Scuola di Dottorato Leonardo da Vinci – a.a. 2012/13

LASER: CARATTERISTICHE, PRINCIPI FISICI, APPLICAZIONI

Version 4.1 – June 13 – http://www.df.unipi.it/~fuso/dida

Part 2
Inadequacy of the classical approach: light/matter interaction and conventional light sources (blackbody)

OUTLOOK

- Light/matter interaction in purely classical terms:
  - the Lorenz oscillator for the Thomson's atom
  - absorption and dispersion in a dielectrics
  - the behavior of a metal (conductors)

- Some anticipations of quantum mecahnics:
  - the photons and their properties

- "Conventional" (thermal) soruces of light:
  - the black-body problem
  - features of conventional (non-laser!) light

Main objectives:

to show that classical approaches cannot explain the Amplification of light, needed for laser operation

To show that conventional light sources have properties far from those of the laser (a laser cannot be built starting from a lamp!)

(Additional objective: to spend a few words on light/matter interaction)

LIGHT/MATTER INTERACTION

1. Most laser applications exploit laser/matter interaction
2. The ability to amplify the radiation inherent to laser operation requires interaction with a material (active medium)

⇩

Light/matter radiation is an unavoidable topic witin our context

In the optical (visible & nearby) range the interaction involves only electrons

Roughly speaking:
Electrons are put in oscillations by the (oscillating) electric field brought by the e.m. wave
Their oscillation is the source for secondary radiation (e.g., reflected and transmitted)

The behavior depends on the nature of the material:
electrons are "free" in metals, "bound" in dielectrics
→ different macroscopic response to radiation for metal (conductors) and dielectrics

In case of metals we will neglect effects due to penetration ("skin effects") and the related Joule dissipation of energy
We will neglect effects relevant in nanosized (or low-dimensionality) systems, e.g., plasmons in noble metals

BACK TO WAVEFUNCTIONS

How to conveniently include the effects of propagation into a material in the wavefunction?

Example: a plane wave moving along *x* and polarized along *y*

$$\vec{E}(x,t) = E_0 e^{i(kx-\omega t+\phi)}\hat{y}$$

ω won't change
(no physical reason producing an inelastic interaction, so far)

k will change
(to account for propagation velocity and dispersion)

Phase velocity in the vacuum (according to Maxwell and Einstein)

$$v_{fase,vacuum} = \frac{1}{\sqrt{\varepsilon_0\mu_0}} = c$$

Definition of wavevector *k*

$$k_{vacuum} = \frac{\omega}{v_{fase,vacuum}} = \frac{\omega}{c} = \frac{2\pi}{\lambda}$$

Accounting for the material (according to Maxwell)

$$v_{fase,mater} = \frac{1}{\sqrt{\varepsilon_R\varepsilon_0\mu_0}} = \frac{c}{\sqrt{\varepsilon_R}} = \frac{c}{n}$$

Wavevector in the material

$$k_{mater} = \frac{\omega}{v_{fase,mater}} = \frac{\omega}{c}n = k_{vacuum}n = \frac{2\pi}{\lambda}n$$

COMPLEX REFRACTIVE INDEX

It is convenient to introduce a **complex** refractive index: *n' = n + iα*

$$k_{mater} = k_{vacuum} n' \rightarrow$$

$$\vec{E}(x,t) = E_0 e^{i(k_{mater}x - \omega t + \phi)} \hat{y} = E_0 e^{i(nk_{vacuum}x - \omega t + \phi)} \hat{y} =$$

$$= E_0 e^{i(nk_{vacuum}x + i\alpha x - \omega t + \phi)} \hat{y} = E_0 e^{-\alpha x} e^{i(nk_{vacuum}x - \omega t + \phi)} \hat{y}$$

The imaginary part of *n* accounts for **absorption**

The real part of *n* accounts for **dispersion**

Accordingly, it is convenient to introduce a **complex** relative dielectric constant:

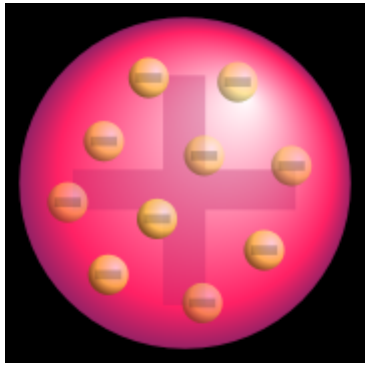
$$\varepsilon_r = n'^2 = n^2 - \alpha^2 + 2in\alpha$$

$$\varepsilon_r = \varepsilon_{r1} + i\varepsilon_{r2}$$

$$\varepsilon_{r1} = n^2 - \alpha^2; \quad \varepsilon_{r2} = 2in\alpha$$

DIELECTRICS

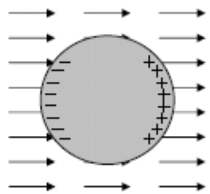
The simplest dielectrics:
"Thomson model" (for a hydrogen atom):
- positive charge (proton) homogeneously distributed in a sphere
- pointlike electron can move in such a positive charge distribution



Assuming a_0 as the radius of the sphere (and using Gauss' theorem):
$\rho = 3e/(4\pi a_0^3) \rightarrow E_{int}(r) = Q_{int}/(4\pi r^2) = er/(4\pi a_0^3)$

Note: an electric dipole is involved in the interaction

Displaced from the equilibrium position (r = 0) the electron feels a **restoring** force
$F(r) = -eE_{int}(r) = -kr$
For instance, applying an external steady electric field $\mathbf{E_0}$
the new equilibrium position is $\mathbf{r_0} = e\mathbf{E_0}/k$
→ an electric **dipole** is produced with $\mathbf{p_0} = e\mathbf{r_0}$
→ a polarization field is produced $\mathbf{P_0} = N\mathbf{p_0} = ce_0\mathbf{E_0}$
→ (static polarizability: $\chi = Ne/(k\varepsilon_0)$)



$$\vec{P} = \chi_e \varepsilon_0 \vec{E}$$

When bound electrons are present in the matter, a restoring (i.e., **elastic**) force is felt by the electrons in the interaction with an external electric field

"LORENZ" MODEL

Due to the elastic force, bound electrons behave like harmonic oscillators with **proper** angular frequency $\omega_0=(k/m)^{1/2}$
It is reasonable to include a friction (viscous) force $-\beta \boldsymbol{v}$ playing against the electron motion

Assuming a driving electric force at frequency ω, the equation of motion reads:

$$m\frac{d^2x}{dt^2}+\beta\frac{dx}{dt}+kx=Fe^{i\omega t}$$

Note: since the atom is much smaller than the wavelength, the space-dependence of the driving force can be neglected, that is within the atom the driving force is uniform at any instant (also called **dipole approximation**)

smorzamento collisionale e quello proporzionale alla sola x il termine armonico. Dividendo per la massa dell'elettrone m, ponendo $b/m=h$ e $k/m=w_0^2$ (w_0 = frequenza propria di risonanza del sistema), si ottiene:

Solution of the equation of motion

$$\frac{d^2x}{dt^2}+\eta\frac{dx}{dt}+\omega_0{}^2x=\frac{F}{m}\exp(i\omega t)$$

La soluzione dell'equazione differenziale deve essere di tipo periodico. Potremo quindi pensare che sia del tipo:

$$x(t) = x_0\exp(i\omega t)$$

Sostituiamo le derivate prima e seconda della nostra soluzione nell'equazione differenziale:

$$-\omega^2x_0\exp(i\omega t)+i\eta\omega x_0\exp(i\omega t)+\omega_0{}^2x_0\exp(i\omega t)=\frac{F}{m}\exp(i\omega t)$$

LORENZ II

$$-\omega_2 + i\eta\omega + \omega_0^{\,2} = \frac{F}{m x_0}$$

da cui:

$$x_0 = \frac{F}{m}\left(\frac{1}{\omega_0^{\,2} - \omega^2 + i\eta\omega}\right)$$

Razionalizzando:

$$x_0 = \frac{F}{m}\left[\frac{\omega_0^{\,2} - \omega^2 - i\eta\omega}{\left(\omega_0^{\,2} - \omega^2\right)^2 + \eta^2\omega^2}\right] = \frac{F}{m}\left[\frac{\omega_0^{\,2} - \omega^2}{\left(\omega_0^{\,2} - \omega^2\right)^2 + \eta^2\omega^2} - i\,\frac{\eta\omega}{\left(\omega_0^{\,2} - \omega^2\right)^2 + \eta^2\omega^2}\right]$$

che rappresenta un'ampiezza d'oscillazione complessa. Se ricordiamo che la forza elettrica F è data dal prodotto della carica e dell'elettrone per il campo elettrico E della radiazione, avremo:

$$x_0 = \frac{eE}{m}\left[\frac{\omega_0^{\,2} - \omega^2}{\left(\omega_0^{\,2} - \omega^2\right)^2 + \eta^2\omega^2} - i\,\frac{\eta\omega}{\left(\omega_0^{\,2} - \omega^2\right)^2 + \eta^2\omega^2}\right]$$

(Complex) amplitude of oscillation
→ Induced dipole (complex) ≈ ex_0

Il vettore polarizzazione P è legato allo spostamento x_0 dalla relazione:

$$P = nex_0$$

essendo n il numero di cariche (elettroni) per unità di volume ed e la carica dell'elettrone. Ed essendo inoltre:

Polarization field
(remember: $\mathbf{P} = N\mathbf{p}$ $\chi\mathbf{E} = (\varepsilon_r - 1)\mathbf{E}$)

$$\overline{P} = (\varepsilon_r - 1)\varepsilon_0\overline{E}$$

si avrà:

$$\varepsilon_r - 1 = \frac{P}{\varepsilon_0 E} = \frac{nex_0}{\varepsilon_0 E}$$

ovvero:

$$\varepsilon_r = 1 + \frac{ne^2}{\varepsilon_0 m}\left[\frac{\omega_0^{\,2} - \omega^2}{\left(\omega_0^{\,2} - \omega^2\right)^2 + \eta^2\omega^2} - i\,\frac{\eta\omega}{\left(\omega_0^{\,2} - \omega^2\right)^2 + \eta^2\omega^2}\right]$$

(Complex) relative dielectric constant

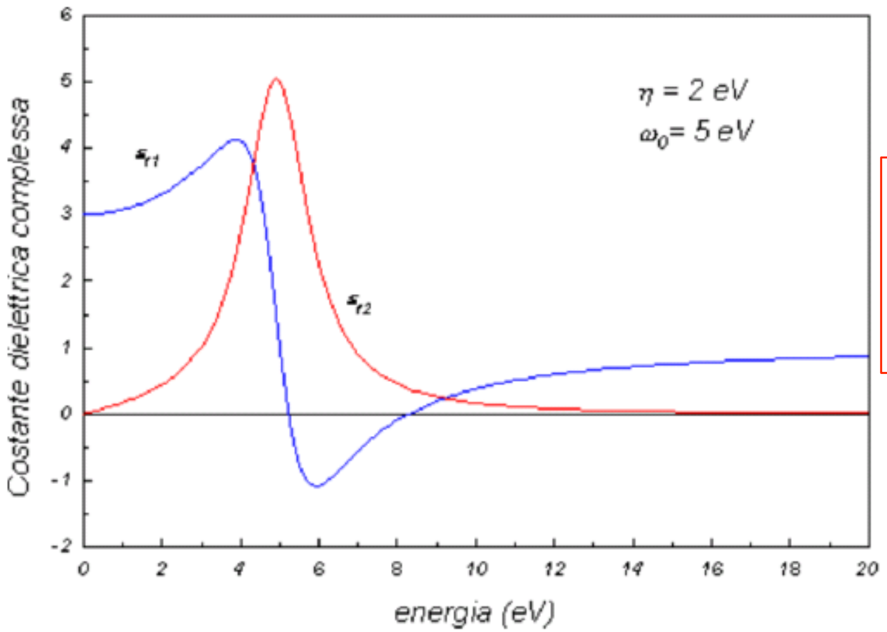
DIELECTRIC CONSTANT IN DIELECTRICS

Costante dielettrica complessa:

$$\varepsilon_r(\omega) = \varepsilon_{r1}(\omega) - i\varepsilon_{r2}(\omega)$$

$$\varepsilon_{r1}(\omega) = 1 + \frac{ne^2}{\varepsilon_0 m}\left[\frac{\omega_0^2 - \omega^2}{(\omega_0^2 - \omega^2)^2 + \eta^2\omega^2}\right]$$

$$\varepsilon_{r2}(\omega) = \frac{ne^2}{\varepsilon_0 m}\left[\frac{\eta\omega}{(\omega_0^2 - \omega^2)^2 + \eta^2\omega^2}\right]$$



"Dispersive" behavior for the real part (ε_{r1})
Lorentzian behavior for the imaginary part (ε_{r2})

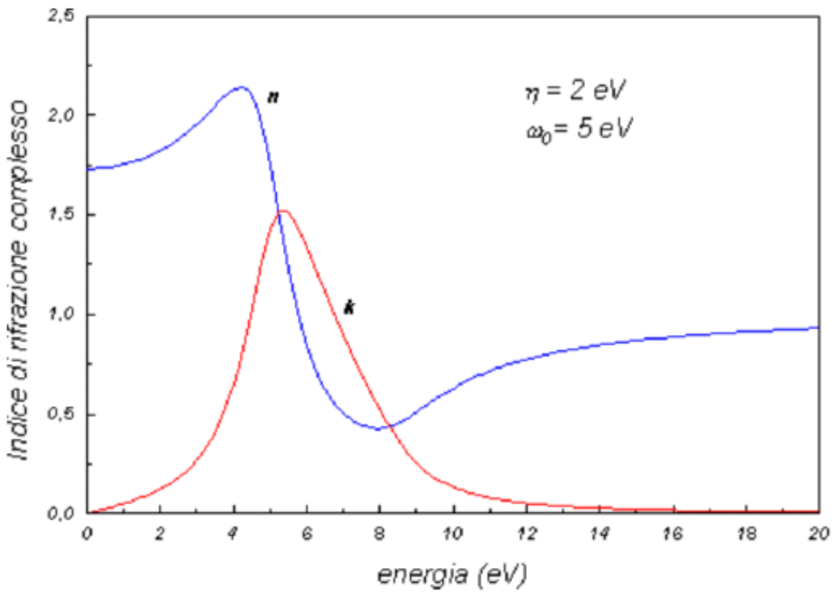
[Resonance at $\omega = \omega_0$]

La funzione dielettrica, qui desunta per una singola frequenza di risonanza del sistema assorbente, può venire generalizzata al caso di un numero qualunque di tali frequenze, diventando:

$$\varepsilon_r = 1 + \sum_i \frac{n_i e^2}{\varepsilon_0 m}\left[\frac{\omega_{0i}^2 - \omega^2}{(\omega_{0i}^2 - \omega^2)^2 + \eta_i^2\omega^2} - j\frac{\eta_i\omega}{(\omega_{0i}^2 - \omega^2)^2 + \eta_i^2\omega^2}\right]$$

REFRACTIVE INDEX IN DIELECTRICS

It is convenient to introduce a <u>complex</u> refractive index $n' = n + i\alpha$
con: $\varepsilon_{r1}=n^2-\alpha^2$ $\varepsilon_{r2}=2n\alpha$



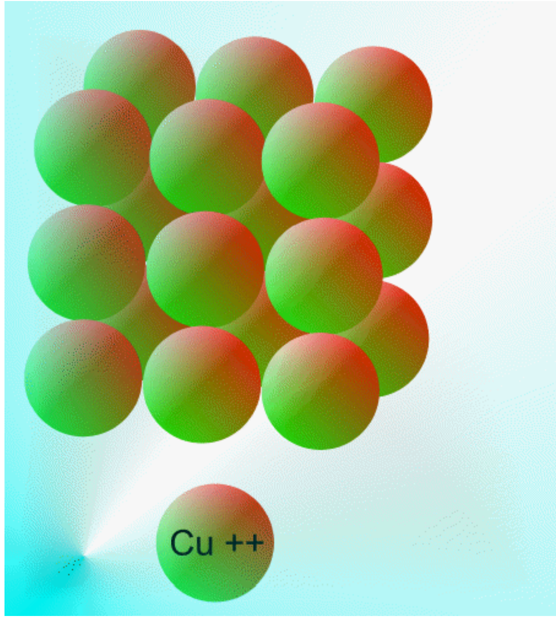
n is responsible for "dephasing" of the wave (so-called dispersion)

α is responsible for **absorption**:
$E(x) \sim E_0 e^{-\alpha x}$

In a classical picture, a dielectrics can only **absorb** radiation (with a resonance at the proper frequency and a Lorentzian-like shape, width depending on η, i.e., on the viscous friction coefficient)

No amplification is possible!!

METALS (CONDUCTORS)



Metals own "free" electrons → not a bound system
→ $\omega_0 = 0$!!

Note: actually electrons are not fully free because of resistivity
→ there is a *friction force due to resistivity*

We found in dielectrics:

$$\varepsilon_r = 1 + \frac{ne^2}{\varepsilon_0 m}\left[\frac{\omega_0^2 - \omega^2}{\left(\omega_0^2 - \omega^2\right)^2 + \eta^2\omega^2} - i\frac{\eta\omega}{\left(\omega_0^2 - \omega^2\right)^2 + \eta^2\omega^2}\right]$$

We can put $\omega_0 = 0$

We can define the **plasma frequency** ω_P

$$\omega_P = \frac{ne^2}{\varepsilon_0 m}$$

$$\varepsilon_r(\omega) = 1 - \frac{ne^2}{\varepsilon_0 m}\left[\frac{1}{\omega^2 + \eta^2} + i\frac{\eta}{\omega\left(\omega^2 + \eta^2\right)}\right] =$$

$$= 1 - \omega_P^2 \frac{1}{\omega}\frac{\omega + i\eta}{(\omega + i\eta)(\omega - i\eta)} = 1 - \omega_P^2 \frac{1}{\omega^2 - i\eta\omega}$$

No resonance occurs in metals

Note: we are not considering here "plasmon" effects occurring in noble metal nanostructures

DIELECTRIC CONSTANT FOR A METAL

We can define a complex dielectric constant ε_r as we did in the dielectrics

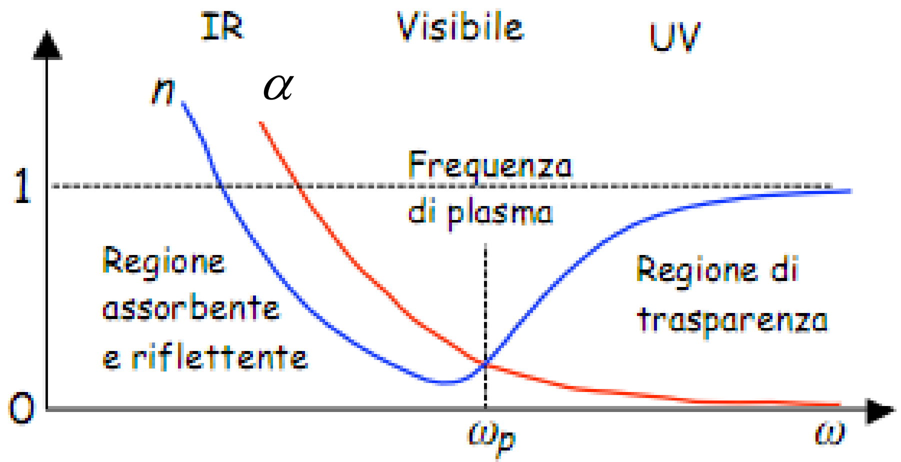
Real part: $\varepsilon_{r1} = n^2 - \alpha^2 = 1 - \frac{\omega_P^2}{\omega^2 + \eta^2}$

This is negative for $\omega < \omega_P$

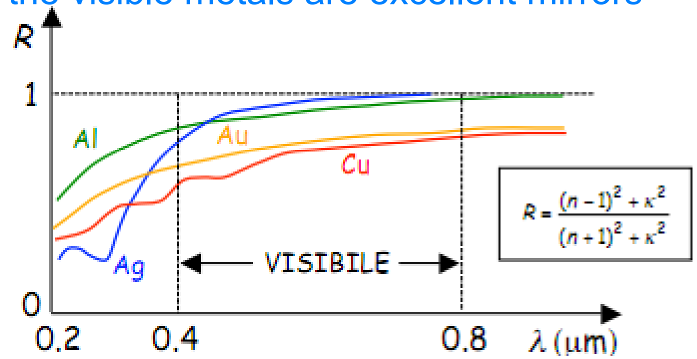
Imag part: $\varepsilon_{r2} = 2n\alpha = \frac{\omega_P^2}{\omega^2 + \eta^2} \frac{\eta}{\omega}$

In typical metals
$\omega_P \approx 10^{15}$-10^{16} rad/s
$\eta \approx 1/\tau_{coll\,(Drude)} \approx 10^{13}$-$10^{14}$ Hz

For $\omega > \omega_P$ metals get transparent
(this however occurs outside the visible range, e.g., in the x-ray frequency range)



In the visible metals are excellent mirrors



No amplification is possible in metals, too!!

“CONVENTIONAL” SOURCES

We have seen that the classical approach does not lead to amplification in the light/ matter interaction

We’ll see now the other side of the topic, that is why a “conventional” (non laser) source cannot produce laser-like light

Unfortunately, this will require to introduce some quantum mechanics (strange, being conventional inherently classical), but this simplifies a lot the treatment)

Our (simple) model for conventional sources: thermal sources of light

Roughly speaking:
temperature → thermal motion → oscillation of electrons → emission

The model fits perfectly with bulb (filament) lamps as well as to heated material (e.g., the sun), but it can be shown that it applies also to other non-laser sources such as, discharge lamps, LEDs, …

⇩

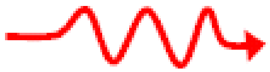
The black-body problem
(very important in the passage from classical to quantum mechanics, across XIX and XX centuries)

THE CONCEPT OF PHOTON

"Particle of light"

Photon

$E=h\nu$



Planck's constant:
$h=6.6\ 10^{-16}$ eV-s

$\lambda\nu=c$
λ=wavelength
ν=frequency

Speed of light:
$c=3\ 10^{8}$ m/s

Wavevector
$k=2\pi/\lambda$

1 eV
$2.4\ 10^{14}$ Hz
1.25 µm
502 cm^{-1}

(photons are the the eigenstates of harmonic oscillators, we'll mention later on)

The photon (a purely QM concept inspired by Einstein) can be seen as a **massless** particle owning:

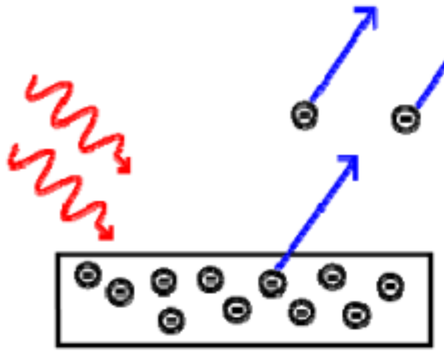
- energy $E = hn$
- impulse $p = hn /c$
- angular momentum $L = \pm \hbar$

Energy is conveniently measured in units of electronvolt
1 eV = $1.6\text{x}10^{-19}$ J

Energy [eV] = 1240/l[nm]
→ In the visible range E ~ 1.5-3 eV

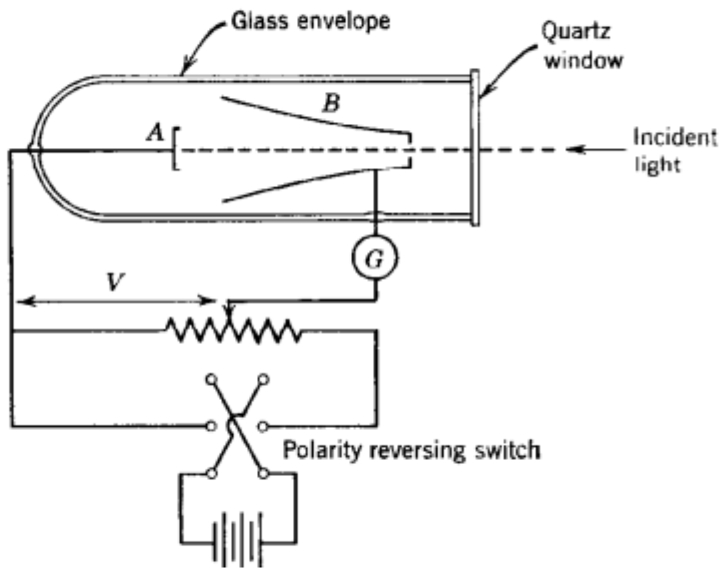
Laser light can be thought as a flux of photons having all similar properties each other

HYSTORICAL REMARKS ON PHOTONS I



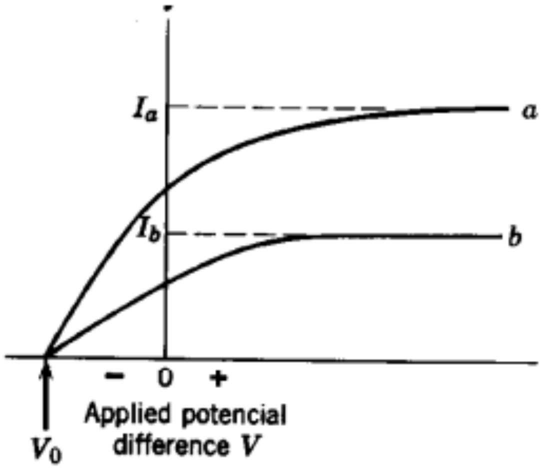
2-2 THE PHOTOELECTRIC EFFECT

It was in 1886 and 1887 that Heinrich Hertz performed the experiments that first confirmed the existence of electromagnetic waves and Maxwell's electromagnetic theory of light propagation. It is one of those fascinating and paradoxical facts in the history of science that in the course of his experiments Hertz noted the effect that Einstein later used to contradict other aspects of the classical electromagnetic theory. Hertz discovered that an electric discharge between two electrodes occurs more readily when ultraviolet light falls on one of the electrodes. Lenard, following up some experiments of Hallwachs, showed soon after that the ultraviolet light facilitates the discharge by causing electrons to be emitted from the cathode surface. The ejection of electrons from a surface by the action of light is called the *photoelectric effect*. It is the phenomenon underlying the operation of the solar cells being developed to convert thermal energy received from the sun directly into electrical energy.



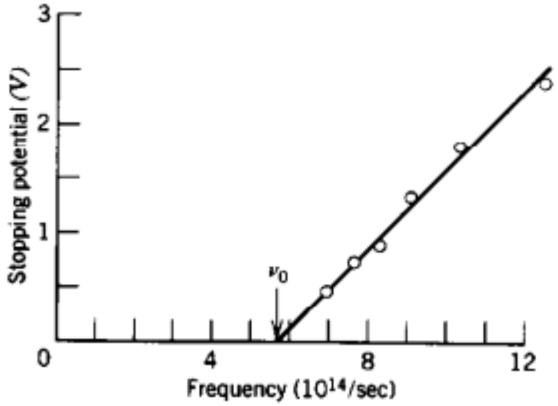
Photoelectric effect

HYSTORICAL REMARKS ON PHOTONS II



Stopping potential independent of radiation intensity

Figure 2-2 Graphs of current *i* as a function of potential difference *V* from data taken with the apparatus of Figure 2-1. The applied potential difference *V* is called positive when the cup *B* in Figure 2-1 is positive with respect to the photoelectric surface *A*. In curve *b* the incident light intensity has been reduced to one-half that of curve *a*. The stopping potential V_0 is independent of light intensity, but the saturation currents I_a and I_b are directly proportional to it.



Stopping potential linearly dependent on radiation frequency

Figure 2-3 The stopping potential at various frequencies for sodium. The points show Millikan's data, except that the correction mentioned in the caption to Figure 2-1 has been recalculated using a recent measurement of the contact potential. The cutoff frequency ν_0 is 5.6×10^{14} Hz.

The photon deliver energy proportional to the radiation frequency

HYSTORICAL REMARKS ON PHOTONS III

There are three major features of the photoelectric effect that cannot be explained in terms of the classical wave theory of light:

1. Wave theory requires that the oscillating electric vector **E** of the light wave increase in amplitude as the intensity of the light beam is increased. Since the force applied to the electron is $e\mathbf{E}$, this suggests that the *kinetic energy* of the photo-electrons should also increase as the light beam is made more intense. However, Figure 2-2 shows that K_{max}, which equals eV_0, *is independent of the light intensity.* This has been tested over a range of intensities of 10^7.

2. According to the wave theory the photoelectric effect should occur for any frequency of the light, provided only that the light is intense enough to give the energy needed to eject the photoelectrons. However, Figure 2-3 shows that there exists, for each surface, *a characteristic cutoff frequency* ν_0. *For frequencies less than* ν_0, *the photoelectric effect does not occur, no matter how intense the illumination.*

3. If the energy acquired by a photoelectron is absorbed from the wave incident on the metal plate, the "effective target area" for an electron in the metal is limited, and probably not much more than that of a circle having about an atomic diameter. In the classical theory the light energy is uniformly distributed over the wave front. Thus, if the light is feeble enough, there should be a measurable time lag, which we shall estimate in Example 2-1, between the time when light starts to impinge on the surface and the ejection of the photoelectron. During this interval the electron should be absorbing energy from the beam until it has accumulated enough to escape. *However, no detectable time lag has ever been measured.* This disagreement is particularly striking when the photoelectric substance is a gas; under these circumstances collective absorption mechanisms can be ruled out and the energy of the emitted photoelectron must certainly be soaked out of the light beam by a single atom or molecule.

The wave approach (alone) cannot explain the experimental findings

EINSTEIN'S INTERPRETATION (1905)

Einstein assumed that such a bundle of energy is initially localized in a small volume of space, and that it remains localized as it moves away from the source with velocity c. He assumed that the energy content E of the bundle, or photon, is related to its frequency ν by the equation

$$E = h\nu \tag{2-2}$$

He also assumed that in the photoelectric process one photon is completely absorbed by one electron in the photocathode.

When the electron is emitted from the surface of the metal, its kinetic energy will be

$$K = h\nu - w \tag{2-3}$$

where $h\nu$ is the energy of the absorbed incident photon and w is the work required to remove the electron from the metal. This work is needed to overcome the attractive fields of the atoms in the surface and losses of kinetic energy due to internal collisions of the electron. Some electrons are bound more tightly than others; some lose energy in collisions on the way out. In the case of loosest binding and no internal losses, the photoelectron will emerge with the maximum kinetic energy, K_{max}. Hence

$$K_{max} = h\nu - w_0 \tag{2-4}$$

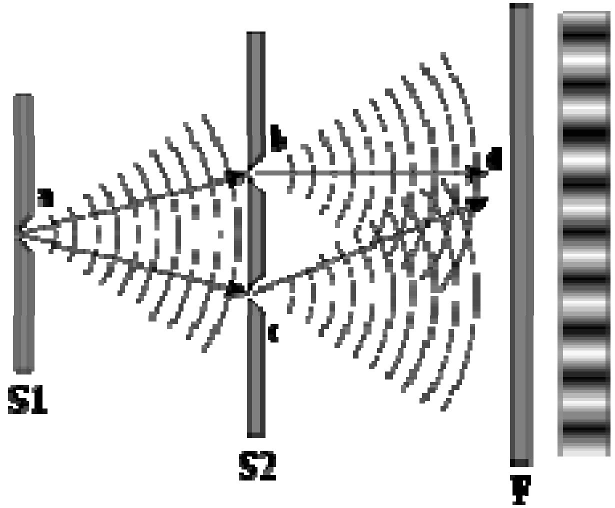
where w_0, a characteristic energy of the metal called the *work function*, is the minimum energy needed by an electron to pass through the metal surface and escape the attractive forces that normally bind the electron to the metal.

Consider now how Einstein's photon hypothesis meets the three objections raised against the wave theory interpretation of the photoelectric effect. As for objection 1 (the lack of dependence of K_{max} on the intensity of illumination), there is complete agreement of the photon theory with experiment. Doubling the light intensity merely doubles the number of photons and thus doubles the photoelectric current; it does *not* change the energy $h\nu$ of the individual photons or the nature of the individual photoelectric process described by (2-3).

The photoelectric effect was one of the most relevant experimental proof of the photon
Also: Compton effect (x-rays) and creation/annihilation (gamma rays)

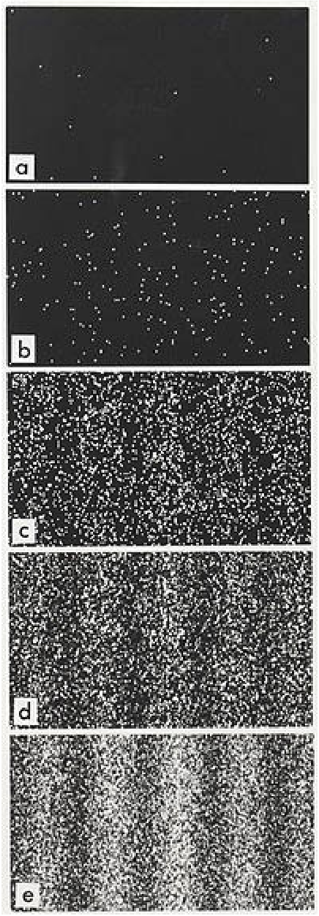
EINSTEIN'S INTERPRETATION (1905)

Double slit (Young) experiment

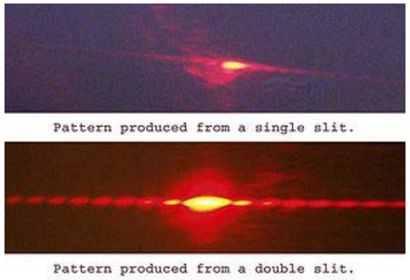


Can be done with photons or particles (e.g. electrons) leading to similar results (characteristic interference fringes)

With electrons

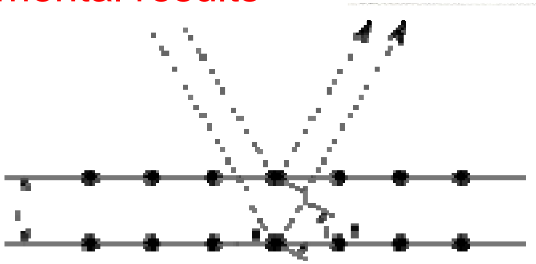


With photons



The dual properties of light (wave and particles) and matter (particle and waves) well confirmed by many experimental results

Also: Bragg diffraction with x-ray and particles, electron microscopy, etc.



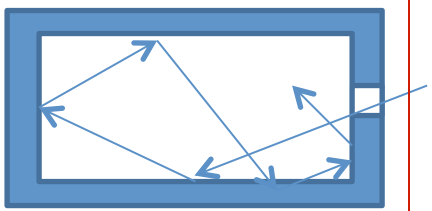
POSING THE BLACKBODY PROBLEM

We have seen that materials can absorb radiation
The absorbed radiation can also be re-emitted (for a general principle of energy balance)

Black-body: a material system able to perfectly absorb and re-emit all the radiation
→ We can assume equilibrium between matter and radiation

A **black body** is an idealized physical body that absorbs all incident electromagnetic radiation, regardless of frequency or angle of incidence.

A black body in thermal equilibrium (that is, at a constant temperature) emits electromagnetic radiation called black-body radiation. The radiation is emitted according to Planck's law, meaning that it has a spectrum that is determined by the temperature alone



The model:
A box with perfectly absorbing walls:
The radiation entering the box cannot come back out
That is
Radiation can leave the box from a hole so small that the systme is not perturbed

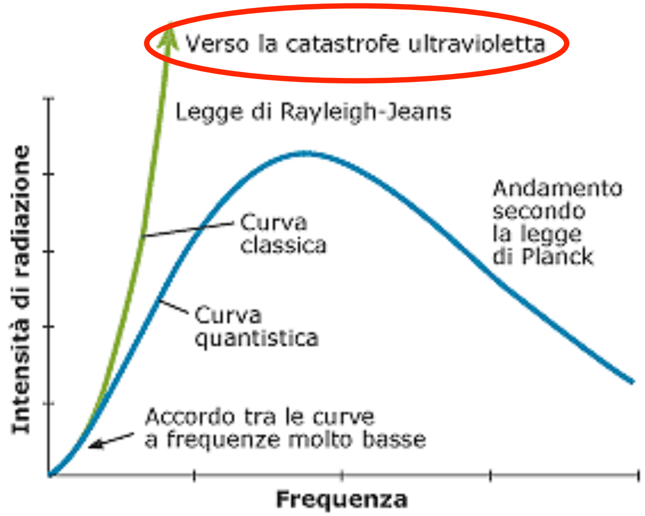
The absorbed spectrum is equal to the emitted spectrum
(this is an ideal blac-body, otherwise we will have a more realistic *grey-body*)

Question: how the radiation/matter equilibrium can be established?

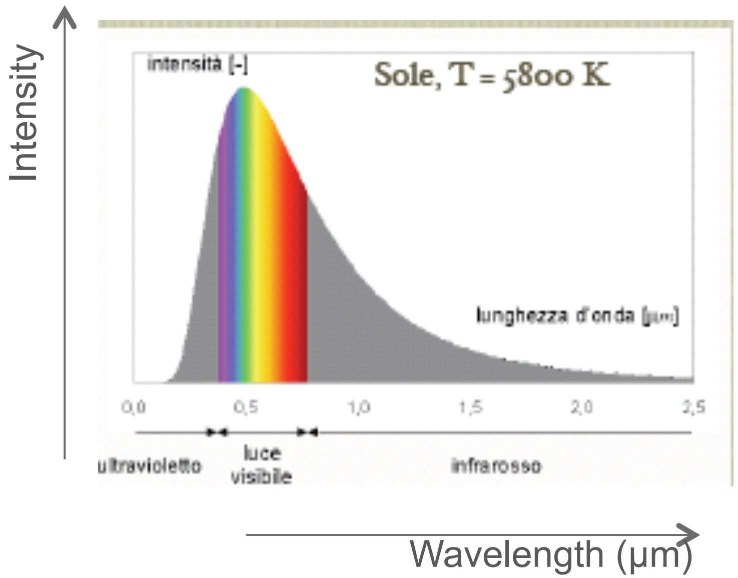
HYSTORICAL REMARKS

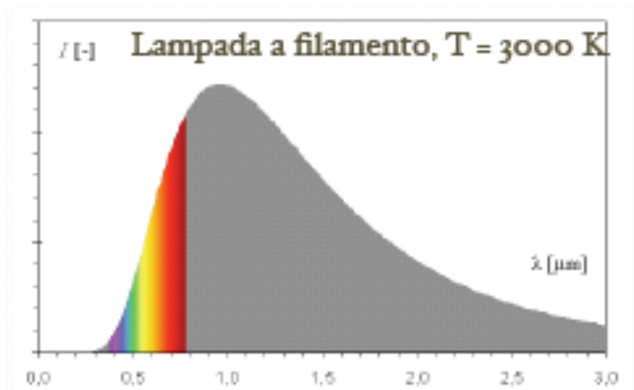
The black-body problem is one of the major conceptual steps leading from classical to quntum mechanics → Kirchoff, Planck, Einstein worked on that

In a more practical view, a bulb (filament) lamp can be rather well approximated by a black-body
→ What is the emission spectrum of an heated material?



Emission spectrum





BASIS OF THE SOLUTION

The easiest way to solve the BB problem is to assume a *gas of photons* inside the box, in thermal equilibrium with the box walls (the matter)

We want to know the energy corresponding to this configuration

The problem is hence similar to that of massive particles (e.g., a gas) in thermal equilibrium with a heat source: in that case, Maxwell-Boltzmann statistics holds and the energy is obtained through the *equipartition theorem*: each gas particle contribute to an energy with a term ≈ $K_B T$

No equipartition holds in this case!!

Many differences exist indeed with respect to massive gas (i.e., classical systems):

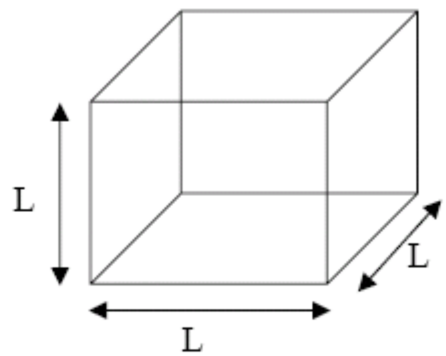
1. When put in a box, radiation must "fit" the box itself → quantized modes
2. The energy of a photon is not $K_B T$ since quantum statistics hold
3. When calculating the total energy inside the box, care must be put into considering the above points

RADIATION (OR PARTICLES) IN A BOX

This is a problem which is at the real basis of many quantum mechanical treatments.

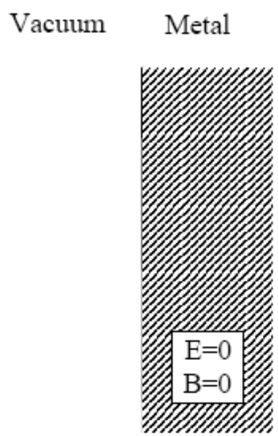
We will see it again many other times!

Problem:
We want to determine the "modes of radiation" which are allowed in a box having perfectly reflecting walls (here, it is our black-body, later on it will be something else)



Boundary conditions at a metal:

Inside a perfect metal, E=0 and B=0 by definition.



In homework, you proved:

$$\vec{E}_{\parallel 2} = \vec{E}_{\parallel 1} \quad \Rightarrow \boxed{\vec{E}_{\parallel} = 0}$$

$$\vec{D}_{\perp 1} - \vec{D}_{\perp 2} = \rho_s$$

$$\vec{H}_{\parallel 2} - \vec{H}_{\parallel 1} = \vec{J}_s$$

$$\vec{B}_{\perp 1} = \vec{B}_{\perp 2} \quad \Rightarrow \boxed{\vec{B}_{\perp} = 0}$$

Standing waves

$$\vec{E}_{incid}(x,t) = E_0 e^{i(kx-\omega t)}\hat{y}$$

$$\vec{E}_{rifl}(x,t) = -E_0 e^{i(-kx-\omega t)}\hat{y}$$

$$\vec{E}_{totale}(x,t) = \vec{E}_{incid}(x,t) + \vec{E}_{rifl}(x,t) =$$

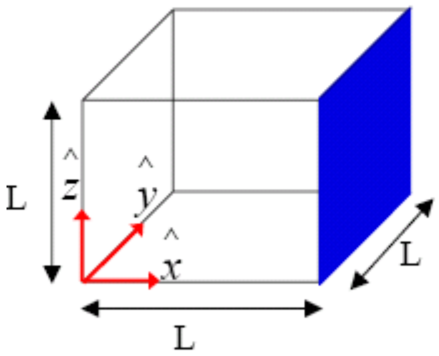
$$= E_0\left(e^{i(kx-\omega t)} - e^{i(-kx-\omega t)}\right)\hat{y} =$$

$$= 2E_0 i e^{-i\omega t}\sin(kx)\hat{y}$$

Note: the real parts are physically relevant…

QUANTIZATION OF MODES

Boundary conditions:



$$\vec{E}(\vec{r},t) = \mathrm{Re}\left[E_0\, \hat{z}\, e^{i(k\hat{x}\cdot\vec{r}-\omega t)}\right] + \mathrm{Re}\left[E_0\, \hat{z}\, e^{i(k\hat{x}\cdot\vec{r}+\omega t)}\right]$$

The plane x=L:

$$\vec{E}(\vec{r},t) = \mathrm{Re}\left[E_0\, \hat{z}\, e^{i(k\hat{x}\cdot\vec{r}-\omega t)}\right] + \mathrm{Re}\left[E_0\, \hat{z}\, e^{i(k\hat{x}\cdot\vec{r}+\omega t)}\right]$$

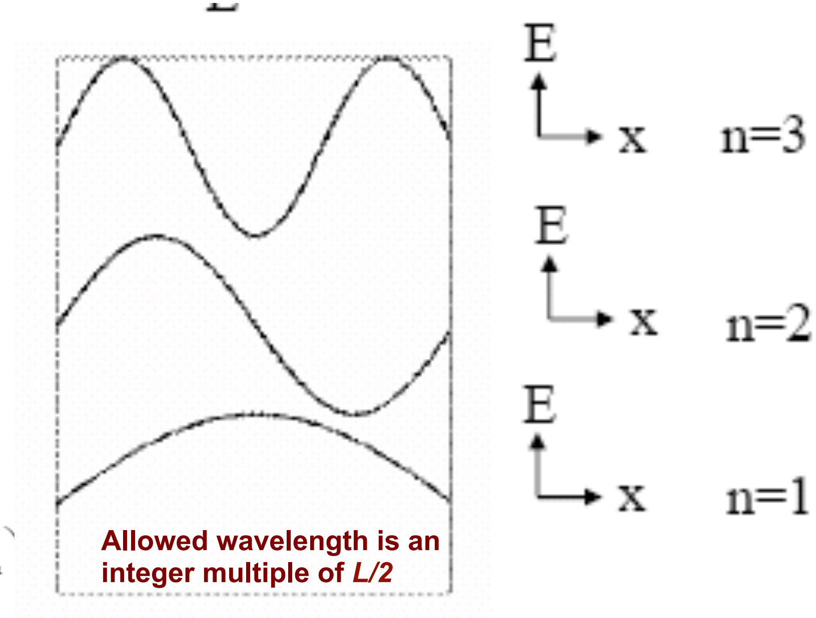
$$= -E_{0imag}\, \hat{z} \sin\left(k\hat{x}\cdot\vec{r}-\omega t\right) - E_{0imag}\, \hat{z}\sin\left(k\hat{x}\cdot\vec{r}+\omega t\right)$$

$$= -E_{0imag}\, \hat{z}\left[\sin(kL-\omega t)+\sin(kL+\omega t)\right]$$

$$= -2E_{0imag}\, \hat{z}\sin(kL)\cos(\omega t) = 0\,?$$

If and only if:

$$k_n = n\pi / L \qquad \omega_n = ck_n = nc\pi / L$$



Boundary conditions ($E_{//}$ = 0 at the walls) → **quantized modes**

Allowed wavevectors are integer multiples of π/L

In the box, only wavelengths integer multiple of $L/2$ can survive

ENERGY IN THE BOX

e.m. energy density *u* (definition)

Instantaneous energy per unit volume:

$$u(\vec{r},t)=\frac{1}{2}\left(\varepsilon_0\left[\vec{E}(\vec{r},t)\right]^2+\mu_0\left[\vec{H}(\vec{r},t)\right]^2\right)$$

Total energy in box:

$$U_{total}=\iiint_{box} u(\vec{r},t)dV$$

It can be shown that for the box:

$$U_{total}=\frac{1}{2}\left(n_x^2+n_y^2+n_z^2\right)\left(E_1^2+E_2^2+E_3^2\right)$$

So, the amount of energy in the box can have any value.
We will show that this leads to a problem and must be wrong.
The energy in the box must be quantized: these are *photons*.

Assuming equipartition a non physical (not acceptable) result is obtained, consisting in diverging *U* for high frequencies (short wavelengths)

Assuming (wrongly!!) equipartition

According to the equipartition theorem from thermodynamics, every mode of the system has an average energy $\langle U\rangle=(1/2)k_BT$.

Note: This is already a problem. Energy infinite.

What is the energy per frequency, then we will integrate over frequencies?
There are many modes per unit frequency. Each has energy k_BT.

$$\varepsilon(\nu)d\nu=\frac{1}{2}kT\cdot N(\nu)d\nu$$

- $\varepsilon(\nu)d\nu$ is the energy between ν and $\nu+d\nu$.
- (This is the *spectrum* of the blackbody radiation.)
- $N(\nu)d\nu$ is the number of modes between ν and $\nu+d\nu$.

NUMBER OF ALLOWED MODES (DENSITY OF STATES)

Modes per frequency $N(\nu)d\nu$

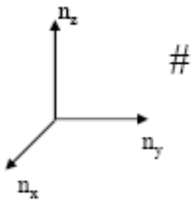
Recall:

$$\left(\frac{\pi}{L}\right)^2 (n_x^2 + n_y^2 + n_z^2) = \left(\frac{2\pi\nu}{c}\right)^2$$

How many modes have $n_x^2 + n_y^2 + n_z^2 < N^2$

I.E. $\nu < cN/2L$

Easy:



$$\#\text{ of modes} = 2 \cdot \frac{1}{8}\left(\frac{4}{3}\pi N^3\right)$$

$$\#\text{ of modes} = \int_0^{cN/2L} N(\nu)d\nu$$

$$\Rightarrow N(\nu)d\nu = \frac{1}{c^3}(8\pi)\nu^2 L^3$$

Again assuming equipartition

$$N(\nu)d\nu = \frac{1}{c^3}(8\pi)\nu^2 L^3$$

$$\varepsilon(\nu)d\nu = kT \cdot N(\nu)d\nu$$

$$\Rightarrow \varepsilon(\nu)d\nu = \frac{8\pi}{c^3} \cdot kT \cdot \nu^2 \cdot L^3$$

Rayleigh-Jeans law.
Experiments confirm at low frequencies only.

$$\Rightarrow \int_0^\infty \varepsilon(\nu)d\nu = \infty$$

Otherwise written, the number of allowed modes in the frequency interval ν, ν+dν is:

$$g(\nu)d\nu = 2\frac{4\pi}{c^3}\nu^2 V$$

That is, being $p = \hbar k = h/\lambda = h\nu/c$:

$$g(p)dp = 2\frac{4\pi}{h^3}p^2 V$$

That is, being $E = h\nu$:

$$g(E)dE = 2\frac{4\pi}{c^3 h^3}E^2 V$$

STATISTICS OF PHOTONS

1. We have found that not all frequencies (or energies, or wavelengths) can stay in the box
2. The density of states *g(ν)dν* dictates the number of allowed states for a certain frequency (a frequency interval ν,ν+dν)
3. Moreover, we know that each photon (at frequency ν) will bring an energy *hν*

We have gathered almost all the ingredients needed for the solution, but one: we must establish the link with the temperature, that is we have to found the statistics

Classically, i.e., for massive particles (e.g., gas atoms) in equilibrium at temperature *T*, we should use the Maxwell-Boltzmann statistics:

$$\bar{n}(E) \sim e^{-\frac{E}{K_B T}} = e^{-\frac{h\nu}{K_B T}}$$

Average number, or density, of particles having energy *E, E+dE* (frequency ν,ν+dν in case of photons)

kT = 1/40 eV @ temp amb (*T* ~ 300 K)

For particles obeying **quantum mechanics**, it can be shown that two statistics hold, depending on the "spin" of the considered particle:

$$\bar{n}_{FD}(E) = \frac{1}{e^{\frac{E-E_F}{K_B T}} + 1}$$

Fermi-Dirac statistics for half integer spin particles (notably, electrons)

$$\bar{n}_{BE}(E) = \frac{1}{e^{\frac{E-\mu_0}{K_B T}} - 1}$$

Bose-Einstein statistics for integer spin particles ("bosons")

ENERGY OF THE BLACK-BODY

- The use of classical or quantum statistics depends on various parameters, the most important being the "density" (in the real space) of the particles
- Photons do not interact each other, hence their density can be huge → *quantum statistics required*
- Moreover, photons are "bosons" since their angular momentum can only get integer values L = ± $\hbar$ (this is related to right and left-handed circular polarization states)

$$\bar{n}(\nu) = \frac{1}{e^{\frac{h\nu}{K_B T}} - 1}$$

Now, the total e.m. energy within the box, expressed for the frequency interval ν,ν+dν, will be:

energy of a mode (*hν*) x number of modes (*g(ν)dν*) x average occupation number of mode

$$U(\nu)d\nu = h\nu\, g(\nu)\bar{n}(\nu)d\nu$$

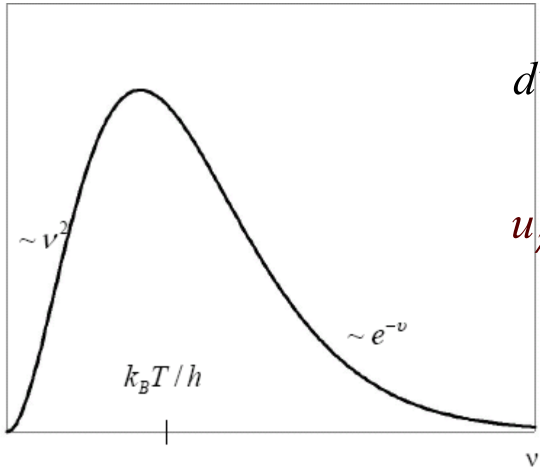
$$U(\nu)d\nu = h\nu \frac{8\pi\nu^2}{c^3} V \frac{1}{e^{\frac{h\nu}{K_B T}} - 1} d\nu$$

or $U(\nu)d\nu = u_\nu V d\nu$, with:

$$u_\nu = \frac{8\pi h\nu^3}{c^3} \frac{1}{e^{\frac{h\nu}{K_B T}} - 1}$$

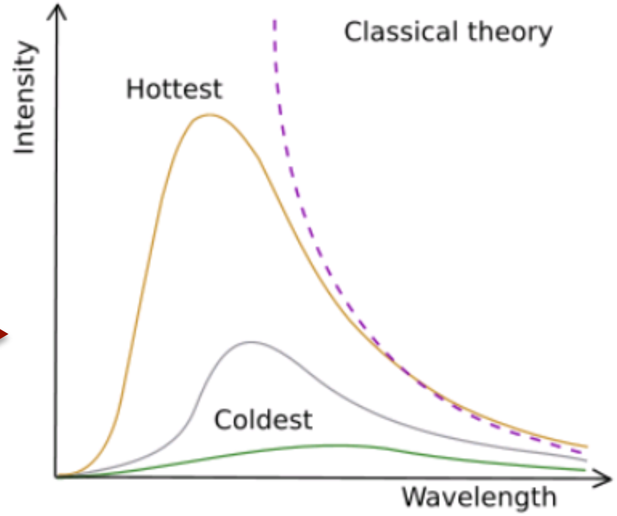
Black-body spectrum (after Planck, ~1900)

BLACK-BODY SPECTRUM



$$d\nu = d\left|\frac{c}{\lambda}\right| = \frac{c}{\lambda^2} d\lambda$$

$$u_\lambda d\lambda = \frac{8\pi hc^3}{\lambda^5} \frac{1}{e^{\frac{hc}{\lambda(K_B T)}} - 1} d\lambda$$



Interesting consequences:

Wien's law: the peak of the black-body spectrum is at wavelength λ_{max} so that

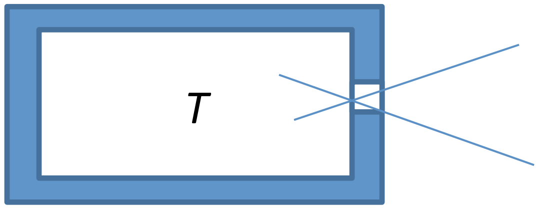
$\lambda_{max} T$ = *constant* = 2898 μm K

(we all emit at a wavelength peaked around 10 μm!)

Stefan-Boltzmann law: integrating over all frequencies one gets that the emitted intensity *I(T)* is proportional to T^4:

$I = \sigma T^4$, with $\sigma = 5.7 \times 10^{-8}$ W/(m^2 K^4)

BLACK-BODY vs LASER



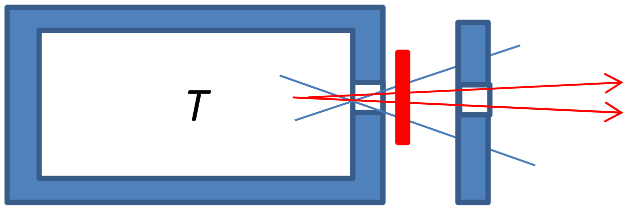
Main features of the light emitted from a balck-body:

- ✓ Non monochromatic: huge spectrum (over all the visible range and more)
- ✓ Non directional: the beam of photons emerges from the box much like a gas flux from a tiny hole (almost isotropically in half the space)
- ✓ Intensity: this depends on *T*
- ✓ Spatial and temporal coherence are not expected, as they are absent in a gas

The light stemming from a conventional (thermal) source has rather nothing to do with that produced by a laser …

Note: this is true also for practically all non-laser sources, including discharge lampsm LEDs, systems where spontaneous emission (we'll see later) is used, and, obviously, the sun

BLACK-BODY vs LASER



Since it is possible to build spectral and spatial filters, one might attempt to emulate the laser emission by using **filters**:

- Select (restrict) the wavelength range → monochromaticity
- Select (restrict) the spread in wavevector directions → collimation

Rough estimation of the result:
Assuming **1 m^3** of black-body at ***T*** **= 4500 K**, assuming a spectral filter with $\Delta\nu = 1$ GHz (still huge, indeed), one would achieve $U \approx 10^{-8}$ J
Assuming a spatial filter able to reduce the divergence down to 1 mrad (still huge, indeed), one would get $I = P/A \approx uc\Delta\Omega/4\pi \approx$ **0.1 mW**

… not that efficient (filters select, they do not amplify)!!

There is no way to conceive a laser within the classical frame!!

HIGHLIGHTS

- ✓ The classical approach (Maxwell, e.m. waves, classical matter) is able to interpret many experimental findings, but there is no way to retrieve amplification

- ✓ On the other hand, the conventional (non laser) sources, like the prototypal black-body, emit in a broad spectrum and their light has no possibility to be coherent

- ✓ Quantum Mechanics is needed in order to:
  - Account for the discreteness of energy in material systems
  - Account for the diverse light/matter interaction paths which are possible
  - Finally account for the possibility of amplification, that is one of the main ingredients of lasing action

- ✓ Before proceeding, we'll need to revise a few, very basic, QM mechanisms

FONTI

G. Mondio , modello di Lorentz, http://www.mi-ros.it/duplomiros/miro/mondio/lorentz.htm

E. Arimondo, Struttura della Materia (ETS, 2005)

http://www.wikipedia.org

P.Burke, Lasers and Photonics, winter 2002 (uci.edu)