7

NONLINEAR EVOLUTION EQUATIONS AND SOLITONS

7.1 HISTORICAL BACKGROUND

A principal theme of the preceding chapters has been that nonlinear systems of just a few degrees of freedom can display very complex behavior. A natural question to ask is, what happens to all this dynamics in the limit of the number of degrees of freedom becoming infinite? In this limit, the degrees of freedom, or modes, are treated as a continuum with a continuous label x rather than a discrete index $i = 1, \ldots, N$. Thus the description of a system in terms of a finite number of ordinary differential equations (o.d.e.'s), with time as the only independent variable, goes over to a partial differential equation (p.d.e.) with both x and t as the independent variables. If only a few nonlinear o.d.e.'s can display complex behavior, it might be thought that a continuum of them (i.e., a nonlinear p.d.e.) could only display more complicated behavior. In many cases this is indeed so, and nonlinear p.d.e.'s will display chaos in both time and space. However, there is also an important class of nonlinear p.d.e.'s whose behavior is remarkably regular, that is, they are, in effect, integrable. The properties of these systems and the behavior of their solutions has given rise to what is generally considered to be one of the most significant advances in post-war mathematical physics.

7.1.a Russell's Observations

The story begins over 150 years ago with the now famous observations made by the Scottish engineer John Scott Russell while riding (on horseback) near the Union Canal outside Edinburgh. His report (Russell, 1844) reads as follows:

> I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped—not so the mass of the water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished and after a chase of two miles I lost it in the windings of the channel. Such in the month of August 1834 was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation . . . .

Russell carried out many experiments on solitary waves and concluded that the persistence of their form was genuine and that the speed of propagation in a channel of uniform depth was

$$c = \sqrt{g(h+\eta)} \tag{7.1.1}$$

where η is the amplitude of the wave, h is the depth of the (undisturbed) channel, and g is the gravitational constant.

Russell's results were controversial since it was not believed at that time that such a wave could be stable. The Astronomer Royal, Sir John Herschel, dismissed it as "merely half of a common wave that has been cut off." There was also a dispute with Airy, who had developed a shallow-water wave theory in which such waves were not stable. The controversy was resolved in 1895 by Korteweg and de Vries (1895), who derived an equation governing weakly nonlinear shallow-water waves of the form

$$\frac{\partial \eta}{\partial t} = \frac{3}{2}\sqrt{\frac{g}{h}}\left(\eta\frac{\partial \eta}{\partial x} + \frac{2}{3}\frac{\partial \eta}{\partial x} + \frac{1}{3}\sigma\frac{\partial^3 \eta}{\partial x^3}\right) \tag{7.1.2}$$

in which $\sigma = h^3/3 - Th/g\rho$, where T is the surface tension of the liquid of density ρ. This equation was found to have solitary wave solutions of permanent form. After Korteweg and de Vries's work, the problem disappeared and it was not until the early 1960s that Eq. (7.1.2) reappeared in certain plasma physics problems.

At this stage we note that introduction of the scaled variables

$$t' = \frac{1}{2}\sqrt{\frac{g}{h\sigma}}\,t, \qquad x' = \frac{-x}{\sqrt{\sigma}}, \qquad u = -\frac{1}{2}\eta - \frac{1}{3}\alpha$$

reduces (7.1.2) to the form

$$u_t - 6uu_x + u_{xxx} = 0 \tag{7.1.3}$$

7.1.b The FUP Experiment

A motivation for studying the Korteweg–de Vries equation (hereafter referred to as the KdV equation) was provided by the work of Fermi, Ulam, and Pasta (FUP) in 1955 (Fermi et al., 1955). Recall (from Chapter 3) that here the physical question was one of energy distribution in a chain of nonlinear oscillators. The speculation was that as the number of oscillators tended to infinity (the "statistical limit"), the energy would distribute itself uniformly among all the modes, implying ergodicity on the energy shell. Their model consisted of a one-dimensional nonlinear chain, of equal masses, with nearest neighbors connected by a force law of the form $F(\Delta) = k(\Delta + \alpha\Delta^2)$. This gave the following system of coupled nonlinear o.d.e.'s:

$$m\ddot{y}_i = k(y_{i+1} + y_{i-1} - 2y_i) + k\alpha[(y_{i+1} - y_i)^2 - (y_i - y_{i-1})^2] \tag{7.1.4}$$

where $y_i = y_i(t)$ $(i = 1, 2, \ldots, N-1)$ and $y_0 = y_n = 0$. Initial conditions were typically chosen to be $y_i(0) = \sin(i\pi/N)$, $\dot{y}_i(0) = 0$. Working with $N = 64$ the system of equations (7.1.4) was integrated numerically on the Los Alamos MANIAC computer. (It is worth noting that this was one of the first peacetime uses of the computer for scientific research.†) Their results showed that the bulk of the energy tended to cycle periodically through the initially populated modes and that there was little energy sharing—an unexpected result at the time.

7.1.c Discovery of the Soliton

The scene now changes to Princeton, 1965, for the work of Kruskal and Zabusky. They were interested in the continuum limit of the Fermi–Ulam–Pasta chain, which they derived in the following way (Zabusky and Kruskal (1965)). Setting the distance between the springs to be h and introducing

†In 1977, at the First International Conference on Stochastic Behavior in Classical and Quantum Systems held in Como, Italy, Dr. Pasta reminisced about those computations. The program was, of course, punched on cards. A "DO" loop was executed by the operator feeding in the deck of cards over and over again until the loop was completed!

the variables $t' = \omega t$, $\omega = \sqrt{k/m}$, and $x' = x/h$ (where $x = ih$), they showed that by expanding the $y_{i\pm 1}$ in Taylor series to fourth order in h, (7.1.4) becomes (dropping primes)

$$y_{tt} = y_{xx} + \epsilon y_x y_{xx} + \frac{h^2}{12} y_{xxxx} + O(\epsilon h^2, h^4) \tag{7.1.5}$$

where $\epsilon = 2\alpha h$. The next stage is to look for an asymptotic solution of the form

$$y \sim \phi(x, T)$$

where $T = \epsilon t/2$ and $X = x - t$, that is, a right moving wave. Noting that $y_t = -\phi_X + \frac{1}{2}\epsilon\phi_T$, one obtains

$$\phi_{TX} + \phi_X\phi_{XX} + \delta^2\phi_{XXXX} = 0 \tag{7.1.6}$$

where $\delta = h^2/12$. Finally, setting $u = \phi_X$ yields

$$u_T + uu_X + \delta^2 u_{XXX} = 0 \tag{7.1.7}$$

which is, within trivial scalings, just the reduced form of the KdV equation (7.1.3)!

Zabusky and Kruskal (1965) studied the KdV equation (7.1.7) numerically, imposing the periodic boundary conditions $u(L, t) = u(0, t)$, $u_x(L, t) = u_x(0, t)$, and $u_{xx}(L, t) = u_{xx}(0, t)$. (We immediately comment that this choice of periodic boundary conditions was for numerical convenience and does not affect the fundamental result.) Working with an initial condition of the form $u(x, 0) = \cos(2\pi x/L)$, $0 \le x \le L$, they found that the solution broke up into a train of (eight) solitary waves of successively larger amplitude. The larger waves traveled faster than the smaller ones, and, remarkably, the former traveled "through" the latter and emerged from the "collisions" apparently unscathed! This behavior is akin to the superposition principle for linear waves, although in this case the waves are highly nonlinear. The term *soliton* was introduced by Zabusky and Kruskal (1965) to describe these remarkably stable nonlinear solutions. Their numerical results were followed by the development of a remarkable new solution technique by Kruskal and co-workers (Miura et al., 1968) which led to the development of a whole new area of mathematical physics—which might loosely be termed *soliton mathematics*. To begin with, though, we first investigate some of the more elementary properties of the KdV equation.

7.2 BASIC PROPERTIES OF THE KdV EQUATION

There are two basic forces at work in the KdV equation—which from now on we consider in the form (7.1.3). These are (1) the nonlinearity, that is, the term uu_x, which tends to "sharpen" the wave up, and (2) the dispersion, due to the term u_{xxx}, which tends to "spread" the wave out.

7.2.a Effects of Nonlinearity and Dispersion

For a smooth initial condition, such as the one considered by Zabusky and Kruskal (1965), the term u_{xxx} is relatively small compared to the nonlinear term and one can consider the initial evolution to be governed by

$$u_t - 6uu_x = 0 \tag{7.2.1}$$

This is a standard, quasi-linear, first-order p.d.e which is capable of developing shock solutions. Briefly, this can be seen as follows.† Consider the solution to (7.2.1) to be of the form

$$u(s) = u(x(s), t(s)) \tag{7.2.2}$$

where s parameterizes certain paths, termed *characteristics*, in the (x, t)-plane. Then, from the differential equation

$$\frac{du}{ds} = \frac{dx}{ds} u_x + \frac{dt}{ds} u_t \tag{7.2.3}$$

we deduce

$$\frac{dt}{ds} = 1 \tag{7.2.4a}$$

$$\frac{dx}{ds} = -6u \tag{7.2.4b}$$

$$\frac{du}{ds} = 0 \tag{7.2.4c}$$

The set of equations (7.2.4) are easily integrated (ignoring constants of

†A fuller discussion can be found in any standard text on partial differential equations, such as Carrier and Pearson (1976).

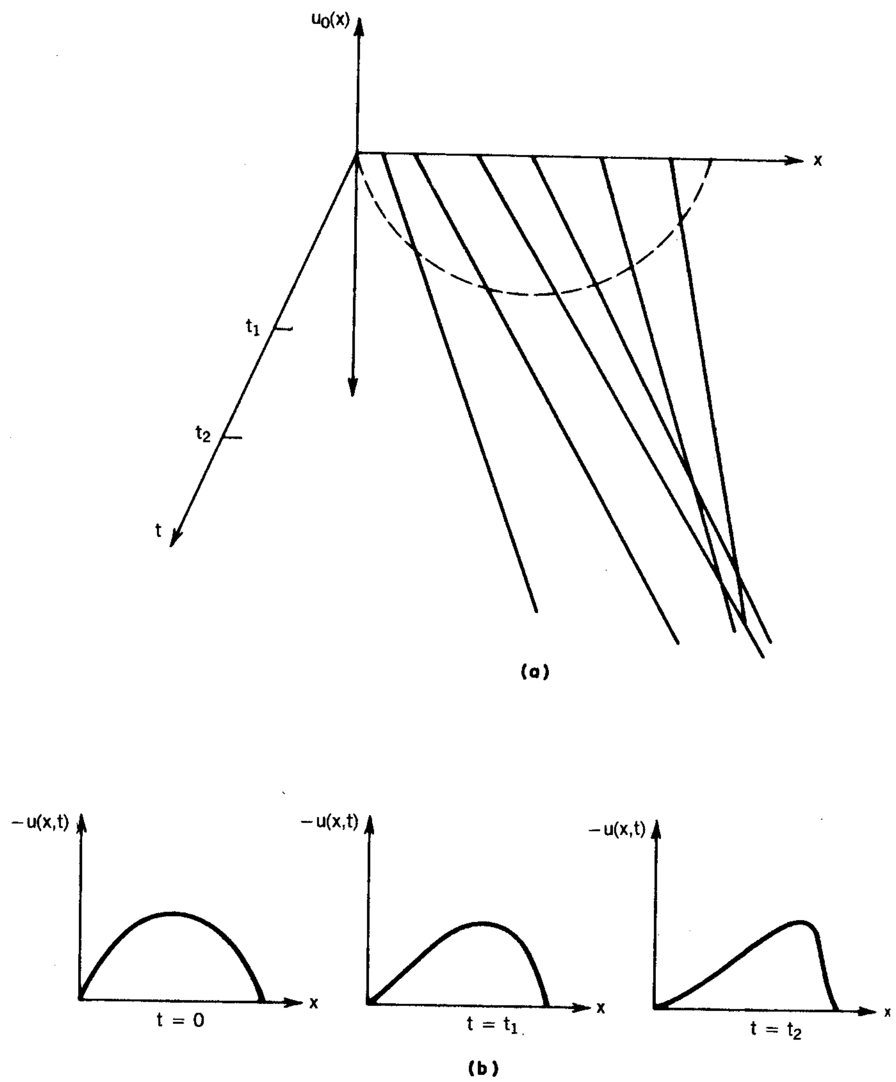


Figure 7.1 (a) Straight-line characteristics associated with Eq. (7.2.1) for smooth initial condition $u_0(x)$ (dotted line). (b) Sequence showing sharpening of $u(x, t)$ as solution evolves toward crossing characteristics.

integration) to give

$$t(s) = s \tag{7.2.5a}$$

$$x(s) = -6su_0(x) \tag{7.2.5b}$$

$$u(s) = u_0(x) \tag{7.2.5c}$$

where $u_0(x) = u(x, 0)$ is the initial condition for (7.2.1). Thus along the characteristics defined by (7.2.5a) and (7.2.5b), the solution $u(s)$ is constant, that is, maintaining the initial amplitude given at $s = 0$. However, the characteristics—which are straight line paths—have a slope proportional to $u_0(x)$, and, depending on the precise shape of this initial data, the characteristics can cross. It is not difficult to see (see Figure 7.1) that this will lead to a steepening of the wave and hence to a wave-breaking-like phenomenon known as *shock formation*.

As the wave steepens, the term u_{xxx} in (7.1.3) will become significant and we must now consider how the evolution of the wave will be affected by the linear part of the equation

$$u_t + u_{xxx} = 0 \tag{7.2.6}$$

Such an equation always admits a solution of the form

$$u(x, t) = e^{i(kx-\omega t)} \tag{7.2.7}$$

and, by direct substitution into (7.2.6), one obtains the "dispersion relation"

$$\omega(k) = -k^3 \tag{7.2.8}$$

Thus longer wavenumbers travel with faster phase velocities, given by $c = \omega(k)/k = -k^2$, and the wave (7.2.7) will spread out as it evolves. Thus, in some sense, this dispersive effect will balance the nonlinear effects and lead to the formation of the stable solitary waves.

7.2.b A Traveling Wave Solution

A simple form of solitary wave solution can be obtained as follows. One assumes a right traveling wave solution of the form

$$u(x, t) = f(x - ct) \equiv f(z) \tag{7.2.9}$$

where $z = x - ct$, and, by direct substitution into (7.1.3), one obtains the ordinary differential equation (prime denotes differentiation with respect to z)

$$f''' - 6ff'' - cf' = 0 \tag{7.2.10}$$

A first integration with respect to z yields

$$f'' = 3f^2 + cf + d \tag{7.2.11}$$

where d is a constant of integration, which is the o.d.e. for Weierstrass elliptic functions described in Chapter 1. A second integration yields

$$(\tfrac{1}{2}f') = f^3 + \tfrac{1}{2}cf^2 + df + e \tag{7.2.12}$$

where e is a second constant of integration. Equation (7.2.12) can now be integrated by quadratures to yield the elliptic integral

$$z - z_0 = \int \frac{df}{\sqrt{2(f^3 + \frac{1}{2}cf^2 + df + e)}} \tag{7.2.13}$$

If (7.1.3) is defined on the infinite domain and one takes the boundary conditions $f, f', f'' \to 0$ as $z \to \pm\infty$, it is easy to deduce from (7.2.11) and (7.2.12) that both the constants of integration d and e are zero. In this case the quadrature (7.2.13) reduces to

$$z - z_0 = \int \frac{df}{f\sqrt{2f + c}} \tag{7.2.14}$$

which is easily integrated and inverted to yield

$$f(z) = -\tfrac{1}{2}c\,\mathrm{sech}^2(\tfrac{1}{2}\sqrt{c}(z - z_0)) \tag{7.2.15}$$

Owing to the minus sign, the solution is of the form of a negative amplitude traveling wave (the sign would be positive if (7.1.3) were $u_t + 6uu_x + u_{xxx} = 0$), with the amplitude proportional to wave speed; that is, larger waves travel faster. The sech^2 form of (7.2.15) gives the wave its localized "lump"-like structure as first seen by Russell (1844). In the numerical experiments of Zabusky and Kruskal (1965), each member of the family of traveling waves was also observed to have a sech^2-like shape. However, the reason for the appearance of such a family and their stability cannot be explained by a simple traveling wave analysis. A much deeper theory is required.

7.2.c Similarity Solutions

Another type of solution to (7.1.3) can also be found by what is termed a *similarity transformation*. If the variables x, t, u are scaled according to $x \to k^\alpha x$, $t \to k^\beta t$, $u \to k^\gamma u$, direct substitution into the KdV equation shows that it is invariant to these scalings if $\beta = 3\alpha$ and $\gamma = -2\alpha$. Any choice of α

may be made, for example, $\alpha = 1$. Thus (7.1.3) is invariant to the scalings

$$x \to kx, \qquad t \to k^3 t, \qquad u \to k^{-2} u$$

Furthermore, the combinations of variables

$$x/t^{1/3}, \qquad ut^{2/3}$$

are also scale invariant. These results suggest a change of variable, namely,

$$z = x/(3t)^{1/3} \tag{7.2.16a}$$

$$u(x, t) = -(3t)^{-2/3} f(z) \tag{7.2.16b}$$

where the factor 3 in the scalings has been chosen for convenience. Noting that

$$\frac{\partial}{\partial t} = \frac{\partial z}{\partial t}\frac{\partial}{\partial z} = -\frac{z}{3t}\frac{\partial}{\partial z}$$

and

$$\frac{\partial}{\partial x} = \frac{\partial z}{\partial x}\frac{\partial}{\partial z} = \frac{1}{(3t)^{1/3}}\frac{\partial}{\partial z}$$

the KdV equation transforms to

$$f''' + (6f - z)f' - 2f = 0 \tag{7.2.17}$$

Another type of similarity reduction is obtained by setting

$$z = x + 3t^2 \tag{7.2.18a}$$

and

$$u(x, t) = t + f(z) \tag{7.2.18b}$$

This leads (after one integration) to

$$f'' = 3f^2 - t + c \tag{7.2.19}$$

where c is a constant of integration. This equation is a special ordinary differential equation known as the *first Painlevé transcendent*. Equation (7.2.17) is related to another of these special equations, namely, the *second Painlevé transcendent*. The significance of the appearance of these special ordinary differential equations on making similarity reductions of the KdV equation will be discussed in Chapter 8.

7.2.d Conservation Laws

If we think of a p.d.e., such as the KdV equation, to be a dynamical system with an infinite number of degrees of freedom, it is natural to ask, in keeping with our preceding discussions, if the equations have any integrals of motion. For p.d.e.'s the notion of integrals of motion is replaced by the notion of *conservation laws*. These are relations of the form

$$T_t + X_x = 0 \tag{7.2.20}$$

where T and X are certain functions of the solution to the p.d.e., u, and its derivatives. T is termed the ***density***, and $-X$ is called the ***flux***. If T and X are connected by a gradient relationship (i.e., $T = F_x$) and hence from (7.2.20), $X = -F_t$, the conservation law is trivial since

$$(F_x)_t + (-F_x)_t = 0 \tag{7.2.21}$$

If, for systems defined on the infinite interval $(-\infty \le x \le \infty)$, the flux X decays to zero as $x \to \infty$, integrating both sides of (7.2.20) with respect to x yields

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty} T\,dx = \int_{-\infty}^{\infty} X_x\,dx = \Big|_{-\infty}^{\infty} X\Big| = 0 \tag{7.2.22}$$

This has the consequence that

$$\int_{-\infty}^{\infty} T\,dx = \text{constant} \tag{7.2.23}$$

Thus we can think of these quantities as the p.d.e. analogues to the integrals of motion for o.d.e.'s.

For the KdV equation (7.1.3) the equation itself is in conservation law form, that is,

$$(u)_t + (-3u^2 + u_{xx})_x = 0$$

From this conservation law we see that

$$\int_{-\infty}^{\infty} u\,dx = \text{constant} \tag{7.2.24}$$

which represents the conservation of mass. Multiplying the KdV equation through by u, it is not difficult to obtain the second conservation law

$$(u^2)_t + (-2u^3 - \tfrac{1}{2}u_x^2 + uu_{xx})_x = 0$$

In this case we see that

$$\int_{-\infty}^{\infty} u^2\, dx = \text{constant} \qquad (7.2.25)$$

which represents the conservation of momentum. A certain amount of experimentation yields a third conservation law of the form

$$(u^3 + \tfrac{1}{2}u_x^2)_t = (\tfrac{9}{2}u^4 - 3u^2 u_{xx} + 6uu_x^2 - u_x u_{xxx} + \tfrac{1}{2}u_{xx}^2)_x \qquad (7.2.26)$$

and hence

$$\int_{-\infty}^{\infty} (u^3 + \tfrac{1}{2}u_x^2)\, dx = \text{constant} \qquad (7.2.27)$$

Later on we shall describe how the integral (7.2.27) actually represents the Hamiltonian for the KdV equation.

Having found three conservation laws, one naturally asks if there are any more and, indeed, if there could be an infinite number corresponding to the infinite number of degrees of freedom? This latter result would imply, in some sense, a type of complete "integrability." Kruskal and co-workers (Miura et al., 1968) found, at first, by little more than brute-force pencil-and-paper computations, nine conservation laws. A heroic effort by Miura (1968) produced a tenth, which, at that time, strongly suggested that there were indeed an infinite number of conserved quantities.

7.2.e The Miura Transformation

An important part of Miura's investigation of conservation laws for the KdV equation was the simultaneous study of a closely related equation, called the modified KdV (mKdV) equation, which takes the form

$$u_t + 6u^2 u_x + u_{xxx} = 0 \qquad (7.2.28)$$

This equation can be derived, in a similar manner to the KdV equation, from the Fermi–Ulam–Pasta lattice if the nonlinearity is taken to be cubic rather than quadratic. Miura found a set of conservation laws for the mKdV equation parallel to the set for the KdV equation. His crucial observation was that the two sets of conservation laws were connected by the "Miura transformation"

$$u = v_x + v^2 \qquad (7.2.29)$$

where u and v denote solutions to the KdV and mKdV equations, respectively. Furthermore, if we make the notation

$$P(u) = u_t - 6uu_x + u_{xxx} = 0 \qquad (7.2.30a)$$

and

$$K(v) = v_t + 6v^2 v_x + v_{xxx} = 0 \qquad (7.2.30b)$$

then one finds, using (7.2.29), that

$$P(u) = \left(2v + \frac{\partial}{\partial x}\right) K(v) \qquad (7.2.31)$$

These results subsequently motivated Miura et al. (1968) to introduce a slightly different transformation of the form

$$u = w + \epsilon w_x + \epsilon^2 w^2 \qquad (7.2.32)$$

where ϵ is some (small) parameter. With this transformation, one finds that

$$P(u) = \left(1 + \epsilon \frac{\partial}{\partial x} + 2\epsilon^2 w\right) Q(w) \qquad (7.2.33)$$

where

$$Q(w) = w_t - 6(w + \epsilon^2 w^2) w_x + w_{xxx} = 0 \qquad (7.2.34)$$

is known as the *Gardner equation*. Note that $Q(w)$ can also be written in conservation law form, that is,

$$(w)_t + (-3w^2 - 2\epsilon^2 w^3 + w_{xx})_x = 0 \qquad (7.2.35)$$

The idea is to expand w in a small ϵ power series, that is,

$$w = \sum_{j=0}^{\infty} \epsilon^j w_j \qquad (7.2.36)$$

The individual w_j are easily found by solving (7.2.32) recursively, giving

$$w_0 = u \qquad (7.2.37a)$$

$$w_1 = -u_x \qquad (7.2.37b)$$

$$w_2 = u_{xx} - u^2 \qquad (7.2.37c)$$

and so on. The conservation laws are found by substituting (7.2.36) into (7.2.35) and equating powers of ϵ. This result follows from the fact that

(7.2.36) is itself in conservation form. The first few laws are

$$O(\epsilon^0):\quad (w_0)_t = (3w_0^2 - w_{0_{xx}})_x \tag{7.2.38a}$$

$$O(\epsilon^1):\quad (w_1)_t = (6w_0w_1 - w_{1_{xx}})_x \tag{7.2.38b}$$

$$O(\epsilon^2):\quad (w_2)_t = (3w_1^2 + 6w_0w_2 + 2w_0^2 - w_{2_{xx}})_x \tag{7.2.38c}$$

The reader will be able to verify that (7.2.38a) and (7.2.38c) correspond to the conservation laws (7.2.24) and (7.2.25), respectively. (Derivation of the latter requires use of (7.2.38a) to rearrange the left-hand side of (7.2.38c).) The law found at $O(\epsilon^1)$ is just the differential of the law found at $O(\epsilon^0)$. This is a general result: The conservation laws found at the odd powers of ϵ are just derivatives of those found at the preceding even powers.

7.2.f Galilean Invariance

Gardner's transformation provides an algorithm to compute an infinity of conserved densities for the KdV equation. As discussed in Chapter 2 the existence of an integral—here a conserved density—implies the existence of some special symmetry or invariance. That the KdV equation should possess an infinity of such symmetries suggests that it must have some very special properties.

One basic invariance possessed by the KdV equation is Galilean (or translation) invariance. If one makes the change of variables, namely,

$$t' = t, \qquad x' = x - ct, \qquad u'(x', t') = u(x, t) + \tfrac{1}{6}c$$

which corresponds to transforming to a frame of reference moving to the right, the KdV equation becomes

$$u'_{t'} - 6u'u'_{x'} + u'_{x'x'x'} = 0$$

that is, it is invariant to such a transformation. By contrast, the mKdV equation is easily seen not to be Galilean invariant. However, if the above change of variables is applied to the Miura transformation (7.2.29), one obtains the Gardner transformation (7.2.32) on setting $c = \frac{3}{2}\epsilon^2$.

7.3 THE INVERSE SCATTERING TRANSFORM: BASIC PRINCIPLES

So far the basic facts we have learned about the KdV equation are: (1) it exhibits (numerically) solitons, (2) it possesses a variety of special solutions, (3) it is Galilean invariant, and (4) it possesses an infinite number of

conservation laws which are connected to those of the mKdV equation through the Miura transformation

$$v_x + v^2 = u \tag{7.3.1}$$

This was, approximately, the information at the disposal of Gardner, Greene, Kruskal, and Miura (GGKM) in 1967 (Gardner et al., 1967). Their observation was that (7.3.1) is a Riccati equation for v which can be linearized (see Chapter 1) by making the substitution

$$v = \frac{\psi_x}{\psi} \tag{7.3.2}$$

to yield

$$\psi_{xx} = u(x, t)\psi \tag{7.3.3}$$

Furthermore, since the KdV equation is Galilean invariant, u can be replaced by $u - \lambda$, where λ is any (at this stage) constant. On making this shift, (7.3.3) becomes

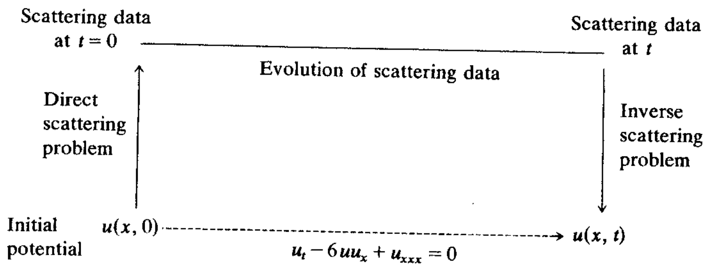
$$\psi_{xx} - (u(x, t) - \lambda)\psi = 0 \tag{7.3.4}$$

which is just the one-dimensional time-independent Schrödinger equation for a "potential" $u(x, t)$ with eigenvalues λ.

7.3.a The Connection with Quantum Mechanics

GGKM then made a remarkable intuitive leap by proposing that the time evolution of $u(x, t)$, according to the KdV equation, could be studied through the properties of the quantum mechanical problem (7.3.4). Their idea was as follows. Given an initial condition $u = u(x, 0)$, solve the "*direct scattering*" problem, that is, treat $u(x, 0)$ as a potential in the Schrödinger equation (7.3.4) and find all the associated eigenvalues and eigenfunctions. As u evolves, or deforms as a function of t, these associated quantum mechanical properties—termed the *scattering data*—will also evolve. At this point it is most important to emphasize that the variable t in $u(x, t)$ should be thought of as some *deformation parameter* in the KdV equation and should in no way be confused with the time variable that apppears in the traditional time-dependent Schrödinger equation. The idea of GGKM was that the evolution of the scattering data, initially obtained from $u(x, 0)$, might somehow be obtained without having to solve the KdV equation directly. If this could be achieved, the scattering data, thereby obtained at some later value of t, could be used to "reconstruct" the "potential" $u(x, t)$.

This latter step involves solving the quantum mechanical *inverse scattering* problem, that is, going from the scattering data to the potential, in contrast to the direct scattering problem of going from the potential to the scattering data. This indirect route to solving the KdV equation is sketched below:



7.3.b Analogy with Fourier Transforms

Such a scheme is not as far fetched as it might at first sound and is, in fact, closely analogous to the use of Fourier transforms in solving linear evolution equations. Consider such an equation, on the interval $-\infty \le x \le \infty$, of the form

$$u_t = \mathscr{L}\left(\frac{\partial}{\partial x}\right)u \tag{7.3.5}$$

where $\mathscr{L}(\partial/\partial x)$ is a polynomial in $\partial/\partial x$, that is, a linear operator. A simple example would be $\mathscr{L} = \partial^2/\partial x^2$, in which case (7.3.5) is just the standard diffusion equation. Now define the Fourier transform of $u(x, t)$:

$$\tilde{u}(k, t) = \int_{-\infty}^{\infty} u(x, t)e^{ikx}\, dx \tag{7.3.6}$$

Also define the "inverse" Fourier transform:

$$u(x, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{u}(k, t)e^{-ikx}\, dk \tag{7.3.7}$$

Fourier transforming the p.d.e. (7.3.5) gives the evolution equation for $\tilde{u}(k, t)$, that is,

$$\frac{d\tilde{u}}{dt} = \mathscr{L}(ik)\tilde{u}$$

This linear equation has the simple solution

$$\tilde{u}(k, t) = \tilde{u}(k, 0)e^{\mathscr{L}(ik)t} \tag{7.3.8}$$

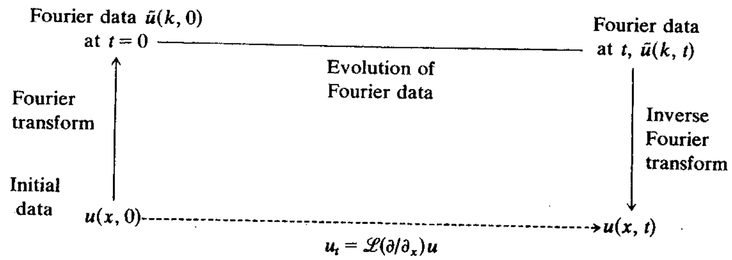
where the initial Fourier transform data $\tilde{u}(k, 0)$ is determined from the given initial data $u(x, 0)$, that is,

$$\tilde{u}(k, 0) = \int_{-\infty}^{\infty} u(x, 0)e^{ikx}\, dx \tag{7.3.9}$$

The evolution of the "Fourier data" $\tilde{u}(k, t)$ is governed by the trivial relation (7.3.8) and can be inverted at any subsequent value of t to give the desired $u(x, t)$, namely,

$$u(x, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{u}(k, 0)e^{\mathscr{L}(ik)t}\, e^{-ikx}\, dk \tag{7.3.10}$$

Thus the solution route is analogous to that proposed by GGKM, that is,



Of course the GGKM method is more complicated since the evolution is nonlinear. To understand how it works, we must first discuss in more detail the quantum mechanical direct and inverse scattering problems.

7.3.c The Direct Scattering Problem

Depending on its precise shape, a given potential $u_0(x) = u(x, 0)$ can support bound states. The Schrödinger equation (7.3.4) will then admit a corresponding set of discrete eigenvalues, $\lambda_n = -k_n^2$ $(n = 1, \ldots, N)$ corresponding to "negative energy" bound states with associated eigenfunctions $\psi_n(x)$, that is,

$$\psi_{n,xx} = (u_0(x) + k_n^2)\psi_n = 0 \tag{7.3.11}$$

Bound-state eigenfunctions are required to be square integrable and normalized to unity, that is,

$$\int_{-\infty}^{\infty} |\psi_n(x)|^2\, dx = 1 \tag{7.3.12}$$

The normalizing constant, c_n, that ensures (7.3.12) is defined by

$$\lim_{x\to\infty} e^{k_n x}\psi_n(x) = c_n \tag{7.3.13}$$

This follows from the assumption that $u_0(x)$ decays to zero sufficiently rapidly as $x \to \pm\infty$ such that (7.3.11) reduces to $\psi_{n,xx} - k_n^2\psi_n = 0$. One may also define, equivalently, the c_n by

$$c_n = \left[\int_{-\infty}^{\infty} |\psi_n(x)|^2\, dx\right]^{-1/2} \tag{7.3.14}$$

The set of eigenvalues λ_n $(n = 1, \ldots, N)$ is termed the bound-state spectrum.

At positive energy the Schrödinger equation for $u_0(x)$ exhibits a continuous spectrum and we set $\lambda = k^2$. As is well known, quantal wave-functions can exhibit reflection above a potential barrier. Thus the asymptotic form of $\psi(x)$ in the limit $x \to \infty$ is

$$\lim_{x\to\infty} \psi(x) = e^{-ikx} + b(k)e^{+ikx} \tag{7.3.15}$$

where the first term on the right-hand side represents an incoming wave and the second term represents the reflected wave with *reflection coefficient* $b(k)$. In the limit $x \to -\infty$, we have

$$\lim_{x\to-\infty} \psi(x) = a(k)e^{-ikx} \tag{7.3.16}$$

which represents the transmitted wave with *transmission coefficient* $a(k)$.

7.3.d The Inverse Scattering Problem

The term *scattering data* is used to mean, for a given potential, the set of all bound-state eigenvalues, λ_n, normalizing constants, c_n, and the continuum functions $a(k)$ and $b(k)$. A remarkable result of Gelfand, Levitan, and Marchenko in the 1950s (quite independent of anything to do with solitons!) was to show how the scattering data could be used to uniquely find the associated potential function $u_0(x)$ (Gelfand and Levitan, 1955; Marchenko, 1955). Assuming that $u_0(x)$ satisfies the boundedness condition

$$\int_{-\infty}^{\infty} (1 + |x|)|u_0(x)|\, dx < \infty \tag{7.3.17}$$

one defines the following quantity

$$B(\zeta) = \sum_{n=1}^{N} c_n^2 e^{-k_n\zeta} + \frac{1}{2\pi}\int_{-\infty}^{\infty} b(k)e^{ik\zeta}\, dk \tag{7.3.18}$$

which can be thought of as a sort of Fourier transform of the scattering data. The next step is to solve the following linear integral equation

$$K(x, y) + B(x + y) + \int_{-\infty}^{\infty} B(x + z)K(z, y)\, dz = 0 \tag{7.3.19}$$

for the function $K(x, y)$. If $K(x, y)$ can be found, then it is possible to show that the potential $u_0(x)$ giving rise to the scattering data used in (7.3.18) is given by

$$u_0(x) = -2(d/dx)K(x, x) \tag{7.3.20}$$

A truly remarkable result!

In order to use all this to solve the KdV equation, we have to:

(i) find out how the scattering data "evolves" as $u_0(x)$ is "deformed" into $u(x, t)$ and
(ii) be able to solve the Gelfand–Levitan–Marchenko equation (Eq. (7.3.19)).

As it turns out, (i) can be solved relatively easily. The problem, in keeping with the conservation of difficulty principle, is (ii). Unfortunately, (7.3.19) can only be solved exactly for rather special cases, but these include the case required to explain the appearance of solitons. These results are discussed in the next section.

7.4 THE INVERSE SCATTERING TRANSFORM: THE KdV EQUATION

To begin with, we again emphasize that the variable t appearing in the KdV equation

$$u_t - 6uu_x + u_{xxx} = 0 \tag{7.4.1}$$

should not be thought of as "real" time but rather as a deformation parameter. Thus in the Schrödinger equation we can assign a t dependence to the eigenvalue λ without confusion, that is,

$$\psi_{xx} - (u(x, t) - \lambda(t))\psi = 0 \tag{7.4.2}$$

where $\psi = \psi(x, t)$.

7.4.a The Isospectral Deformation

Using (7.4.2) to express u as a function of ψ, that is,

$$u = \frac{\psi_{xx}}{\psi} + \lambda \tag{7.4.3}$$

it follows that

$$u_t = \frac{\psi_{xxt}}{\psi} - \frac{\psi_{xx}\psi_t}{\psi^2} + \lambda_t \tag{7.4.4}$$

with analogous expressions obtainable for uu_x and u_{xxx}. For these latter terms it is convenient to eliminate third and higher derivatives (with respect to x) of ψ by repeated use of (7.4.2). In this manner the KdV equation (7.4.1) can be reexpressed as

$$\lambda_t \psi^2 + (\psi M_x - \psi_x M)_x = 0 \tag{7.4.5}$$

where

$$M = \psi_t - 2(u + 2\lambda)\psi_x + u_x \psi \tag{7.4.6}$$

For bound-state eigenfunctions the ψ are square integrable, so integrating both sides of (7.4.5) from $-\infty$ to $+\infty$ yields

$$\begin{aligned}\lambda_t \int_{-\infty}^{\infty} \psi^2\, dx &= -\int_{-\infty}^{\infty} (\psi M_x - \psi_x M)_x\, dx \\ &= -|\psi M_x - \psi_x M|_{-\infty}^{\infty} \\ &= 0\end{aligned} \tag{7.4.7}$$

Since $\int_{-\infty}^{\infty} \psi^2\, dx$ is just a nonzero constant, (7.4.7) implies that

$$\lambda_t = 0 \tag{7.4.8}$$

This is an immensely significant result since it tells us that for a potential $u(x, t)$, deformed according to the KdV equation, the bound-state eigenvalues $\lambda_n(t)$ $(n = 1, \ldots, N)$ remain *unchanged*! This is an example of what is termed an *isospectral deformation*. In the continuum (i.e., $\lambda > 0$), there is a solution to the Schrödinger equation for every value of λ. Thus we can simply argue that at every positive energy, λ is fixed and hence $\lambda_t = 0$. Either way, (7.4.5) gives

$$M_{xx}\psi - M\psi_{xx} = 0 \tag{7.4.9}$$

and by using (7.4.2) for ψ_{xx} we write this as a second-order differential equation for M, that is,

$$M_{xx} - (u - \lambda)M = 0 \tag{7.4.10}$$

The general solution to (7.4.10) is of the standard form

$$M = A\psi + B\varphi \tag{7.4.11}$$

where ψ and φ are the two linearly independent solutions. Obviously one of these solutions is just the eigenfunction ψ (just compare (7.4.2) and (7.4.10)). It is a standard result to show that the second solution φ can be computed from

$$\varphi = \psi \int^{x} \frac{dx'}{\psi^2} \tag{7.4.12}$$

which is easily verified by checking that the Wronskian $\varphi_x\psi - \psi_x\varphi = 1$. However, it is not difficult to show from the asymptotic properties of (7.4.10) that $B = 0$ for both the bound states and the continuum. (In the limit $x \to \pm\infty$, (7.4.10) becomes $M_{xx} + \lambda M = 0$; and using the asymptotic forms of ψ in (7.4.11), this can only be satisfied for nontrivial φ if $B = 0$.) Thus, overall, we have

$$M = \psi_t - 2(u + 2\lambda)\psi_x + u_x\psi = A\psi \tag{7.4.13}$$

For bound states, we can go further and also show that $A = 0$. Multiply both sides of (7.4.13) by ψ to obtain

$$\psi\psi_t - 2(u + 2\lambda)\psi^2 + u_x\psi^2 = A\psi^2 \tag{7.4.14}$$

and rewrite this as

$$\tfrac{1}{2}(\psi^2)_t + (u\psi^2 - 2\psi_x^2 - 4\lambda\psi^2)_x = A\psi^2 \tag{7.4.15}$$

For square integrable bound-state eigenfunctions, we can now integrate over x to obtain

$$\frac{1}{2}\left(\int_{-\infty}^{\infty} \psi^2\, dx\right)_t + \left| u\psi^2 - 2\psi_x^2 - 4\lambda\psi^2 \right|_{-\infty}^{\infty} = A\int_{-\infty}^{\infty} \psi^2\, dx \tag{7.4.16}$$

The second term on the left-hand side of (7.4.16) is zero—as is the first term by the constancy of the normalization integral. Thus $A = 0$ and we have

$$\psi_t - 2(u + 2\lambda)\psi_x + u_x\psi = 0 \tag{7.4.17}$$

7.4.b Evolution of the Scattering Data

Equation (7.4.17) can now be used to derive the "evolution" equation for the normalization constants $c_n(t)$. In the limit $x \to \infty$, for suitably decaying u and u_x, (7.4.17) reduces to

$$\psi_{n,t} + 4k_n^2\psi_n = 0 \tag{7.4.18}$$

where we have set $\lambda = -k_n^2$ for the eigenfunction ψ_n. By definition,

$$\lim_{x\to\infty} \psi_n(x, t) = c_n(t)e^{-k_n x} \qquad (7.4.19)$$

so by direct substitution of (7.4.19) into (7.4.18) we obtain the first-order ordinary differential equation for c_n, namely,

$$\frac{dc_n}{dt} = 4k_n^3 c_n \qquad (7.4.20)$$

This has the simple solution

$$c_n(t) = c_n(0)e^{4k_n^3 t} \qquad (7.4.21)$$

where the initial value $c_n(0)$ is the bound-state normalization constant for the nth eigenfunction of $u_0(x) = u(x, 0)$.

In the case of the continuum, we still have to work with (7.4.13), but setting $\lambda = k^2$ and again taking the limit $x \to \infty$, (7.4.13) reduces to

$$\psi_t - 4k^2\psi_n = A\psi \qquad (7.4.22)$$

Recalling the asymptotic form

$$\lim_{x\to\infty} \psi(x, t) = e^{-ikx} + b(k, t)e^{ikx} \qquad (7.4.23)$$

direct substitution into (7.4.22) and the choice $A = 4ik^3$ yields the evolution equation for $b(k, t)$, namely,

$$\frac{db}{dt} = 8ik^3 b \qquad (7.4.24)$$

This has the solution

$$b(k, t) = b(k, 0)e^{8ik^3 t} \qquad (7.4.25)$$

where $b(k, 0)$ is the reflection coefficient for $u_0(x)$. Using the same arguments for the limit $x \to -\infty$, it is easy to show that

$$\frac{da}{dt} = 0 \qquad (7.4.26)$$

and hence

$$a(k, t) = a(k, 0) \qquad (7.4.27)$$

The reader will now see that the desired miracle has occurred—namely, that under deformation according to the KdV equation the scattering data $\lambda_n(0)$, $c_n(0)$, $a(k, 0)$, $b(k, 0)$, associated with the initial potential $u_0(x)$, evolve according to simple linear equations. Since the deformation is isospectral, we have $\lambda_n(t) = \lambda_n(0)$. Using (7.4.21) and (7.4.25), we can construct the corresponding quantity

$$B(\zeta; t) = \sum_{n=1}^{\infty} c_n^2(t)e^{-k_n\zeta} + \frac{1}{2\pi}\int_{-\infty}^{\infty} b(k, t)e^{ik\zeta}\, dk \qquad (7.4.28)$$

and, on solving (if we are lucky!) the Gelfand–Levitan–Marchenko equation for $K(x, y; t)$, we obtain $u(x, t)$ from

$$u(x, t) = -2(d/dx)K(x, x; t) \qquad (7.4.29)$$

This whole procedure, called the ***inverse scattering transform***, or IST, is, in effect, an indirect linearization of the KdV equation with a strikingly close analogy to the Fourier transform method. Indeed, IST is often referred to as a ***nonlinear Fourier transformation***.

As was hinted earlier on, the real problem lies with solving the integral equation for $K(x, y; t)$. It turns out, to date, that it can only be solved in closed form for those scattering problems which are *reflectionless*, that is, those for which $b(k, t) = b(k, 0) = 0$.

7.4.c A Two-Soliton Solution

A standard illustration of the IST for the KdV equation is provided by the study of potentials of the form $u_0(x) = -V\,\mathrm{sech}^2 x$, where V is a constant. Here we work with the particular case

$$u(x, 0) = -6\,\mathrm{sech}^2 x \qquad (7.4.30)$$

The associated Schrödinger equation

$$\psi_{xx} + (6\,\mathrm{sech}^2 x + \lambda)\psi = 0 \qquad (7.4.31)$$

can be solved exactly. (Details are given in the excellent account by Drazin (1983)). There are just two bound-state eigenfunctions:

$$\psi_1 = \tfrac{1}{4}\mathrm{sech}^2(x), \quad \text{with } \lambda_1 = -k_1^2 = -4 \text{ and } c_1^2(0) = 12$$

$$\psi_2 = \tfrac{1}{2}\tanh(x)\,\mathrm{sech}(x), \quad \text{with } \lambda_2 = -k_2^2 = -1 \text{ and } c_2^2(0) = 6$$

Fortunately, all sech^2 potentials and reflectionless, so $b(k) = 0$. Thus

$$B(\zeta, t) = \sum_{n=1}^{2} c_n^2(t)e^{-k_n\zeta}$$
$$= \sum_{n=1}^{2} c_n^2(0)e^{8k_n^3t-k_n\zeta}$$
$$= 12e^{64t-2\zeta} + 6e^{8t-\zeta} \tag{7.4.32}$$

For reflectionless potentials the Gelfand–Levitan–Marchenko equation can be solved by assuming the kernel to be of the form

$$K(x, y; t) = \sum_{n=1}^{N} p_n(x, t)e^{-k_n y} \tag{7.4.33}$$

Using this separation for the problem at hand, one may show that

$$K(x, y; t) = \frac{-3(2e^{28t+x-2y} + 2e^{36t-x-2y} - e^{36t-2x-y} + e^{-24t+2x-y})}{(3\cosh(x-28t) + \cosh(3x-36t))} \tag{7.4.34}$$

Then, using (7.4.29), the solution can (eventually) be expressed as

$$u(x, t) = -\frac{12(3 + 4\cosh(2x-8t) + \cosh(4x-64t))}{(3\cosh(x-28t) + \cosh(3x-36t))^2} \tag{7.4.35}$$

Note that $u(x, 0) = -6\,\text{sech}^2\,x$. To understand the properties of this solution we follow the analysis given by Drazin (1983). This involves introducing the variables $x_1 = x - 4k_1^2t = x - 16t$ and $x_2 = x - 4k_2^2t = x - 4t$. Expressing the arguments of the cosh terms in (7.4.35) in terms of x_1 we obtain

$$u(x, t) = -\frac{12(3 + 4\cosh(2x_1 + 24t) + \cosh(4x_1))}{(3\cosh(x_1 - 12t) + \cosh(3x_1 + 12t))^2} \tag{7.4.36}$$

Now, for fixed x_1, take the limit $t \to \infty$ and discard the exponentially decaying portions of the cosh terms, that is,

$$\lim_{t\to\infty} u(x, t) \simeq \frac{-96e^{2x_1+24t}}{(3e^{-x_1+12t} + e^{3x_1+12t})^2}$$
$$= \frac{-32}{((1/\sqrt{3})e^{2x_1} + \sqrt{3}e^{-2x_1})^2}$$
$$= \frac{-32}{(e^{2x_1-\ln\sqrt{3}} + e^{-2x_1+\ln\sqrt{3}})^2}$$
$$= -8\,\text{sech}^2(2x_1 + \delta) \tag{7.4.37}$$

where $\delta = \ln\sqrt{3}$. Similarly, by expressing (7.4.35) in terms of x_2, we obtain

$$u(x, t) = \frac{-12(3 + 4\cosh(2x_2) + \cosh(4x_2 - 48t))}{(3\cosh(x_2 - 24t) + \cosh(3x_2 - 24t))^2}$$

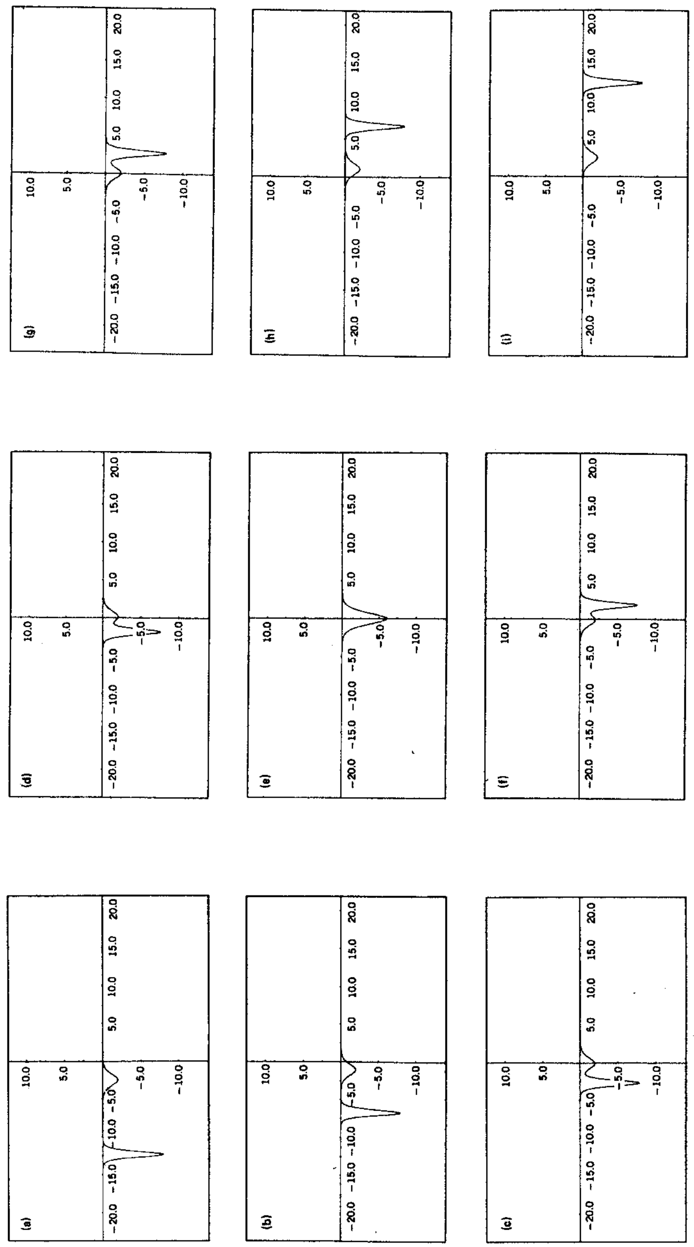


Figure 7.2 Sketch of evolution of two-soliton solution (7.4.35) for (*a*) $t = -0.75$, (*b*) $t = -0.4$, (*c*) $t = -0.15$, (*d*) $t = -0.1$, (*e*) $t = 0.0$, (*f*) $t = 0.1$, (*g*) $t = 0.15$, (*h*) $t = 0.4$, and (*i*) $t = 0.75$ showing the two separate solitons of depths 8 and 2, respectively, merging at $t = 0$ to give the initial potential $u(x, 0) = -6\,\text{sech}^2\,x$ and then separating again with the deeper wave overtaking the smaller one. The "mass" $\int_{-\infty}^{\infty} u(x, t)\,dx$ is conserved throughout. (Computation by Mr. E. Dresselhaus (private communication).)

and, on taking the limit $t \to \infty$, we have

$$\begin{aligned}\lim_{t\to\infty} u(x, t) &\simeq \frac{-24e^{-4x_2+48t}}{(3e^{-x_2+24t} + e^{-3x_2+24t})^2} \\ &= \frac{-8}{(\sqrt{3}e^{x_2} + (1/\sqrt{3})e^{-x_2})^2} \\ &= -2\,\text{sech}^2(x_2 + \delta) \end{aligned} \tag{7.4.38}$$

In the same spirit, one can, in fact, show that in the limit $t \to \pm\infty$ for either fixed x_1 or x_2, $u(x, t)$ behaves as

$$\lim_{t\to\pm\infty} u(x, t) = -2\,\text{sech}^2(x_2 \pm \delta) - 8\,\text{sech}^2(2x_1 \mp \delta) \tag{7.4.39}$$

From the above analysis we can now understand the behavior observed by Zabusky and Kruskal (1965). The solution represents the interaction of two solitary traveling waves (cf discussion of traveling wave solutions in the previous section). At $t = -\infty$ the deeper wave lies to left of the shallow one (see Figure 7.2). As $t \to 0$, the deeper wave catches up with the shallower one and at $t = 0$ they merge into the original potential $u(x, 0) = -6\,\text{sech}^2 x$. Remarkably, as $t \to \infty$ they separate again, with the deeper wave moving ahead of the shallower one. At $t = +\infty$ the solution is again just the sum of two separate solitary waves, with the only consequence of the collision being the small phase shift δ.

7.4.d More General Solutions

The result obtained for this "two-soliton" potential is fairly easily generalized to any potential of the form $u(x, 0) = -V\,\text{sech}^2 x$. For such a potential supporting N bound states with eigenvalues $\lambda_n = -k_n^2$ $(n = 1, \ldots, N)$, the asymptotic form of solution is

$$\lim_{t\to+\infty} u(x, t) = \sum_{n=1}^{N} -2\lambda_n^2\,\text{sech}^2(k_n(x - 4k_n^2 t - \delta_n)) \tag{7.4.40}$$

where the phase shift δ_n is given by

$$\delta_n = \frac{1}{2k_n}\ln\left\{\frac{c_n^2(0)}{2k_n}\prod_{m=1}^{N-1}\left(\frac{k_n - k_m}{k_n + k_m}\right)^2\right\} \tag{7.4.41}$$

Such a solution is called an ***N*-soliton solution**. As $t \to \infty$ the initial condition takes up into a train of solitary traveling waves, with the deepest at the front and the shallowest at the rear. As t goes from $-\infty$ to $+\infty$, the only effect of the interactions is just the phase shift δ_n.

Our identification of the solitons has come about through considering the limit $t \to \pm\infty$. In fact, it is possible to show that the N-soliton solutions can be represented exactly in the general form

$$u(x, t) = \sum_{n=1}^{N} -4k_n\psi_n^2(x, t) \tag{7.4.42}$$

where the ψ_n are the bound-state eigenfunctions with eigenvalues $\lambda_n = -k_n^2$.

For potentials which are not reflectionless, the continuum portion of the quantal spectrum renders the Gelfand–Levitan–Marchenko equation intractable to exact solution. However, the same basic picture holds, and in the limit $t \to \infty$ the initial condition separates into a procession of isolated traveling waves. The continuum has the effect of introducing an oscillatory portion to the solution which dies out dispersively as $t \to \infty$. This phenomenon is sometimes termed *radiation*. Little is known about the exact nature of this part of the solution, although some asymptotic estimates are available. A discussion of some of these results can be found, for example, in Ablowitz and Segur (1981) or Drazin (1983). (The latter also gives a simple example.)

7.4.e The Lax Pair*

The reader will have noticed that the quantum mechanical problem boils down (for the bound state spectrum) to the pair of linear equations

$$\psi_{xx} = (u - \lambda)\psi \tag{7.4.43a}$$

and

$$\psi_t = 2(u + 2\lambda)\psi_x - u_x\psi \tag{7.4.43b}$$

In order for these equations to be consistent with each other, they must satisfy the "integrability condition"

$$\psi_{xxt} = \psi_{txx} \tag{7.4.44}$$

Differentiating (7.4.43a) with respect to t and using (7.4.43b) yields

$$\psi_{xxt} = (u_t - uu_x + \lambda u_x - \lambda_t)\psi + 2(u + 2\lambda)(u - \lambda)\psi_x \tag{7.4.45}$$

where, for now, we are not assuming $\lambda_t = 0$. Similarly, differentiating (7.4.43b) with respect to x finally gives

$$\psi_{txx} = (5uu_x + \lambda u_x - u_{xxx})\psi + 2(u + 2\lambda)(u - \lambda)\psi_x \tag{7.4.46}$$

In order to satisfy condition (7.4.44), we immediately see that the following two conditions must be imposed, namely,

$$u_t - 6uu_x + u_{xxx} = 0 \tag{7.4.47}$$

and

$$\lambda_t = 0 \tag{7.4.48}$$

The pair of equations (7.4.43) is termed a *Lax pair* after Lax, who showed (immediately following GGKM) that the KdV equation and other closely related nonlinear evolution equations are equivalent to the isospectral integrability condition for pairs of linear operators.

7.5 OTHER SOLITON SYSTEMS

For a while it was thought that the IST technique developed by GGKM was only applicable to the KdV equation. However, within a few years a whole host of other, physically important, nonlinear evolution equations were found to have soliton solutions and suitable ISTs developed. For these systems the quantal problem does not, typically, involve the time-independent Schrödinger equation any more and a different, but related, eigenvalue problem has to be solved. Here we just introduce a few of these equations and some of their simple solutions and briefly describe the basic IST procedure. Fuller accounts can be found in the cited texts.

7.5.a The Modified KdV Equation

An evolution equation that we have already mentioned is the mKdV equation, namely,

$$v_t + 6v^2 v_x + v_{xxx} = 0 \tag{7.5.1}$$

The first step is to look for traveling wave solutions of the form

$$v(x, t) = f(z) \tag{7.5.2}$$

where $z = x - ct$. Direct substitution into (7.5.2), followed by two integrations, yields the quadrature

$$z - z_0 = \int \frac{df}{\sqrt{cf^2 - f^4 + \frac{1}{2}df + e}} \tag{7.5.3}$$

where d and e are the first two integration constants. The general solution to this problem is in terms of Jacobi elliptic functions, but for the choice of

boundary conditions $f, f' \to 0$ as $z \to \pm\infty$ the quadrature reduces to

$$z - z_0 = \int \frac{df}{f\sqrt{c - f^2}} \tag{7.5.4}$$

This is easily integrated and inverted to give the solitary wave solution

$$f(z) = -\sqrt{c}\,\text{sech}(\sqrt{c}(z - z_0)) \tag{7.5.5}$$

Note that the solution is a sech rather than the analogously obtained sech^2 solution (7.2.15) for the KdV equation.

Another simple solution can be obtained by a similarity transformation. Following the arguments given in Section 7.2, it is not difficult to show that (7.5.1) is invariant to the scalings $x \to kx$, $t \to k^3 t$, $u \to k^{-1}u$. This suggests the change of variables,

$$z = x/t^{1/3}, \qquad u(x, t) = t^{-1/3} f(z) \tag{7.5.6}$$

which gives

$$f'' + 6ff' - \tfrac{1}{3}zf' - \tfrac{1}{3}f = 0 \tag{7.5.7}$$

This can be integrated once to yield

$$f'' + 2f^3 - \tfrac{1}{3}zf + c = 0 \tag{7.5.8}$$

which is the special ordinary differential equation known as the second Painlevé transcendent.

7.5.b The Sine–Gordon Equation

A very important nonlinear p.d.e. is the Sine–Gordon equation

$$u_{tt} - u_{xx} + \sin u = 0 \tag{7.5.9}$$

This equation, as well as various solution techniques, was already used in the last century, where it appeared in various problems of differential geometry. A more contemporary use for it is in relativistic field theory. It is often convenient to study (7.5.9) in the variables

$$\xi = \tfrac{1}{2}(x - t), \qquad \eta = \tfrac{1}{2}(x + t) \tag{7.5.10}$$

which transforms it to

$$u_{\xi\eta} = \sin u \tag{7.5.11}$$

The periodicity of the sine introduces some interesting properties. If (7.5.9) is linearized (i.e., expanded to first order) about the solution $\psi = 0$, one obtains

$$u_{tt} - u_{xx} + u = 0 \tag{7.5.12}$$

for which the dispersion relations are easily seen to be

$$\omega = \sqrt{k^2 + 1} \tag{7.5.13}$$

This is real for all real k, implying that $\psi = 0$ is a stable equilibrium point. This should hardly be too surprising since if the space-dependent part of (7.5.9) is dropped, one is just left with the simple pendulum equation $u_{tt} + \sin u = 0$. On the other hand, if (7.5.9) is expanded about the solution $u = \pi$, the result is

$$u_{tt} - u_{xx} - u = 0 \tag{7.5.14}$$

which has the dispersion relations

$$\omega = \sqrt{k^2 - 1} \tag{7.5.15}$$

This demonstrates that the solution $u = \pi$ is unstable for $0 < k < 1$. This is again consistent with the properties of the space-independent problem.

A traveling wave analysis (i.e., setting $u(x, t) = f(z)$), yields after one integration the quadrature

$$z - z_0 = (c^2 - 1)^{1/2} \int \frac{df}{\sqrt{2(d - 2\sin^2(\frac{1}{2}f))}} \tag{7.5.16}$$

where d is the first constant of integration. For the particular choice $d = 0$, (7.5.16) is easily solved to give

$$z - z_0 = \pm\sqrt{1 - c^2}\,\ln(\pm\tan(\tfrac{1}{4}f)) \tag{7.5.17}$$

and hence

$$f(z) = \pm 4\tan^{-1}(e^{\pm(z - z_0)/\sqrt{1 - c^2}}) \tag{7.5.18}$$

This solution takes on different shapes, depending on the choice of signs. For the case of both signs being positive, $f(z)$ rises, from left to right, from zero to a height of 2π (see Figure 7.3). Such a solution is called a *kink*. The solutions whose amplitude decays from 2π down to zero are termed *antikinks*. Although, at first sight, these solutions look quite different from solitons, their derivative has the characteristic sech shape, that is,

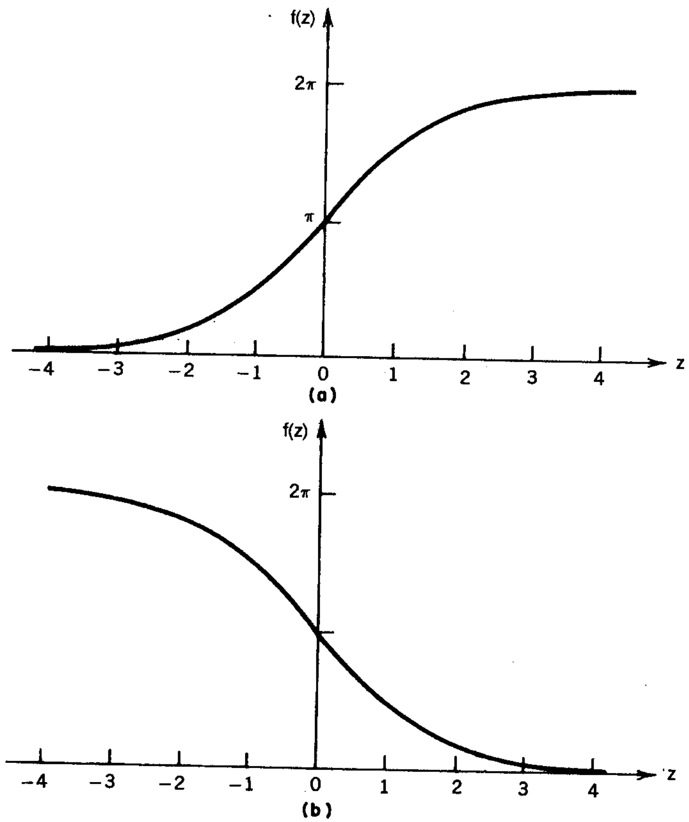


Figure 7.3 (*a*) Kink solution $f(z) = 4\tan^{-1}[\exp(z)]$ and (*b*) anti-kink solution $f(z) = 4\tan^{-1}[-\exp(z)]$.

$$u_x(x, t) = f'(z) = \frac{2}{\sqrt{1 - c^2}}\,\text{sech}((z - z_0)/\sqrt{1 - c^2}) \tag{7.5.19}$$

Kinks (and antikinks) show all the collisional properties of solitons; that is, they emerge unscathed from collision, suffering only a phase shift. A two-kink solution that demonstrates this property was derived, using standard separation of variables techniques, by Perring and Skyrme (1962). (This predates the work of Kruskal and co-workers, but at that time the full significance of their result was not appreciated.) The solution (see Drazin (1983) for details) takes the form

$$u(x, t) = 4\tan^{-1}\left[\frac{c\sinh(x/\sqrt{1 - c^2})}{\cosh(ct/\sqrt{1 - c^2})}\right] \tag{7.5.20}$$

The limits $t \to \pm\infty$ yield

$$\lim_{t\to-\infty} u(x,t) = 4\tan^{-1}[e^{(x+ct-\delta)/\sqrt{1-c^2}} - e^{-(x-ct+\delta)/\sqrt{1-c^2}}] \tag{7.5.21}$$

and

$$\lim_{t\to+\infty} u(x,t) = 4\tan^{-1}[-e^{-(x+ct+\delta)/\sqrt{1-c^2}} + e^{(x-ct-\delta)/\sqrt{1-c^2}}] \tag{7.5.22}$$

where we have introduced the phase shift

$$\delta = \sqrt{1-c^2}\,\ln\left(\frac{1}{c}\right) \tag{7.5.23}$$

The behavior of this solution is sketched in Figure 7.4.

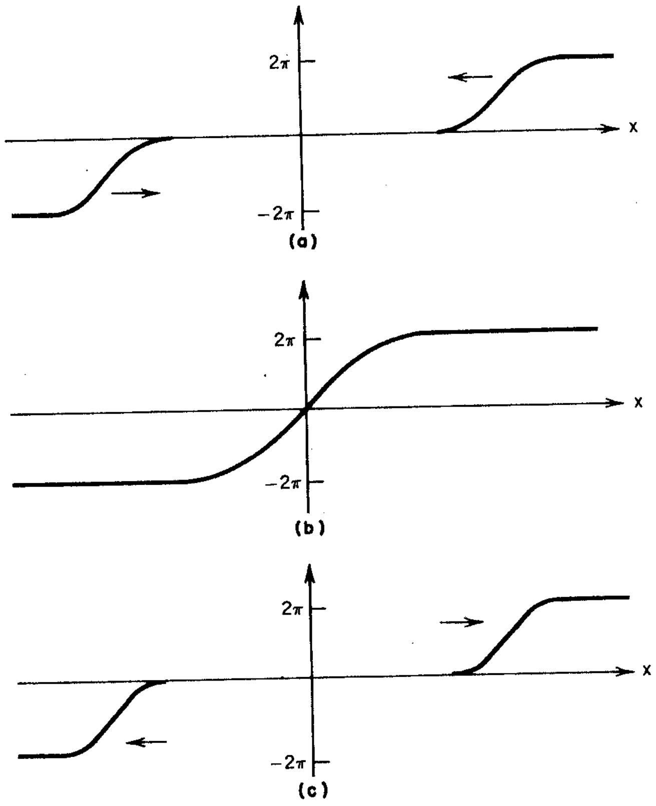


Figure 7.4 Evolution of Perring–Skyrme solution (7.5.20) for (*a*) $t<0$, (*b*) $t=0$, and (*c*) $t>0$.

Another special solution to the Sine–Gordon equation can be obtained by a similarity transformation. Working with (7.5.11), one notes that it is invariant to the scalings $\xi \to k\xi$ and $\eta \to \eta/k$. Making the substitution $u(\xi,\eta) = f(z)$, where $z = \xi\eta$, yields

$$zf'' + f' = \sin(f) \tag{7.5.24}$$

The change of variable $g = e^{if}$ then gives

$$g'' - \frac{(g')^2}{g} + \frac{(2g' - g^2 + 1)}{2z} = 0 \tag{7.5.25}$$

which is a special case of the *third Painlevé transcendent.*

7.5.c The NLS Equation

Another significant nonlinear p.d.e. is the nonlinear Schrödinger equation (NLS), which takes the form

$$iu_t = u_{xx} + 2u|u|^2 \tag{7.5.26}$$

where u represents the amplitude of an almost monochromatic wave train. Note that this is a complex equation. Thus $|u|^2$ gives the (real) amplitude of the wavetrain envelope, which is also found to be of sech^2 form.

7.5.d A General IST Scheme*

As was mentioned at the beginning of the section, the IST for the above equations is different from that used for the KdV equation. The relevant schemes were developed by Zakharov and Shabat (1979) and Ablowitz et al. (1974). Here the associated eigenvalue problem is the two-component system of equations

$$v_{1x} = -i\zeta v_1 + qv_2 \tag{7.5.27a}$$

$$v_{2x} = i\zeta v_2 + rv_1 \tag{7.5.27b}$$

where $q = q(x,t)$ and $r = r(x,t)$ are the potentials and ζ is the eigenvalue. Note that if $r = -1$, (7.5.27) reduces to the Schrödinger equation $v_{2xx} + (q + \zeta^2)v = 0$. The associated time-dependent part of the problem takes the general form

$$v_{1t} = Av_1 + Bv_2 \tag{7.5.28a}$$

$$v_{2t} = Cv_1 - Av_2 \tag{7.5.28b}$$

where A, B, and C are various functions of q and r and the spectral parameter ζ. The derivation of the precise forms of A, B, and C for the particular equations considered here is not, in fact, all that difficult. However, we omit the details and refer the reader to the account given in Ablowitz and Segur (1981).

For the Sine–Gordon equation, one sets $q = -r = -u_x/2$ and obtains the scattering problem

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}_x = \begin{bmatrix} -i\zeta & -\frac{1}{2}u_x \\ \frac{1}{2}u_x & i\zeta \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{7.5.29}$$

and

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}_t = \begin{bmatrix} \dfrac{i}{4\zeta}\cos u & \dfrac{i}{4\zeta}\sin u \\ \dfrac{i}{4\zeta}\sin u & -\dfrac{i}{4\zeta}\cos u \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{7.5.30}$$

The reader may easily verify that these equations are the Lax pair for the Sine–Gordon equation since the "integrability condition"

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}_{xt} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}_{tx} \tag{7.5.31}$$

will only be satisfied if (i) $\zeta_t = 0$ (i.e., the deformation is isospectral) and (ii) $u_{xt} = \sin u$.

For the NLS equation the scattering problem is found to be

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}_x = \begin{bmatrix} -i\zeta & u \\ \pm u^* & i\zeta \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{7.5.32}$$

where the asterisk denotes complex conjugate, and

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}_t = \begin{bmatrix} 2i\zeta^2 \pm iuu^* & 2u\zeta + iu_x \\ \mp 2u^*\zeta \pm iu_x^* & -2i\zeta^2 \mp iuu^* \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{7.5.33}$$

The integrability condition for this system is

$$iu_t = u_{xx} \pm 2u^2u^* \tag{7.5.34}$$

It turns out that the case with the minus sign cannot exhibit soliton solutions, whereas the positive-sign case can.

For the mKdV equation the scattering problem is

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}_x = \begin{bmatrix} -i\zeta & u \\ \mp u & i\zeta \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{7.5.35}$$

and

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}_t = \begin{bmatrix} -4i\zeta^3 \pm 2i\zeta u^2 & 4u\zeta^2 + 2i\zeta u_x - u_{xx} \mp 2u^3 \\ \mp 4u\zeta^2 \pm 2i\zeta u_x \pm u_{xx} + 2u^3 & 4i\zeta^3 \mp 2i\zeta u^2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{7.5.36}$$

for which the integrability condition is

$$u_t \pm 6u^2u_x + u_{xxx} = 0 \tag{7.5.37}$$

In this case, soliton solutions can be found for either sign.

A version of the Gelfand–Levitan–Marchenko equation for these systems has been derived, although its solution is highly nontrivial. In addition, the basic eigenvalue problem (7.5.27) can, unlike the Schrödinger equation (7.4.2), have solutions whose eigenvalues form complex conjugate pairs. This leads to an oscillatory type of soliton solution known as *breathers* or *bions*.

7.6 HAMILTONIAN STRUCTURE OF INTEGRABLE SYSTEMS

One of the most important properties of soliton equations is that they are integrable Hamiltonian systems. Here we give a simple account of this which, in turn, provides a link between the properties of the finite-degree-of-freedom integrable systems described in Chapters 2 and 3 and their continuum counterparts discussed here.

7.6.a The Functional Derivative

An important mathematical technique that we require is the *variational* or *functional derivative*. This is easily understood by recalling the variational principle used in Chapter 2. Consider some *functional*, denoted by $F[u]$, of the form

$$F[u] = \int_{x_1}^{x_2} f(x, u, u_x)\, dx \tag{7.6.1}$$

in which f is some function of x, u, and u_x with $u = u(x)$ and $u_x = du/dx$. An obvious example of (7.6.1) is the action integral where f is the Lagrangian and where $u(x)$ is interpreted as $q(t)$. An ordinary derivative (of, for example, $g(x)$) is evaluated by determining the effect of adding a small deviation to the argument of the function, that is, $g(x+\Delta x)$; the functional derivative is evaluated by determining the effect of a small deviation to the *function* $u(x)$, that is, $u(x)+\delta u(x)$ in $f(x, u, u_x)$. In this way we can evaluate the *first variation* of $F[u]$ in the usual way, that is,

$$\delta F[u] = \int_{x_1}^{x_2} f(x, u + \delta u, (u + \delta u)_x)\, dx - \int_{x_1}^{x_2} f(x, u, u_x)\, dx \quad (7.6.2)$$

where the variation $\delta u(x)$ is assumed to vanish at the end points x_1 and x_2. Expanding the first integrand to first order in δu, one obtains

$$\delta F[u] = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial u}\,\delta u + \frac{\partial f}{\partial u_x}(\delta u)_x\right] dx \quad (7.6.3)$$

where $(\delta u)_x = d(\delta u)/dx$. Evaluating the second term by parts (assuming that $\delta u(x_1) = \delta u(x_2) = 0$) gives the standard result

$$\delta F[u] = \int_{x_1}^{x_2} \delta u\left(\frac{\partial f}{\partial u} - \frac{d}{dx}\left(\frac{\partial f}{\partial u_x}\right)\right) dx \quad (7.6.4)$$

The integrand is termed the *variational derivative* and we denote it by $\delta F/\delta u$, that is,

$$\frac{\delta F}{\delta u} = \frac{\partial f}{\partial u} - \frac{d}{dx}\left(\frac{\partial f}{\partial u_x}\right) \quad (7.6.5)$$

Now consider the more general case in which f is a function of any number of derivatives of u, that is,

$$F[u] = \int_{x_1}^{x_2} f(x, u, u_x, u_{xx}, \ldots, u_{nx})\, dx \quad (7.6.6)$$

where $u_{nx} = d^n u/dx^n$. In this case the variational derivative is easily found to be

$$\frac{\delta F}{\delta u} = \sum_{m=0}^{n} (-1)^m \frac{d^m}{dx^m}\left(\frac{\partial f}{\partial u_{mx}}\right) \quad (7.6.7)$$

where the alternating signs have come from repeated integrations by parts to bring $(\delta u)_{mx}$ down to δu. Some simple examples are as follows: For

$$F[u] = \int_{-\infty}^{\infty} (\tfrac{1}{2}u^2)\, dx \quad (7.6.8)$$

we obtain

$$\frac{\delta F}{\delta u} = u$$

and for

$$F[u] = \int_{-\infty}^{\infty} (\tfrac{1}{2}u_x^2)\, dx \quad (7.6.9)$$

we obtain

$$\frac{\delta F}{\delta u} = -u_{xx}$$

A less trivial example is the conserved density in the KdV equation (7.2.27), that is,

$$F[u] = \int_{-\infty}^{\infty} (u^3 + \tfrac{1}{2}u_x^2)\, dx \quad (7.6.10)$$

for which

$$\frac{\delta F}{\delta u} = 3u^2 - u_{xx} \quad (7.6.11)$$

Thus the KdV equation can be expressed as

$$u_t = \frac{\partial}{\partial x}\left(\frac{\delta F}{\delta u}\right) \quad (7.6.12)$$

where F is the functional given in (7.6.10).

Functional derivatives can also be defined in the following way. Consider the functional (7.6.1) in which u is also a function of some parameter α (i.e., $u = u(x; \alpha)$), so that

$$F[u(x, \alpha)] = \int_{x_1}^{x_2} f(x, u(x, \alpha), u_x(x, \alpha))\, dx \quad (7.6.13)$$

The derivative of F with respect to α, using the chain rule, is

$$\frac{dF}{d\alpha} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial u}\frac{\partial u}{\partial \alpha} + \frac{\partial f}{\partial u_x}\frac{\partial u_x}{\partial \alpha}\right) dx \quad (7.6.14)$$

where $\partial u_x/\partial \alpha = d(\partial u/\partial \alpha)/dx$. Again integrating by parts (and assuming vanishing end-point contributions) gives

$$\frac{dF}{d\alpha} = \int_{x_1}^{x_2} \frac{\partial u}{\partial \alpha}\left(\frac{\partial f}{\partial u} - \frac{d}{dx}\left(\frac{\partial f}{\partial u_x}\right)\right) dx \quad (7.6.15)$$

This can be written as

$$\frac{dF}{d\alpha} = \int_{x_1}^{x_2} \frac{\partial u}{\partial \alpha} \frac{\delta F}{\delta u}\, dx \tag{7.6.16}$$

and can be taken as the definition of $\delta F/\delta u$. (This is obviously generalizable to (7.6.6).)

7.6.b Hamiltonian Structure of the KdV Equation

The Hamiltonian nature of (7.6.12) was first demonstrated by Gardner (1971), whose approach we follow here. In his derivation the solution $u = u(x, t)$ to the KdV equation is assumed to be periodic in the interval $(0, 2\pi)$. In this case, u can be represented by the Fourier series

$$u(x, t) = \sum_{k=-\infty}^{\infty} u_k e^{ikx} \tag{7.6.17}$$

where $u_k = u_k(t)$ are a set of complex coefficients. Thinking of the function $F[u]$ given in (7.6.10) as a function of the set of "parameters" u_k, we can use (7.6.16) to write

$$\begin{aligned} \frac{\partial F}{\partial u_k} &= \int_0^{2\pi} \frac{\delta F}{\delta u} \frac{\partial u}{\partial u_k}\, dx \\ &= \int_0^{2\pi} \frac{\delta F}{\delta u} e^{ikx}\, dx \end{aligned} \tag{7.6.18}$$

where we have used (7.6.17). From this relation we obtain the Fourier representation of $\delta F/\delta u$, that is,

$$\frac{\delta F}{\delta u} = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \frac{\partial F}{\partial u_{-k}} e^{ikx} \tag{7.6.19}$$

Using (7.6.12), the equations of motion for the individual u_k are just

$$\frac{du_k}{dt} = \frac{ik}{2\pi} \frac{\partial F}{\partial u_{-k}} \tag{7.6.20}$$

By defining, for $k > 0$, the variables

$$q_k = \frac{u_k}{k}, \qquad p_k = u_{-k}, \qquad H = \frac{i}{2\pi} F \tag{7.6.21}$$

(7.6.20) is exactly in the form of the Hamiltonian system

$$\frac{dq_k}{dt} = \frac{\partial H}{\partial p_k}, \qquad \frac{dp_k}{dt} = -\frac{\partial H}{\partial q_k} \tag{7.6.22}$$

Using the above definitions, one can then go on to define the *Poisson bracket* of two functionals, F and G, as

$$\begin{aligned} [F, G] &= \frac{i}{2\pi} \sum_{k=1}^{\infty} \left(\frac{\partial F}{\partial q_k} \frac{\partial G}{\partial p_k} - \frac{\partial F}{\partial p_k} \frac{\partial G}{\partial q_k} \right) \\ &= \frac{i}{2\pi} \sum_{k=1}^{\infty} \left(k \frac{\partial F}{\partial u_k} \frac{\partial G}{\partial u_{-k}} - k \frac{\partial F}{\partial u_{-k}} \frac{\partial G}{\partial u_k} \right) \\ &= \frac{i}{2\pi} \sum_{k=-\infty}^{\infty} k \frac{\partial F}{\partial u_k} \frac{\partial G}{\partial u_{-k}} \end{aligned} \tag{7.6.23}$$

Using (7.6.19), one can then show that (7.6.23) can be re-expressed as

$$[F, G] = \int_0^{2\pi} \frac{\delta F}{\delta u} \frac{\partial}{\partial x} \left(\frac{\delta G}{\delta u} \right) dx \tag{7.6.24}$$

which can be taken as the definition of the Poisson bracket $[F, G]$. Such a bracket can be shown to satisfy the Jacobi indentity.

We recall from Section 7.2 that the KdV equation has an infinite number of conserved densities of the form

$$F_n[u] = \int T_n\, dx \tag{7.6.25}$$

where the range of integration is $(0, 2\pi)$ for periodic systems or $(-\infty, \infty)$ for systems on the infinite domain. Since the F_n are conserved (i.e., are constants of motion), their Poisson bracket with the Hamiltonian H (defined by (7.6.21) and (7.6.10)) must vanish, that is,

$$\frac{dF_n}{dt} = [F_n, H] = \int \frac{\delta F_n}{\delta u} \frac{\partial}{\partial x} \left(\frac{\delta H}{\delta u} \right) dx = 0 \tag{7.6.26}$$

One can further go on to show that all the F_n commute with each other, that is,

$$[F_n, F_m] = 0 \tag{7.6.27}$$

for all n and m. This is the continuum analogue of the property (2.5.11) of finite-degree-of-freedom integrable Hamiltonians. Thus the KdV equation can be thought of as a completely integrable, infinite-degree-of-freedom Hamiltonian system. Continuing the analogy with finite systems, the result (7.6.27) furthermore suggests that the KdV flow must, it some sense, be confined to an infinite-dimensional torus. Further, remarkable work by Zakharov and Faddeev (1971) identified the canonical transformation of H to action-angle variables in which these variables are expressible in terms of the IST scattering data.

7.6.c Hamiltonian Structure of the NLS Equation

All the soliton equations discussed in this chapter can be shown to be Hamiltonian systems with an associated Poisson bracket (the original restriction of $(0, 2\pi)$ is not required). To conclude we just mention the case of the NLS equation (7.5.26), which we write in the form

$$iu_t = u_{xx} + 2u^2 v \qquad (7.6.28a)$$

$$-iv_t = v_{xx} + 2v^2 u \qquad (7.6.28b)$$

where $v = u^*$. In this case the Hamiltonian is

$$H = -i \int (u^2 v^2 - u_x v_x)\, dx \qquad (7.6.29)$$

and Eqs. (7.6.28) are immediately given by the canonical relations

$$u_t = \frac{\delta H}{\delta v}, \qquad v_t = -\frac{\delta H}{\delta u} \qquad (7.6.30)$$

7.7 DYNAMICS OF NONINTEGRABLE EVOLUTION EQUATIONS

Integrable partial differential equations exhibiting soliton solutions arise surprisingly often in the derivation of realistic physical models of various wave phenomena occurring in one dimension. (For an introductory review see the article by Gibbon (1985).) Equally important are the host of closely related nonlinear evolution equations which are not integrable and have no IST solution. These equations can exhibit behavior ranging from finite time singularities ("blowup") to spatial chaos. It seems likely that an improved understanding of spatiotemporal chaos (and perhaps even fluid dynamical turbulence) will be provided by the study of some of these model equations. This is an enormous topic, but to complete our picture of chaos and integrability (in a sense, these two opposing concepts are brought together in this context) in dynamical systems we briefly mention some of the "canonical" models and their associated behaviors.

7.7.a Self-Focusing Singularities

The nonlinear Schrödinger equation in two dimensions, namely,

$$iu_t + \Delta u + u|u|^2 = 0 \qquad (7.7.1)$$

where Δ denotes the two-dimensional Laplacian, is not soluble by IST and, furthermore, its solutions can exhibit a finite time blowup or a "self-focusing" singularity. This singularity is usually studied with (7.7.1) cast in radially symmetric coordinates, that is,

$$iu_t + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + u|u|^2 = 0 \qquad (7.7.2)$$

with $u = u(r, t)$. That a singularity is possible can be seen from quite simple mechanical principles (see, for example, the paper by Berkshire and Gibbon (1983)). For Eq. (7.7.2), one can write down integrals corresponding to conservation of mass and energy, respectively, namely,

$$M = 2\pi \int_0^\infty |u|^2 r\, dr \qquad (7.7.3)$$

and

$$E = 2\pi \int_0^\infty \frac{1}{2}\left(\frac{1}{2}\left|\frac{\partial u}{\partial r}\right|^2 - \frac{1}{4}|u|^4\right) r\, dr \qquad (7.7.4)$$

The moment of inertia can also be defined, that is,

$$I = 2\pi \int_0^\infty |u|^2 r^3\, dr \qquad (7.7.5)$$

which, in turn, can be shown to be related to the energy integral through

$$\frac{\partial^2 I}{\partial t^2} = 4E \qquad (7.7.6)$$

For suitable choice of initial conditions $u(r, 0)$, one can have $E < 0$ with the consequence that $\ddot{I} < 0$. This leads to a vanishing moment of inertia in finite time, that is, collapse (blowup).

The precise nature of the singularity is difficult to determine and has stimulated a lot of theoretical work—much of it initiated by Zakharov and co-workers (see, for example, Zakharov and Synakh (1976)). This earlier work suggested that the singularity (in time) was algebraic; that is, the solution behaved like $(t - t_*)^{2/3}$ as it approached the blowup time t_*. Subsequent work suggests that the singularity has a much more complicated logarithmic structure (see, for example, the paper by McLaughlin et al. (1986)). Singularities in the NLS equation are not confined to the two-dimensional case. Indeed they can occur for the general equation

$$iu_t + \Delta_d u + u|u|^{2\sigma} = 0 \qquad (7.7.7)$$

where Δ_d is the d-dimensional Laplacian and σ is the order of the nonlinearity. For each dimension d there will be a σ for which a self-focusing singularity can be found. For example, in one dimension, the quartic NLS equation

$$iu_t + u_{xx} + u|u|^4 = 0 \qquad (7.7.8)$$

exhibits blowup. Early work by Zakharov and Synakh (1976) suggested an algebraic blowup going as $(t - t_*)^{4/7}$—but again it seems that the singularity is more complicated than this.

7.7.b The Zakharov Equations

In many physical contexts, one is interested in modeling the interaction of long waves with short waves. For example, in the theory of Langmuir waves in plasma physics the interaction between a rapidly oscillating electric field (denoted by u) and a slowly varying ion density (denoted by v) takes the form

$$iu_t + u_{xx} + uv = 0 \qquad (7.7.9a)$$

$$v_{xx} - \frac{1}{C^2} v_{tt} = -\beta(|u|^2)_{xx} \qquad (7.7.9b)$$

These are known as the *one-dimensional Zakharov equations*. Equations such as (7.7.9) also arise in models of excitations in idealized DNA chains proposed by Davydov (1979). An important feature of Eqs. (7.7.9) is the limit of large C in (7.7.9b). In this case, one has $v_{xx} = -\beta(|u|^2)_{xx}$; in other words, v is directly proportional to u and (7.7.9a) reduces to the integrable, soliton-bearing, NLS equation. However, the full equations are not integrable and cannot be solved by IST. Numerical studies show them to be capable of very complicated behavior. By contrast, if Eqs. (7.7.9) are reduced to a "one-wave" form by factorizing the operator $(\partial^2/\partial x^2 - (1/C^2)(\partial^2/\partial t^2))$ into $(\partial/\partial x + (1/C)(\partial/\partial t))\,(\partial/\partial x - (1/C)(\partial/\partial t))$, one obtains

$$iu_t + u_{xx} + uv = 0 \qquad (7.7.10a)$$

$$v_x + \frac{1}{C} v_t = -\beta(|u|^2)_x \qquad (7.710b)$$

which are integrable and have an IST solution. Particularly important is the physically more realistic, two-dimensional version of (7.7.9), namely,

$$iu_t + \Delta u + uv = 0 \qquad (7.7.11a)$$

$$\Delta v - \frac{1}{C^2} v_{tt} = -\beta\Delta(|u|) \qquad (7.7.11b)$$

In this case the large C limit reduces (7.7.11a) to the 2-D NLS equation which can display blowup. Numerical simulations of (7.7.11) display a rich behavior including near blowup followed by "burnout." (A good review of these phenomena, in the plasma context, is given by Goldman (1984).)

7.7.c Coherence and Chaos

An important model system for the study of spatiotemporal chaos is the damped and driven one-dimensional Sine–Gordon equation, namely,

$$u_{tt} - u_{xx} + \sin u = \Gamma\cos(\omega t) - \alpha u_t \qquad (7.7.12)$$

This equation is typically studied with periodic boundary conditions, that is, $u(x + L, t) = u(x, t)$. Detailed numerical studies of (7.7.12) have been carried out by Bishop et al. (1983) and others. A rich range of behaviors is found as the driving and damping parameters, Γ and α, respectively, are varied. Typically, though, the spatial structure of the solutions tends to be quite coherent; that is, they just exhibit a few well-defined spatial modes. What is especially striking is that this spatial coherence can be maintained even when the temporal evolution becomes chaotic. It seems clear that the soliton structure of the unperturbed ($\Gamma = \alpha = 0$) system can be quite robust.

In a variety of problems in statistical mechanics and fluid dynamics, two equations frequently occur—these are the Ginzburg–Landau and closely related Newell–Whitehead equations, respectively. A typical form for these equations (in one dimension) is

$$u_t = \alpha u_{xx} + \beta u - \gamma|u|^2 u \qquad (7.7.13)$$

where u usually represents the (complex) amplitude of some unstable mode and α, β, γ are adjustable parameters that can be complex. This equation, which can be generalized to higher dimensions, can exhibit an enormous variety of behaviors, ranging from the coherent to the chaotic, depending on the choice of parameter values. The properties of such equations are a most active area of research.

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