}$. The fact that the curl vector is out of the page agrees with the right-hand rule.

Example

Another field with a counterclockwise circulation can be made by rotating the previous example by 90 degrees, giving $-y\hat{\boldsymbol{x}}$. We could compute its curl straightforwardly by the same method as in the previous example. The rotational invariance of the curl, however, guarantees that the result is the same, $\hat{\boldsymbol{z}}$; if it was different in magnitude, then the curl would apparently have some preference concerning x versus y, but the universe doesn't come equipped with coordinates, and the curl-meter must work the same way when the whole laboratory is rotated.

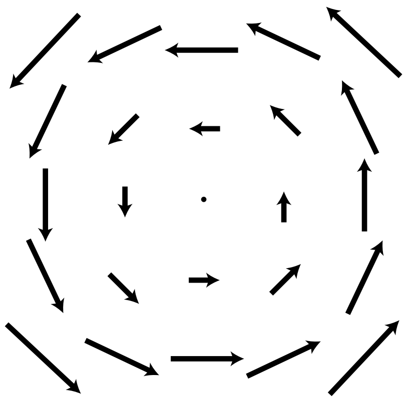
We have only calculated each field's curl at the origin, but each of these fields actually has the same curl everywhere. In the second example, for instance, it is obvious that the curl is constant along any horizontal line. But even if we move along the y axis, there is still an imbalance between the torques on the top and bottom sides of the curl-meter. More formally, suppose we start from the origin and move down by one unit. We find ourselves in a region where the field is very much as it was before, except that all the field vectors have had one unit worth of $\hat{x}$ added to them. But what do we get if we take the curl of $-y\hat{x}+\hat{x}$? The curl, like any god-fearing derivative operation, has the linear property

$$\mathrm{curl}(\boldsymbol{F}+\boldsymbol{G})=\mathrm{curl}(\boldsymbol{F})+\mathrm{curl}(\boldsymbol{G}) \quad ,$$

so

$$\mathrm{curl}(-y\hat{x}+\hat{x})=\mathrm{curl}(-y\hat{x})+\mathrm{curl}(\hat{x}) \quad .$$

But the second term is zero, since the curl is a kind of derivative, and the derivative of a constant is zero.



The field $-y\hat{\boldsymbol{x}}+x\hat{\boldsymbol{y}}$.

Example: A field that goes in a circle

Question: What is the curl of the field $-y\hat{\boldsymbol{x}}+x\hat{\boldsymbol{y}}$?

Solution: Using the linearity of the curl, and recognizing each of the terms as one whose curl we have already computed, we find that this field's curl is a constant $2\hat{\boldsymbol{z}}$.

Example: Magnetic field inside a long, straight wire

We're now ready for our first physics calculation. Suppose we think of the field in the previous example as a magnetic field. In a static situation, where no fields have time derivatives, we have simply curl $\boldsymbol{B} = \mu_o \boldsymbol{J}$. Suppose we are interested in the magnetic field *inside* a long, straight wire, and we assume that the current density is constant across the wire's cross-section.

The wire is of course a three-dimensional object, and we have only done curls of fields in a plane. However, the situation has symmetry both with respect to rotation about the wire's axis and with respect to position along the wire (since it is a long wire, we don't see any get perceptibly closer to or further from either end by moving along it). The latter symmetry means that we can simply analyze the situation in a plane that cuts perpendicularly through the wire; one such plane will be the same as any other. We let the current density be in the z direction, $\boldsymbol{J}=J\hat{\boldsymbol{z}}$.

Since the current density is constant, and curl $\boldsymbol{B}$ equals $\mu_o\boldsymbol{J}$, the field must have a constant curl. We already know of three fields that have constant curls, but only one of them has symmetry with respect to rotation about a point. The magnetic field inside the wire must therefore be $\boldsymbol{B}=\left(\mu_o J/2\right)\left(-y\hat{\boldsymbol{x}}+x\hat{\boldsymbol{y}}\right)$.

The curl in component form

It might seem as though we have only a few highly artificial examples under our belts, notwithstanding the lucky coincidence that one of them happened to have a physical application. However, we are ready to use these simple examples to develop a more powerful technique for computing curls. Consider that, just as the Earth's curvature is not apparent from a couple of meters above its surface, any graph appears like a straight line if you look at it close up near a particular point. Within that tiny neighborhood around that point, all you know and all you need to know are the value of the function and its first derivative. Now since the curl is defined in terms of the behavior of the field very close to the point of interest, the same applies here. Thus, for the purpose of finding a curl, the most general form we need to consider for any field in two dimensions is

$$\boldsymbol{F} = (a+bx+cy)\hat{\boldsymbol{x}} + (d+ex+fy)\hat{\boldsymbol{y}} \quad .$$

Any term with a more complicated form, e.g. $gx^2\hat{\boldsymbol{y}}$, would represent more complicated, nonlinear behavior that disappears in the close-up view. We have already seen that the a and d constant terms don't contribute to the curl, and it is easily verified that the b and f terms are curlless as well, so we have

$$\begin{aligned}\text{curl}\,\boldsymbol{F} &= \text{curl}\left(cy\hat{\boldsymbol{x}}\right) + \text{curl}\left(ex\hat{\boldsymbol{y}}\right)\\ &= c\,\text{curl}\left(y\hat{\boldsymbol{x}}\right) + e\,\text{curl}\left(x\hat{\boldsymbol{y}}\right)\end{aligned}$$

$$= (-c+e)\hat{\boldsymbol{z}}$$

Another way of writing this is

$$(\text{curl } \boldsymbol{F})_z = \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \quad \text{[the curl in component form]}$$

We can still use this equation even if $\boldsymbol{F}$ doesn't have this simple form $(a+bx+cy)\hat{\boldsymbol{x}} + (d+ex+fy)\hat{\boldsymbol{y}}$. Note that the x and y coordinates are treated entirely symmetrically except for the signs; reversing the signs would have meant we had chosen the left-hand rule.

Example: The field outside a wire

Let's check the equation for the magnetic field outside a wire given without proof in section 6.2,

$$B = \frac{\mu_o I}{2\pi r} ,$$

where r is the distance from the wire. We'll show that it has zero curl, which makes sense since there is no current outside the wire. One can also easily check that at the wire's surface, this expression matches the equation for the field *inside* the wire.

The hardest part is simply changing this equation for the magnitude of the field into an equation that gives its components. As raw material, we already have the expression $-y\hat{\boldsymbol{x}}+x\hat{\boldsymbol{y}}$, which has the right direction, but has magnitude r. Dividing by r^2 gives a vector with the right direction and falling off like $1/r$:

$$\boldsymbol{B} = \frac{\mu_o I}{2\pi r^2}\left(-y\hat{\boldsymbol{x}}+x\hat{\boldsymbol{y}}\right)$$

$$= \frac{\mu_o I}{2\pi}\left(-\frac{y}{x^2+y^2}\hat{\boldsymbol{x}}+\frac{x}{x^2+y^2}\hat{\boldsymbol{y}}\right)$$

The rest is straightforward, if tedious, calculus:

$$\text{curl } \boldsymbol{B} = \frac{\mu_o I}{2\pi}\text{curl}\left(-\frac{y}{x^2+y^2}\hat{\boldsymbol{x}}+\frac{x}{x^2+y^2}\hat{\boldsymbol{y}}\right)$$

$$= \frac{\mu_o I}{2\pi}\left[\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right)-\frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right)\right]$$

$$= \frac{\mu_o I}{2\pi}\left[\left(\frac{1}{x^2+y^2}-\frac{2x^2}{\left(x^2+y^2\right)^2}\right)+\left(\frac{1}{x^2+y^2}-\frac{2y^2}{\left(x^2+y^2\right)^2}\right)\right]$$

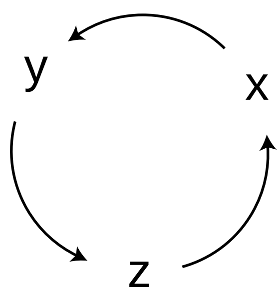
$$= \frac{\mu_o I}{2\pi}\left[\left(\frac{2}{x^2+y^2}-\frac{2x^2+2y^2}{\left(x^2+y^2\right)^2}\right)\right]$$

$$= 0$$

The curl in three dimensions

The original verbal definition of the curl in terms of the curl-meter specified that the direction of the curl was the direction of the axis of the meter that produced the maximum reading. But rather than going back to this definition in order to generalize the curl in three dimensions, it's easier to start with the component form,

$$(\text{curl } \boldsymbol{F})_z = \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \qquad \text{[curl of a field in the } x\text{-}y \text{ plane]}$$



A cyclic renaming of the axes.

and use the rotational invariance and additivity properties of the curl. The rotational invariance of the curl guarantees that we can rename the *x*, *y*, and *z* axes, and as long as the resulting coordinate system is right-handed, everything still works. We can ensure right-handedness by using a cyclic renaming of the axes,

$$(\text{curl } \boldsymbol{F})_x = \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \qquad \text{[curl of a field in the } y\text{-}z \text{ plane]}$$

$$(\text{curl } \boldsymbol{F})_y = \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \qquad \text{[curl of a field in the } z\text{-}x \text{ plane]} \quad .$$

Close up, a field in two dimensions always looked like $\boldsymbol{F} = (a+bx+cy)\hat{\boldsymbol{x}} + (d+ex+fy)\hat{\boldsymbol{y}}$. In three dimensions, the most general form is $\boldsymbol{F} = (a+bx+cy+dz)\hat{\boldsymbol{x}} + (e+fx+gy+hz)\hat{\boldsymbol{y}} + (i+jx+ky+lz)\hat{\boldsymbol{z}}$. Since the curl is additive, this field can be broken up into three pieces, each of which is confined to a plane and can be calculated using the three equations above. Adding the three results together, we we have an equation for the curl that is not confined to any plane,

$$\text{curl } \boldsymbol{F} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\hat{\boldsymbol{x}} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\hat{\boldsymbol{y}} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\hat{\boldsymbol{z}} \quad .$$

[curl of any field ***F*** in three dimensions]

Example: Electromagnetic waves

The story about Maxwell and his wife makes it clear he understood that the existence of electromagnetic waves was by far the most important application of his equations. We now show that a sinusoidal wave with the geometry described in section 6.4 obeys the two equations that have curls in them.

A sinusoidal electromagnetic wave is supposed to be an empty-space solution to the equations, since it can travel through a vacuum. We therefore set the current density equal to zero, and Maxwell's last two equations become

$$\text{curl } \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

$$\text{curl } \boldsymbol{B} = \mu_o \varepsilon_o \frac{\partial \boldsymbol{E}}{\partial t}$$

Note how symmetric these equations are. Also, the quantity $\mu_o\varepsilon_o$ is equal to $1/c^2$. (Presumably people had noticed this numerical coincidence even before Maxwell's work.)

Since the curl is rotationally invariant, we are free to set up a wave with any polarization and direction of motion. If that wave is a solution to Maxwell's equations, then waves moving in other directions must also give be solutions. Likewise the additivity of the curl permits us to choose any overall amplitude for the wave, although the amplitudes of the ***E*** and ***B*** parts will necessarily be related to each other. We choose

$$E = E_o \sin(kx\text{-}\omega t)\ \hat{\boldsymbol{y}}$$

$$B = B_o \sin(kx\text{-}\omega t)\ \hat{\boldsymbol{z}} \quad ,$$

where k and ω are merely the standard notations for the two constants. You should be able to show $k=2\pi/\lambda$ and $\omega=2\pi f$, where radians are assumed for the stuff inside the trig functions.

The two time derivatives terms are

$$-\frac{\partial \boldsymbol{B}}{\partial t} = \omega B_o \cos(kx\text{-}\omega t)\ \hat{\boldsymbol{z}} \quad \text{and}$$

$$\mu_o\varepsilon_o \frac{\partial \boldsymbol{E}}{\partial t} = -\omega\mu_o\varepsilon_o E_o \cos(kx\text{-}\omega t)\ \hat{\boldsymbol{y}} \quad .$$

Each curl has only one nonzero term out of the possible six:

$$\text{curl } \boldsymbol{E} = \frac{\partial E_y}{\partial x}\hat{\boldsymbol{z}}$$

$$= kE_o \cos(kx\text{-}\omega t)\ \hat{\boldsymbol{z}}$$

$$\text{curl } \boldsymbol{B} = -\frac{\partial B_z}{\partial x}\ \hat{\boldsymbol{y}}$$

$$= -kB_o \cos(kx\text{-}\omega t)\hat{\boldsymbol{y}}$$

Inserting all four things into Maxwell's equations, we have

$$kE_o \cos(kx\text{-}\omega t)\ \hat{\boldsymbol{z}} = \omega B_o \cos(kx\text{-}\omega t)\ \hat{\boldsymbol{z}}$$

$$-kB_o \cos(kx\text{-}\omega t)\hat{\boldsymbol{y}} = -\omega\mu_o\varepsilon_o E_o \cos(kx\text{-}\omega t)\ \hat{\boldsymbol{y}} \quad .$$

Things are looking good. All the functional forms agree, all the plus and minus signs work, and the directions work out. We just need to set

$$kE_o = \omega B_o \quad \text{and}$$

$$kB_o = \omega\mu_o\varepsilon_o E_o \quad .$$

Using $\mu_o\varepsilon_o=1/c^2$ and $\omega=2\pi f=2\pi c/\lambda=ck$, these can be reduced to

$$E_o = cB_o \quad \text{and}$$

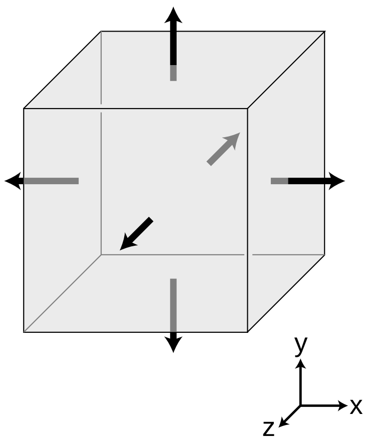
$$B_o = (1/c)E_o \quad .$$

which are two different forms of the relation asserted in section 6.4.

Discussion Question

In the calculation of the curls of the fields in an electromagnetic wave, what about the assumption that the waves had a certain phase?

8.3 The Divergence, Formally



A diverging field. A coordinate system is shown for reference.

The curl-meter was a flat wheel, but the div-meter is three-dimensional, so there is no useful two-dimensional version of the divergence as there was for the curl. We approach the task of formalizing the divergence by dividing space into little cubes. The figure shows a field diverging from all sides of a little cube. If this was an electric field, we'd know that there must be some positive charge inside to make this field pattern. In the case of the curl, it was only the components of the field parallel to the edges that were of interest, but here the thing that is of physical importance is clearly the field's component *perpendicular* to the walls of the cube. There is a quantity called the flux, which is to the divergence as the circulation is to the curl. With the above physical motivation and working by analogy with the circulation, we define each wall's contributiuon to the flux as its area multiplied by the outward component of the field. (A field with an inward component subtracts from the flux.) This can be neatly encapsulated in an equation by defining an area vector, $\boldsymbol{A}$, for each side of the cube, which is perpendicular to the wall and points outward. We then have

$$\text{flux} = \sum \boldsymbol{F} \cdot \boldsymbol{A} \quad . \qquad \text{[definition of the flux of the field } \boldsymbol{F}\text{]}$$

The flux is a scalar, not a vector like the circulation. Again working by analogy, we define the divergence of the field as

$$\text{div}\, \boldsymbol{F} = \lim_{A \to 0} \frac{\text{flux of the field } \boldsymbol{F} \text{ out through the cube}}{\text{volume of the cube}} \quad .$$

The divergence is also a scalar.

This may all seem plausible, but how do we really know this is the right definition? The physicist's answer would be that it matches up correctly with experiments when we use it in Maxwell's equations. The mathematician's answer is that there are only two useful (that is, additive and rotationally invariant) derivative operators that work on a vector field, the curl and the div. One can prove (see homework) that this definition, like the definition of the curl, is rotationally invariant, so it must be the right definition.

A component form is easily derived. Consider the flux on the left and right sides of the cube, which depend only on the x components of the field vectors at the centers of those two walls. If the two fields were the same, say both pointing to the right, then the inward flux through the left side of the cube would exactly cancel the outward flux thrugh the right side. The divergence thus depends on the difference between the x components on the left and right sides of the cube, which is related to the derivative $\partial F_x / \partial x$. The flux through the top and bottom would depend on the corresponding derivative for y, and likewise for the front and back, which relate to z. It is then easily proven that the divergence equals

$$\text{div}\, \boldsymbol{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \quad .$$

Example: Magnetic field inside a wire

Problem: Verify that the expression for the magnetic field inside a wire, $\boldsymbol{B}=(\mu_o J/2)(-y\hat{\boldsymbol{x}}+x\hat{\boldsymbol{y}})$, obeys Maxwell's equation for the divergence of $\boldsymbol{B}$, div $\boldsymbol{B}$=0.

Solution: All three terms in the component form of the divergence vanish individually, so the divergence is zero:

$$\partial B_x / \partial x = -(\mu_o J/2)\partial y / \partial x = 0$$

$$\partial B_y / \partial y = (\mu_o J/2)\partial x / \partial y = 0$$

$$\partial B_z / \partial z = \partial(0) / \partial z = 0$$

This is also plausible visually, since the field pattern goes in a circle, but never converges or diverges at any point.

Example: Electric field of a point charge

The case of a point charge is tricky, because the field behaves badly right on top of the charge, blowing up and becoming discontinuous. At this point, we cannot use the component form of the divergence, since none of the derivatives are well defined. However, a little visualization using the original definition of the divergence will quickly convince us that div $\boldsymbol{E}$ is infinite here, and that makes sense, because the density of charge has to be infinite at a point where there is a zero-size point of charge (finite charge in zero volume). Supplement 4-9 gives methods for taming troublesome infinities like this.

At all other points, we have

$$\boldsymbol{E} = \frac{q}{4\pi\varepsilon_o r^2}\hat{\boldsymbol{r}} \quad ,$$

where we have used $\varepsilon_o=1/(4\pi k)$ instead of expressing the field in terms of the Coulomb constant k, and $\hat{\boldsymbol{r}} = \boldsymbol{r}/r$ is the unit vector pointing radially away from the charge (not the distance from an axis in cylindrical coordinates, as in a previous example). The field can therefore be written as

$$\boldsymbol{E} = \frac{q}{4\pi\varepsilon_o r^3}\boldsymbol{r}$$

$$= \frac{q(x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}})}{4\pi\varepsilon_o(x^2+y^2+z^2)^{3/2}} \quad .$$

The three terms in the divergence are all similar, e.g.

$$\frac{\partial E_x}{\partial x} = \left(\frac{q}{4\pi\varepsilon_o}\right)\frac{\partial}{\partial x}\left[\frac{x}{(x^2+y^2+z^2)^{3/2}}\right]$$

$$= \left(\frac{q}{4\pi\varepsilon_o}\right)\left[\frac{1}{(x^2+y^2+z^2)^{3/2}} - \frac{3}{2}\frac{2x^2}{(x^2+y^2+z^2)^{5/2}}\right]$$

$$= \left(\frac{q}{4\pi\varepsilon_o}\right)\left(r^{-3} - 3x^2 r^{-5}\right) \quad .$$

Straightforward algebra shows that adding in the other two terms results in zero, which makes sense, since there is no charge except at the origin.

8.4 Methods of Attack

Knowing how to compute divs and curls lets us find the sources if we are given the fields. This is usually the opposite of what we want to do, so it may seem that so far we've just developed techniques for checking the answer if we already know it! There are however some straightforward ways of using Maxwell's equations to find the field when we know the sources. We will develop some here that do not require any new mathematical techniques, and supplement 4-9 introduces some approaches that do require new math concepts.

One useful approach is to guess the general form of the field based on experience or physical intuition, and then try to use Maxwell's equations to find what specific version of that general form will be a solution.

Example: The field inside a uniform sphere of charge

Problem: Find the field inside a uniform sphere of charge whose charge density is ρ. (This is very much like finding the gravitational field at some depth below the surface of the earth.)

Solution: By symmetry we know that the field must be purely radial (in and out). We guess that the solution might be of the form

$$\mathbf{E} = b\, r^p \hat{\mathbf{r}} \quad ,$$

where r is the distance from the center, and b and p are constants. A negative value of p would indicate a field that was strongest at the center, while a positive p would give zero field at the center and stronger fields farther out.

As in the problem of the field surrounding a point charge, we rewrite $\hat{\mathbf{r}}$ as $\mathbf{r}/r$, and to simplify the writing we define $n=p-1$, so

$$\mathbf{E} = b\, r^n \mathbf{r} \quad .$$

Maxwell's equation for the divergence of $\mathbf{E}$ is

$$\text{div}\, \mathbf{E} = \frac{1}{\varepsilon_o}\rho \quad ,$$

so we want a field whose divergence is constant. For a field of the form we guessed, the divergence has terms in it like

$$\frac{\partial E_x}{\partial x} = \frac{\partial}{\partial x}\left(br^n x\right)$$

$$= b\left(nr^{n-1}\frac{\partial r}{\partial x}x + r^n\right)$$

The partial derivative $\partial r / \partial x$ is easily calculated to be x/r, so

$$\frac{\partial E_x}{\partial x} = b\left(nr^{n-2}x^2 + r^n\right)$$

Adding in similar expressions for the other two terms in the divergence, and making use of $x^2+y^2+z^2=r^2$, we have

$$\text{div}\, \mathbf{E} = b(n+3)r^n \quad .$$

This can indeed be constant, but only if $n=0$, i.e. $p=1$, so the field is directly proportional to r. Equating the coefficient in front to the one in Maxwell's equation, the field is

$$\mathbf{E} = \frac{\rho}{3\varepsilon_o} r\, \hat{\mathbf{r}} \quad .$$

A second technique that is commonly used is to stitch together solutions to Maxwell's equations covering different regions of space.

Example: The field outside a uniform sphere of charge
Problem: Find the field outside a uniformly charged sphere whose radius is a.
Solution: In the region outside the sphere, we need a field that has zero divergence, and that has the correct value at the surface. From an example in section 8.3, we know that the field with zero divergence and spherical symmetry is the one that is proportional to $1/r^2$, so all we need to do is find the proportionality constant. Starting from the result of the previous example, we find that the field at the surface is

$$\begin{aligned} \boldsymbol{E} &= \frac{\rho}{3\varepsilon_o} a\,\hat{\boldsymbol{r}} \\ &= \frac{q}{3\varepsilon_o \frac{4}{3}\pi a^3} a\,\hat{\boldsymbol{r}} \\ &= \frac{q}{4\pi\varepsilon_o a^2}\hat{\boldsymbol{r}} \\ &= \frac{kq}{a^2}\hat{\boldsymbol{r}} \end{aligned}$$

To splice this onto a $1/r^2$ solution, joining the two solutions at $r=a$, we need

$$\mathrm{E} = \frac{kq}{r^2}\hat{\boldsymbol{r}} \quad .$$

This is the same as the field of a point charge, which is what we would have expected from the shell theorem.

Finally, superposition is an important technique. Suppose we have solved one instance of Maxwell's equations: we know the electric and magnetic fields that constitute a solution to Maxwell's equations for a given charge density and current density distribution. Now suppose are able to solve Maxwell's equations in a second situation. Since the derivative operators in Maxwell's equations are additive, we can now form a third solution to Maxwell's equations by adding the $\boldsymbol{E}$ fields to find a new $\boldsymbol{E}$ field, and likewise for $\boldsymbol{B}$, ρ, and $\boldsymbol{J}$. In particular, we know that if there are no charges or currents, then the solutions to Maxwell's equation are electromagnetic waves. Thus any fields that are a solution to Maxwell's equations for a particular ρ and $\boldsymbol{J}$ can have one or more plane waves added onto them, and they will still be solutions.

Example: Waves bouncing off of charges?
Question: A person claims he can block radio waves by holding a positively charged piece of rubber between the receving and transmitting antennas. He says the radio waves hit the charge and bounce off. Should you believe him?
Solution: No. When there is no radio wave, the piece of rubber will make a diverging elecric field, which is a solution of Maxwell's equations. A second solution can be made by adding this field pattern to the electric field pattern of the radio wave. The wave travels right through the rubber as if it didn't exist.

An interesting thing about this example is that it makes us wonder why electromagnetic waves interact with matter at all! The answer is that if a charge is not fixed firmly in place, it will respond to an en elecromagnetic wave by vibrating. This happens, for example, with the free electrons in a shiny piece of metal when it is hit by some visible light. These electrons will then radiate electromagnetic waves at the same frequency, and it is these reradiated waves that are really the reflected wave.

Homework Problems

1. Show that the divergence of the magnetic field in the region outside a long, straight wire is zero.

2. Suppose that a long cylinder contains a uniform charge density ρ. (a) Use one of the methods in section 8.4 to find the electric field inside. (b) Extend your solution to the outside region.

3. Find an example of a field whose divergence and curl are both constant everywhere (and are not zero).

4. The purpose of this homework problem is to prove that the div is invariant with respect to translations. That is, it doesn't matter where you choose to put the origin of your coordinate system. Suppose we have a field of the form $\boldsymbol{F} = ax\hat{\boldsymbol{x}} + by\hat{\boldsymbol{y}} + cz\hat{\boldsymbol{z}}$. As discussed in the main text, this is the most general field we need to consider in any small region as far as the div is concerned. Define a new set of coordinates (u,v,w) related to (x,y,z) by

$$x = u + p$$

$$y = v + q$$

$$z = w + q \quad ,$$

where p, q, and r are constants. Show that the field's divergence is the same in these new coordinates. Note that $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{u}}$ are identical, and similarly for the other coordinates.

5. This problem is similar to the preceding one, but here you'll prove that the curl is translationally invariant (in two dimensions). This time the most general form we need to consider for the field is $\boldsymbol{F} = ay\hat{\boldsymbol{x}} + bx\hat{\boldsymbol{y}}$.

6. Using techniques similar to those in the two preceding problems, prove that the curl is rotationally invariant (in two dimensions). Suppose we rotate to a new (u,v) coordinate system, whose axes are rotated by an angle θ with respect to those of the (x,y) system. The coordinates are related by

$$x = u \cos\theta + v \sin\theta$$

$$y = -u \sin\theta + v \cos\theta$$

Find how the $\hat{\boldsymbol{u}}$ and $\hat{\boldsymbol{v}}$ components the field $\boldsymbol{F}$ depend on u and v, and show that its curl is the same in this new coordinate system.

7. Using a techniques similar to that of the three preceding problems, show that the div is rotationally invariant (in three dimensions).

S A solution is given in the back of the book.
✓ A computerized answer check is available.
★ A difficult problem.
∫ A problem that requires calculus.