Scuola di Dottorato Leonardo da Vinci – a.a. 2012/13

LASER: CARATTERISTICHE, PRINCIPI FISICI, APPLICAZIONI

Version 4 – July 13 – http://www.df.unipi.it/~fuso/dida

Part 4

Light/matter interaction according to Quantum Mechanics (simplified)

OUTLOOK

- We have derived a description of the matter according to a (simplified) picture in general agreement with QM

- Matter is made of quantized energy levels, which can be connected thorugh the release or absorption of energy in the form of photons

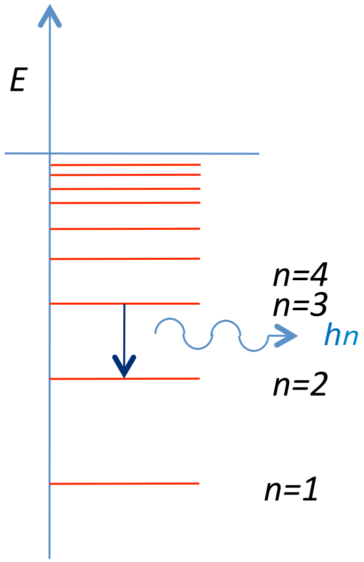
- Different interaction pathways are allowed (or predicted):
    1. Spontaneous emission (à la Bohr)
    2. Absorption
    3. *Stimulated emission*

- We will here investigate the main features of such processes

- We will focus mostly on simple two level quantum systems following a procedure originally developed by Einstein *(unfortunately we will find that a two level system cannot give rise, by itself, a lasing medium…)*

Objectives:
✓to understand the basic mechanims involving stimulated emission that is behind amplification of light
✓To write rate equations able to describe the processes of interest

SPONTANEOUS EMISSION I



A system in an excited state will *soon or after* decay to the ground state (the one with the lowest possible energy)

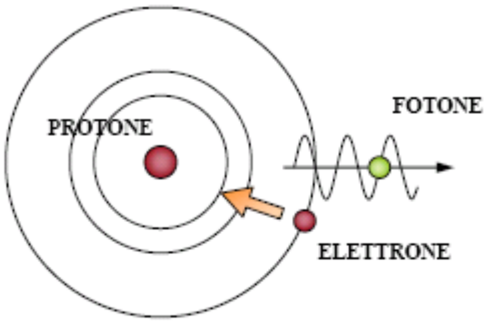
The ground state will be stationary

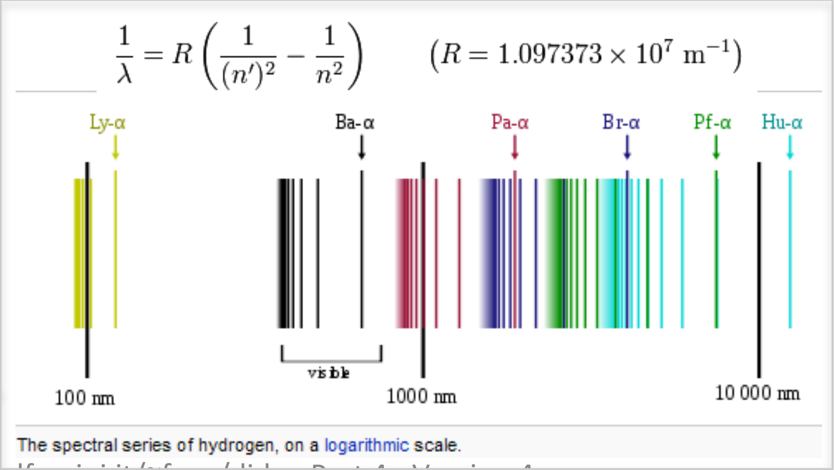
The decay is (or, better, can be) accompanied with the release of excess energy through a photon emission!

Conservation rules must hold:

Energy conservation → $h\nu = -(E_{fin}-E_{in})$

[furthermore linear and angular momentum must be conserved, but this is by far less relevant here]





$$\frac{1}{\lambda} = R\left(\frac{1}{(n')^2} - \frac{1}{n^2}\right) \qquad (R = 1.097373 \times 10^7\ \mathrm{m}^{-1})$$

The spectral series of hydrogen, on a logarithmic scale.

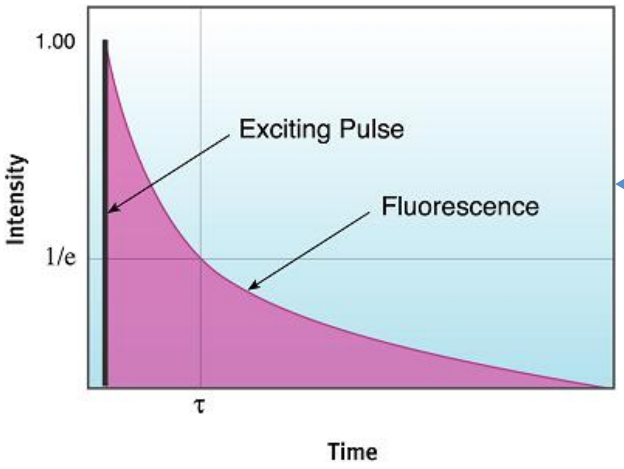
Note: historically the analysis of the hydrogen fluorescence (spontaneous) spectrum (from sun, e.g.) gave strong impulse to the Bohr's hypothesis, being interpreted according to his energy level predictions

SPONTANEOUS EMISSION II

- The process is “naturally” included in the Bohr’s approach and is in agreement with the well known behavior of any (isolated) system, spontaneously leading to min energy
- Its full QM interpretation requires “relativistic” QM, which we will not consider (luckily!)

A few **main features** of spontaneous emission:

- It is a **spontaneous** process → no incoming photons are needed to trigger it!
- No possibility to define a preferential direction → process naturally **isotropic**
- The **rate** will (ultimately) depend only on the internal properties of the system

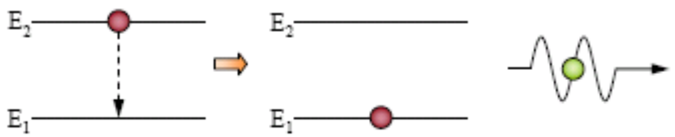


The spontaneous emission is responsible of the so-called fluorescence emission

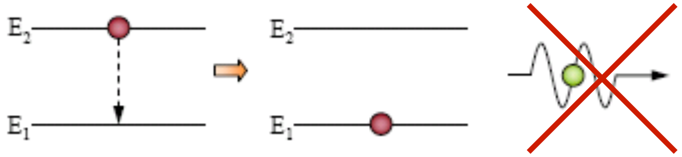
For instance, an atomic gas sample contained in a cell and “excited” by pulse emits (isotropically) radiaition

The temporal behavior shows an exponential decay due to the event probability constant in time

SPONTANEOUS EMISSION RATE



SPONTANEOUS EMISSION



NON RADIATIVE DECAY

The energy pertaining to the excited state can be lost also through processes **non involving** the emission of a photon!

Non radiative processes can take place, for instance triggered by collisions

The rate for the non radiative decay depends on the experimental parameters (e.g., temperature, pressure)

In typical experiments, the effective lifetime (t_{eff}) measured is affected by both spontaneous emission and non radiative decay:

$$1/\tau_{eff} = 1/\tau_{spont} + 1/\tau_{nonrad}$$

(in any case, the lifetime cannot be zero in agreement with uncertainty principle, Dn ~ 1/t)

The rate of the spontaneous emission is denoted as

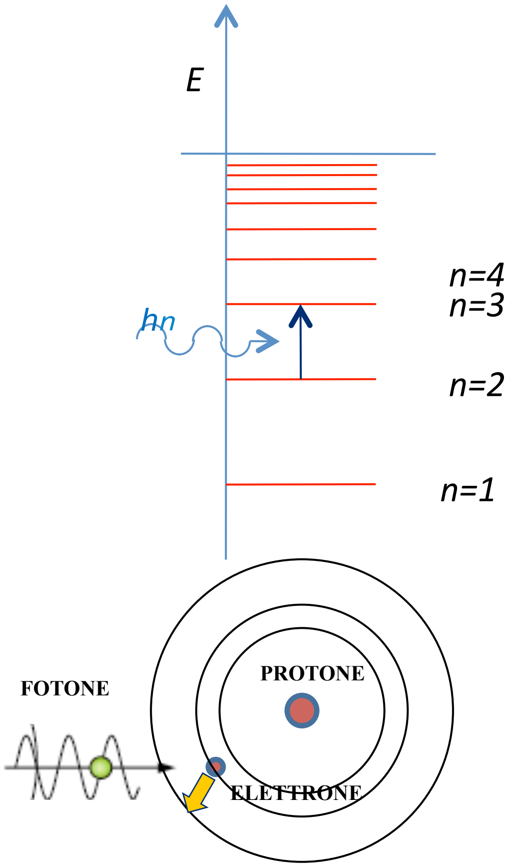
$\boldsymbol{A_{ij}}$

(A is the reciprocal of the "natural lifetime" τ_{spont})

It depends only on the "internal" properties of the system and can span over a wide range

Example: $\tau_{spont} \approx$ 5-50 ns for "simple" atomic transitions, but τ_{spont}>100 ms for ***metastable*** states

ABSORPTION (IN THE BOHR’S ATOM SCHEME)



Absorption, which is well interpreted in classical terms, must be retrieved also in the QM picture

It can be seen as a kind of collision between a photon and a quantum system, with the energy of the photon transferred to the internal degrees of freedom of the quantum system

Conservation rules must hold:

Energy conservation → $h\nu = (E_{fin} - E_{in})$

[furthermore linear and angular momentum must be conserved, but this is by far less relevant here]

Many experimental proofs, for instance absorption spectroscopy

LIGHT/MATTER INTERACTION ACCORDING TO SCHROEDINGER

According to Schroedinger **stationary states** y_n (eigenfunctions) exist with eigenvalues E_n

$$dP(\vec{r},\vec{r}+d\vec{r}) = |\Psi_n|^2\, d^3\vec{r} = |\psi_n|^2 |e^{-i\frac{E_n}{\hbar}t}|^2\, d^3\vec{r} = \text{constant}$$

The density of probability for an eigenfunction does not depend on time

<u>But</u> if a "mixed state" (superposition of eigenstates) is considered:

$$dP(\vec{r},\vec{r}+d\vec{r}) = |A\Psi_n + B\Psi_m|^2\, d^3\vec{r} =$$

$$= (|A|^2|\psi_n|^2|e^{-i\frac{E_n}{\hbar}t}|^2 + |B|^2|\psi_m|^2|e^{-i\frac{E_m}{\hbar}t}|^2 +$$

$$+ A^* B\psi^*_n\psi_m e^{-i\frac{E_m-E_n}{\hbar}t} + B^* A\psi^*_m\psi_n e^{-i\frac{E_n-E_m}{\hbar}t})d^3\vec{r} =$$

$$\neq \text{constant}$$

In other words:
temporal evolution of the probability density (proportional to the module squared) requires superposition of different states which must be *coupled*

ABSORPTION AND EMISSION

- Mixing between different eigenstates can be accomplished through an e.m. field, that is by sending photons!
- **Note**: assuming a two level system, *two distinct time dependent* parts are found
- According to the energy conservation, we can interpret $+A^* B\psi^*_n \psi_m e^{-i\frac{E_m - E_n}{\hbar}t}$ $B^* A\psi^*_m \psi_n e^{-i\frac{E_n - E_m}{\hbar}t}$

ABSORPTION

THE OPPOSITE (STIMULATED EMISSION)

Coupling radiation with quantum systems → **absorption** and **stimulated emission**

Stimulated emission is radically different from spontaneous emission because here irradiation (an incoming photon) is needed

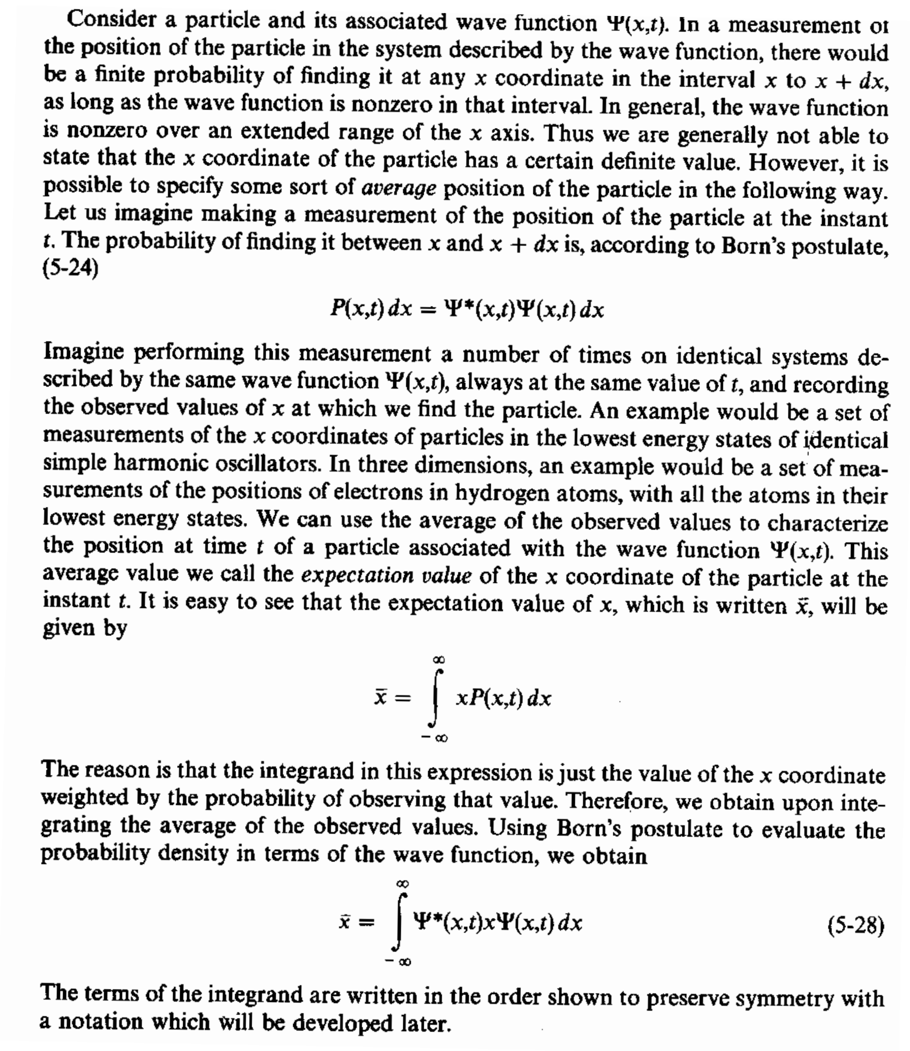
Let's now find the same result in a slightly different way which starts from classics
Classically, oscillating dipoles are responsible for absorption and emission
(as we have seen with dielectrics)

EXPECTATION VALUES IN QM

We know that QM relies on a probabilistic view

We know that the probability of finding a particle is proportional to $|\Psi|^2$

The problem (here): ***how to measure a quantity in QM****?*



Consider a particle and its associated wave function $\Psi(x,t)$. In a measurement of the position of the particle in the system described by the wave function, there would be a finite probability of finding it at any x coordinate in the interval x to $x + dx$, as long as the wave function is nonzero in that interval. In general, the wave function is nonzero over an extended range of the x axis. Thus we are generally not able to state that the x coordinate of the particle has a certain definite value. However, it is possible to specify some sort of *average* position of the particle in the following way. Let us imagine making a measurement of the position of the particle at the instant t. The probability of finding it between x and $x + dx$ is, according to Born's postulate, (5-24)

$$P(x,t)\,dx = \Psi^*(x,t)\Psi(x,t)\,dx$$

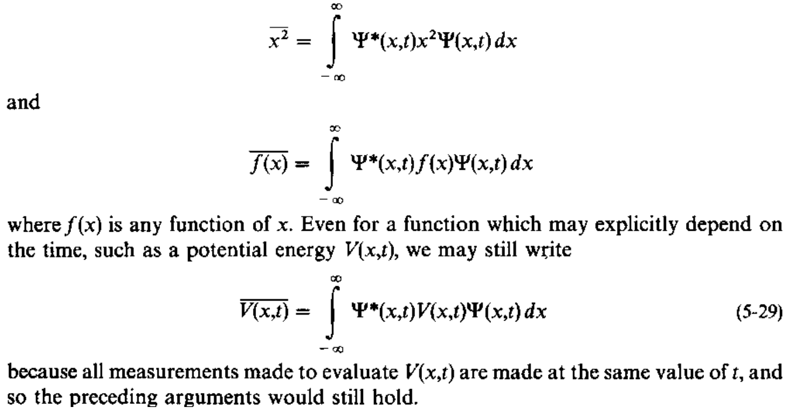
Imagine performing this measurement a number of times on identical systems described by the same wave function $\Psi(x,t)$, always at the same value of t, and recording the observed values of x at which we find the particle. An example would be a set of measurements of the x coordinates of particles in the lowest energy states of identical simple harmonic oscillators. In three dimensions, an example would be a set of measurements of the positions of electrons in hydrogen atoms, with all the atoms in their lowest energy states. We can use the average of the observed values to characterize the position at time t of a particle associated with the wave function $\Psi(x,t)$. This average value we call the *expectation value* of the x coordinate of the particle at the instant t. It is easy to see that the expectation value of x, which is written $\bar{x}$, will be given by

$$\bar{x} = \int_{-\infty}^{\infty} xP(x,t)\,dx$$

The reason is that the integrand in this expression is just the value of the x coordinate weighted by the probability of observing that value. Therefore, we obtain upon integrating the average of the observed values. Using Born's postulate to evaluate the probability density in terms of the wave function, we obtain

$$\bar{x} = \int_{-\infty}^{\infty} \Psi^*(x,t)x\Psi(x,t)\,dx \qquad (5\text{-}28)$$

The terms of the integrand are written in the order shown to preserve symmetry with a notation which will be developed later.



$$\overline{x^2} = \int_{-\infty}^{\infty} \Psi^*(x,t)x^2\Psi(x,t)\,dx$$

and

$$\overline{f(x)} = \int_{-\infty}^{\infty} \Psi^*(x,t)f(x)\Psi(x,t)\,dx$$

where $f(x)$ is any function of x. Even for a function which may explicitly depend on the time, such as a potential energy $V(x,t)$, we may still write

$$\overline{V(x,t)} = \int_{-\infty}^{\infty} \Psi^*(x,t)V(x,t)\Psi(x,t)\,dx \qquad (5\text{-}29)$$

because all measurements made to evaluate $V(x,t)$ are made at the same value of t, and so the preceding arguments would still hold.

The average value of a (measurable) quantity f on a quantum system is $< f >$:

$$< f >= \int \Psi^*(\bar{r},t) f \Psi(\bar{r},t) d^3\bar{r} =<\Psi^* \mid f \mid \Psi >$$

(clearly depends on the *spatial superposition* of the wavefunction)

Example:
<*x*> = 0 a free particle (plane wave function)

ELECTRIC DIPOLE MOMENT

The electric dipole moment can be defined (classical definition) as ***p*** = -*e**r***
Let's evaluate its average (expectation) value on a quantum system

$$<\vec{p}>=<\Psi|(-e\vec{r})|\Psi>=(-e)\int\Psi^*\vec{r}\Psi d^3\vec{r}$$

On a eigenstate (eigenfunction Y=$y_n e^{-i(E_n/\hbar)t}$):

$$<\vec{p}>\sim\int\Psi^*\vec{r}\Psi d^3\vec{r}=\int|\psi_n|^2|e^{-i\frac{E_n}{\hbar}t}|^2\,\vec{r}d^3\vec{r}=\int\vec{r}d^3\vec{r}=0$$

Note: ***p*** is an "odd" operator
→ average is zero for any symmetrical wavefunction

On a mixed state (wavefunction Y=$Ay_n e^{-i(E_n/\hbar)t}$) + $By_m e^{-i(E_m/\hbar)t}$) :

$$<\vec{p}>\sim\int\Psi^*\vec{r}\Psi d^3\vec{r}=A\int|\psi_n|^2\cancel{|e^{-i\frac{E_n}{\hbar}t}|^2}\,\vec{r}d^3\vec{r}+B\int|\psi_m|^2\cancel{|e^{-i\frac{E_m}{\hbar}t}|^2}\,\vec{r}d^3\vec{r}+$$

$$+A^*B\left(\int\psi^*_n\vec{r}\psi_m d^3\vec{r}\right)e^{-i\frac{E_m-E_n}{\hbar}t}+B^*A\left(\int\psi^*_m\vec{r}\psi_n d^3\vec{r}\right)e^{-i\frac{E_n-E_m}{\hbar}t}$$

We find also in this case two non constant, oscillating terms at the frequency $\omega=|E_m-E_n|/\hbar$

ELECTRIC DIPOLE AND TRANSITIONS

The occurrence of a (optical) transition implies an oscillatory character in the wavefunction evolution

This corresponds to a non null, transient electric dipole and requires coupling between differerent states

Such a coupling is due to the interaction with an el.m. field, that is to the arrival of photons (**Note**: energy conservation implies resonant conditions!)

Absorption is a process we can easily figure out based on a classical picture

Stimulated emission is much less straightforward to be interpreted in a classical frame

Note: we are resticting here to "electric dipole" transitions which, while being the most common, are not the only possible transitions

RATE OF STIMULATED PROCESSES

Contrary to spontaneous emission, whose rate depends only on internal parameters of the system, we expect here a dependence on the parameters of the impinging radiation

Obviously, the **frequency** must be resonant in order for the processes to occur

Also **intensity**, that is the number of photons arriving to a unit area of the system in the unit time, should be relevant to define the rate of absorption and stimulated emission

Remember: $I = P/A$
$N = P/(E_{phot} A)$
Example: 1 mW on 1 mm^2 @600 nm (red=
$E_{phot} = 1240/600$ eV ≈ $3\text{x}10^{-19}$ J → $N \approx 3\text{x}10^{15}$ photon/(s mm^2)

Classically: the power radiated by an electric dipole (space-integrated) is given by the **Larmor’s formula**:

$$W = \frac{4\pi^2\nu^4}{3\varepsilon_0 c^3} p^2$$

p : dipole moment
ν : frequency

We can define a rate dividing the radiated power by the frequency

$$R = \frac{W}{\omega} = \frac{4\pi^2\nu^3}{3\varepsilon_0 c^3 h} p^2$$

Since $p \approx E$ one may expect $R \approx E^2$ hence $R \approx I$
(rate depending on the intensity of the e.m. radiation interacting with the system)

A FEW WORDS ON THE QM TREATMENT

Basic assumption: *the interaction can be described as a* ***perturbation*** *of the quantum system*

The potential entering Schroedinger's equation can therefore be written as $V_{tot} = V_0(\mathbf{r}) + V'(\mathbf{r},t)$, with V_0 unperturbed potential (e.g., atomic) and V' perturbation

Moreover the space-dependence of the potential is neglected, assuming interaction with a system much smaller than the wavelength: $V'(\mathbf{r},t) = V'(t)$
(it is the *electric dipole approximation* we have already mentioned)

According to QM, the system is described by a linear superposition of eigenfunctions (orthonormal base)

$$\Psi(x,t) = \sum_n \boxed{a_n(t)}\,\psi_n(x)\,e^{-i\frac{E_n}{\hbar}t}$$

(for the ske of clarity, a one-dimensional problem is considered here)

The time dependence is expressed through the time dependence of the coefficients $a_n(t)$

Recall: Schroediger's equation

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi(x,t)}{\partial x^2} + V_0(x)\Psi(x,t) + V'(t)\Psi(x,t) = i\hbar\frac{\partial\Psi(x,t)}{\partial t}$$

PERTURBATIVE ITERATION I

$$-\frac{\hbar^2}{2m}\sum_n a_n(t)e^{-i\frac{E_n}{\hbar}t}\frac{\partial^2\psi_n(x)}{\partial x^2}+V_0(x)\sum_n a_n(t)\psi_n(x)e^{-i\frac{E_n}{\hbar}t}+$$

$$+V'(t)\sum_n a_n(t)\psi_n(x)e^{-i\frac{E_n}{\hbar}t}=i\hbar\sum_n a_n(t)\frac{\partial\psi_n(x)e^{-i\frac{E_n}{\hbar}t}}{\partial t}+i\hbar\sum_n\frac{da_n(t)}{dt}\psi_n(x)e^{-i\frac{E_{nn}}{\hbar}t}$$

Terms in blue satisfy the unperturbed Schroedinger' s equation, hence they are not of interest to us

Both members are multiplied by ψ_m and subsequently space-integrated:

$$\int\psi^*{}_m(x)V'(t)\sum_n a_n(t)\psi_n(x)e^{-i\frac{E_n}{\hbar}t}dx=i\hbar\int\psi^*{}_m(x)\sum_n\frac{da_n(t)}{dt}\psi_n(x)e^{-i\frac{E_{nn}}{\hbar}t}$$

$$\Downarrow$$

$$\sum_n a_n(t)e^{-i\frac{E_n-E_m}{\hbar}t}\int\psi^*{}_m(x)V'(t)\psi_n(x)dx=i\hbar\frac{da_m(t)}{dt}$$

Since, due to the choice of an
orthonormal basis of eigenfunctions:.

$$\sum_n\int\psi^*{}_m(x)\psi_n(x)dx=\delta_{nm}$$

PERTURBATIVE ITERATION II

A system of interconnected differential equations for $a_n(t)$ is obtained
The perturbative approach consists in placing at the first members of such equations the unperturbed value of the coefficients
For instance, assuming that only one $a_n \neq 0$:

$$a_n(t)e^{-i\frac{E_n-E_m}{\hbar}t} < \psi_m | V' | \psi_n > = i\hbar \frac{da_m(t)}{dt}$$

The $n \rightarrow m$ *transition probability* is then $P_{nm} = |a_m(t)|^2$

Upon reaonable approximations, one gets a probability linear with *t*:

$$P_{nm} = \frac{2\pi}{\hbar} |< \psi_m | V' | \psi_n >|^2 \, t\,\delta(\hbar\omega \pm E_n - E_m = 0)$$

(where the Dirac-δ function accounts for energy conservation)

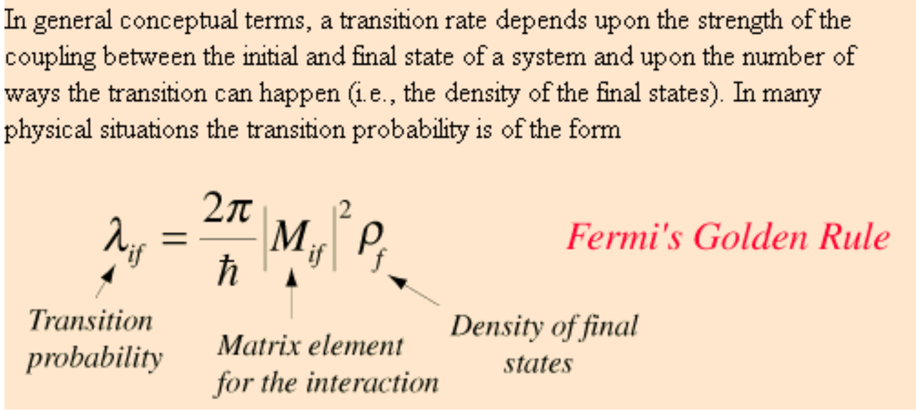
The **transition rate** (probabilityper unit time) is thus:

$$R_{nm} = \frac{2\pi}{\hbar} |< \psi_m | V' | \psi_n >|^2 \, \delta(\hbar\omega \pm E_n - E_m = 0) = \frac{2\pi}{\hbar} |M_{nm}|^2 \rho(E)$$

(where M_{nm} is called *matrix element* of the transition and $\rho(E)$ represents the *density of final states*, which is not actually described by Dirac-δ functions due to “broadening of states”, as we will see in the following)

FERMI' S GOLDEN RULE

What we found (with a very few words!) is a rather general result, called Fermi' s golden rule



In general conceptual terms, a transition rate depends upon the strength of the coupling between the initial and final state of a system and upon the number of ways the transition can happen (i.e., the density of the final states). In many physical situations the transition probability is of the form

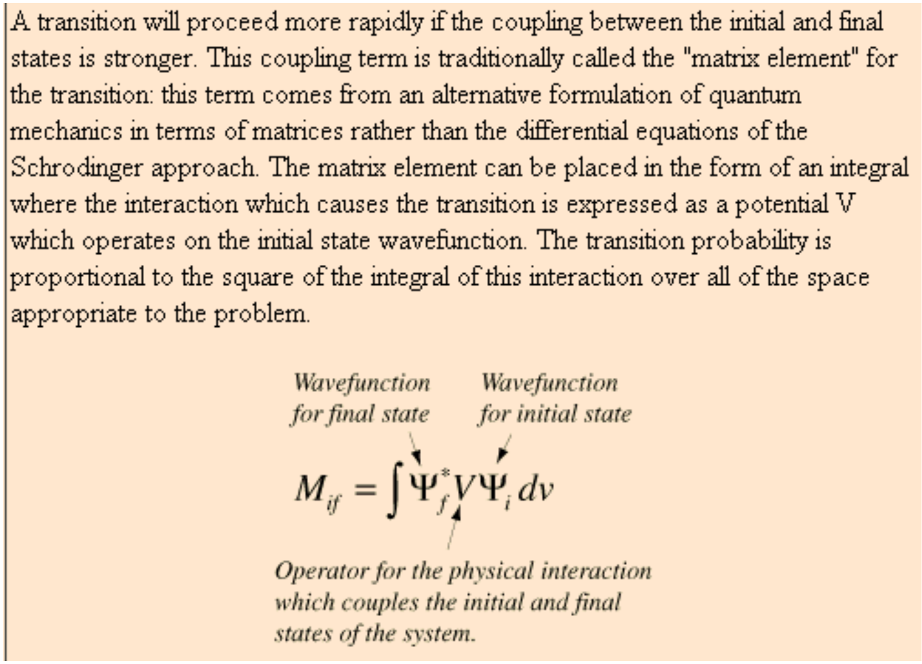
$$\lambda_{if} = \frac{2\pi}{\hbar}\left|M_{if}\right|^2 \rho_f$$

Fermi's Golden Rule

Transition probability

Matrix element for the interaction

Density of final states



A transition will proceed more rapidly if the coupling between the initial and final states is stronger. This coupling term is traditionally called the "matrix element" for the transition: this term comes from an alternative formulation of quantum mechanics in terms of matrices rather than the differential equations of the Schrodinger approach. The matrix element can be placed in the form of an integral where the interaction which causes the transition is expressed as a potential V which operates on the initial state wavefunction. The transition probability is proportional to the square of the integral of this interaction over all of the space appropriate to the problem.

Wavefunction for final state

Wavefunction for initial state

$$M_{if} = \int \Psi_f^* V \Psi_i \, dv$$

Operator for the physical interaction which couples the initial and final states of the system.

Note: $\rho_f = d(E_f - E_i)$ for ideally sharp levels

We define the probability per unit time:

$$R_{if} = \frac{2\pi}{h} | M_{if} |^2 \rho_f$$

which we call **transition rate**

Periodical perturbations → constant transition rate

MATRIX ELEMENTS FOR DIPOLE TRANSITIONS

If, as in the case of our interest, perturbation is due to e.m. field, e.g.:

$\mathbf{E}(\mathbf{r}, t) = E_0\, e^{i(kx-wt)}\, \mathbf{y} \rightarrow E_0\, e^{-iwt}\, \mathbf{y}$ (dipole approx.)

the perturbation potential is given by the dipole interaction energy:
$V' = -\mathbf{p} \bullet \mathbf{E}$, with $\mathbf{p} = -e\mathbf{r}$ (dipole moment)

$$|M_{if}|^2 = E_0^2 \left| \int \psi_f^* (\hat{y} \bullet \vec{p}) \psi_i d^3\vec{r} \right|^2 = E_0^2 < \psi_f | (\hat{y} \bullet \vec{p}) | \psi_i >|^2 = \frac{2up_{if}^2}{c\varepsilon_0}$$

with $u = c\varepsilon_0 E_0^2/2$ energy density of the e.m. field (prop to intensity)

The rate for a dipole transition is:

$$R_{if} = u \frac{4\pi}{\text{h} c\varepsilon_0} | p_{if} |^2 \rho_f$$

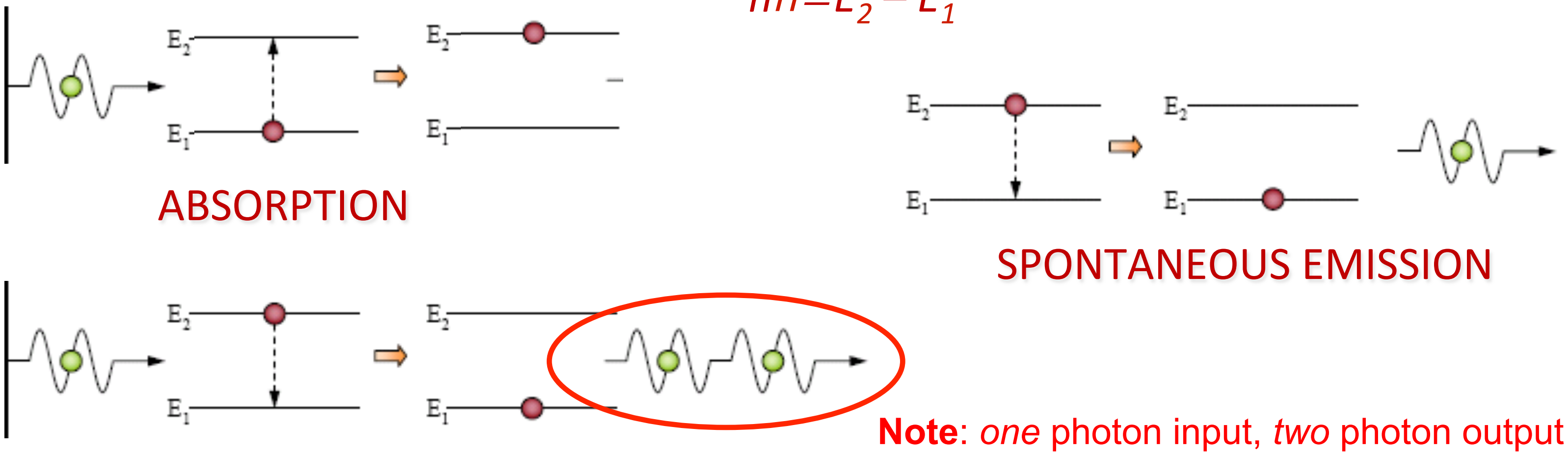
which can be written as: $R_{if} = uB_{if}$

Note: the matrix element determines also the **selection rules** → not all states are equally coupled by the e.m. field in the case of dipole transitions

SUMMARY OF POSSIBLE PROCESSES

Light/matter interaction (according to Einstein)

$hn=E_2-E_1$



ABSORPTION

SPONTANEOUS EMISSION

STIMULATED EMISSION

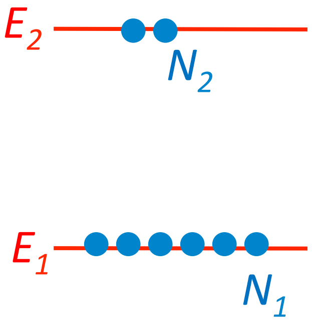
Note: *one* photon input, *two* photon output

Two processes (absorption and stimulated emission) depend on the presence of (resonant) radiation
→ their rate $B_{if}u_\nu$ depends (also) on the e.m. field energy (i.e., intensity)

One process is spontaneous, therefore independent of e.m. field
→ its rate A_{if} depends only on the internal parameters of the quantum system

RATE EQUATIONS FOR A TWO LEVEL SYSTEM

The goal is to describe the temporal evolution of the number of systems at the excited and ground states
For the sake of simplicity, a two level system (ideal) is considered here



N_1, N_2 populations of the levels (number or density of quantum systems in a sample being at a certain energy level

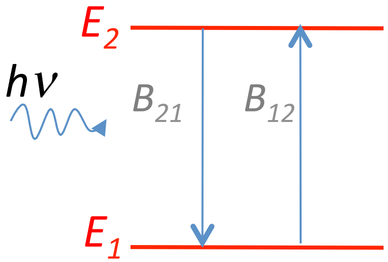
At equilibrium (thermal) Boltzmann statistics holds:

$$\frac{N_2}{N_1} = \exp\left({}^{E_1 - E_2}\!\Big/_{kT} \right)$$

Note: kT ~ 1/40 eV @ room temp, whereas DE ~ eV

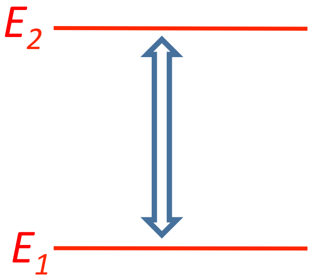
→ $N_2 << N_1$ in ordinary conditions (e.g., thermal equilibrium)

- Resonant radiation with $h\nu = E_2 - E_1$ (and energy density u_ν)
- For the moment (Einstein's procedure) only:
  Absorption 1→ 2 (with rate R_{12})
  Stimulated emission 2 → 1 (with rate R_{21})
- For simplicity: $B_{12}=B_{21}$
  Note: $B_{12}/B_{21} = g_1/g_2$ with g_j ***level degeneracy***



$$\frac{dN_2}{dt} = -B_{21}u_\upsilon N_2 + B_{12}u_\upsilon N_1 \qquad \frac{dN_1}{dt} = B_{21}u_\upsilon N_2 - B_{12}u_\upsilon N_1$$

DETAILED BALANCE AND NEED FOR SPONTANEOUS EMISSION



It must be $R_{12}=R_{21}$ ("detailed balance" for, e.g., chemical reactions and we set $B_{12}=B_{21}$

→ At equilibrium $N_1 = N_2$! Incompatible with Boltzmann!!

but considering also spontaneous emission with rate A_{21}

$$\frac{dN_2}{dt} = -B_{21}u_\upsilon N_2 + B_{12}u_\upsilon N_1 - A_{21}N_2$$

$$\frac{dN_1}{dt} = B_{21}u_\upsilon N_2 - B_{12}u_\upsilon N_1 + A_{21}N_2$$

Assuming $B_{12}=B_{21}$, at equilibrium (stationary solution):

$$N_2^0 = N_1^0 \frac{B_{21}u_\nu}{B_{21}u_\nu + A_{21}}$$

For Boltzmann it must be:

$$\frac{B_{21}u_\nu}{B_{21}u_\nu + A_{21}} = \exp\left(\frac{-h\nu}{kT}\right)$$

Assuming black-body radiation

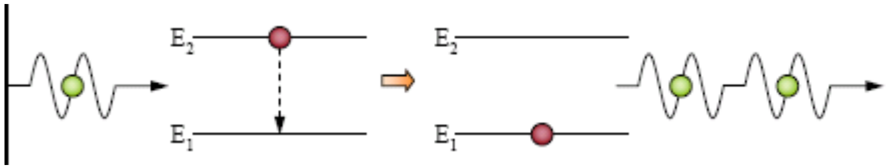
$$u_\nu = \frac{8\pi h\nu^3}{c^3}\frac{1}{\exp({}^{h\nu}\!/_{kT})-1}$$

hence:

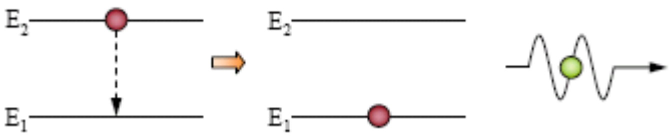
$$\boxed{A_{21} = B_{21}\frac{8\pi h\nu^3}{c^3}}$$

The relationship between *A* and *B* (spontaneous and stimulated rates) contains the cube of frequency

“COMPARISON” OF PROCESSES I



STIMULATED EMISSION



SPONTANEOUS EMISSION

Stimulated emission enables photon amplification!!

$$A_{21} = B_{21}\frac{8\pi h\nu^3}{c^3}$$

$$\Downarrow$$

$$\frac{B_{21}u_\nu}{A_{21}} = u_\nu \frac{c^3}{8\pi h\nu^3} = \frac{1}{\exp\left(h\nu/kT\right) - 1} = \bar{n}_\nu$$

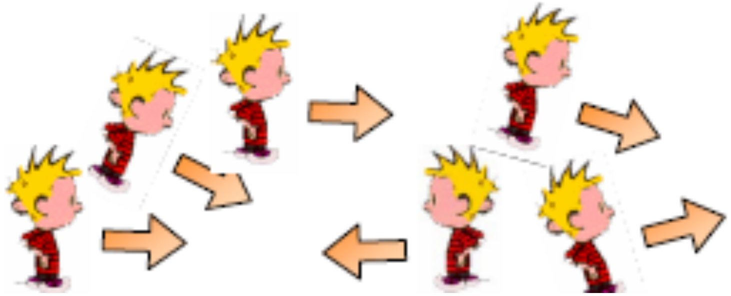
Assuming black-body radiation the ratio between stimulated and spontaneous rates indicates that spontaneous largely prevails (unless *T* < *0* !!!)

$\Downarrow$

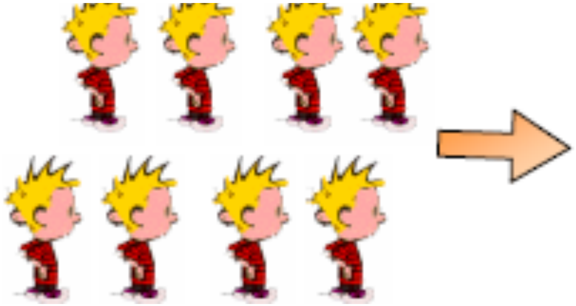
Specific approaches are needed to exploit stimulated emission (as we will see in the following) in practical lasers

“COMPARISON” OF PROCESSES II

Because of conservation rules (energy, but also linear and angular momentum) the stimulated emitted photons are ideally indistinguishable from the incoming photon



SPONTANEOUS EMISSION



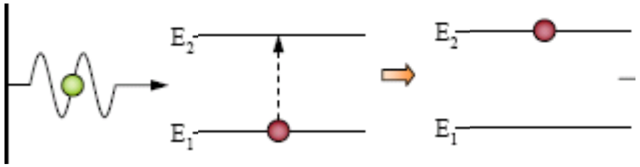
STIMULATED EMISSION

In classical terms, the phenomenon can be understood by considering an oscillating dipole, which gets further forcing from the (resonant) e.m. field
→ the dipole will see an “enhancement” of its radiated energy (therefore, the emission will have the same properties of the excitation)

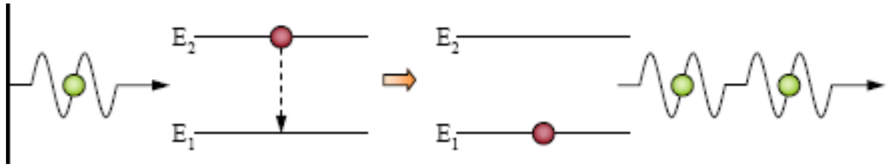
This is not all the case for spontaneous emission (for instance, because of the isotropic character of the emitted radiation)

Stimulated emission is not only able to amplify, but also to do it preserving “coherence”

MATHEMATICAL TREATMENT

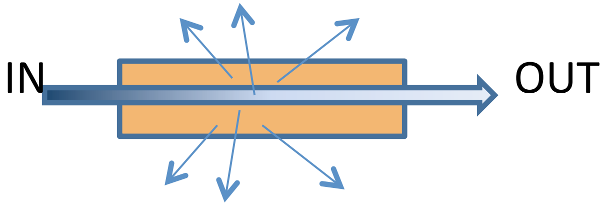


Absorption: one photon disappears



Emission: one (more) photon appears

Let’s simulate now this configuration



In the first approx spontaneous emission can be neglected, being the photons emitted isotropically and hence being difficult for them to be collected by the detector

In the medium: $u_n = Fh\nu$
(with F photon density)

$$\frac{dN_2}{dt} = -B_{21}Fh\nu N_2 + B_{12}Fh\nu N_1 - \cancel{A_{21}N_2}$$

$$\frac{dN_1}{dt} = B_{21}Fh\nu N_2 - B_{12}Fh\nu N_1 + \cancel{A_{21}N_2}$$

$$\frac{dF}{dt} = B_{21}Fh\nu N_2 - B_{12}Fh\nu N_1 + \cancel{A_{21}N_2}$$

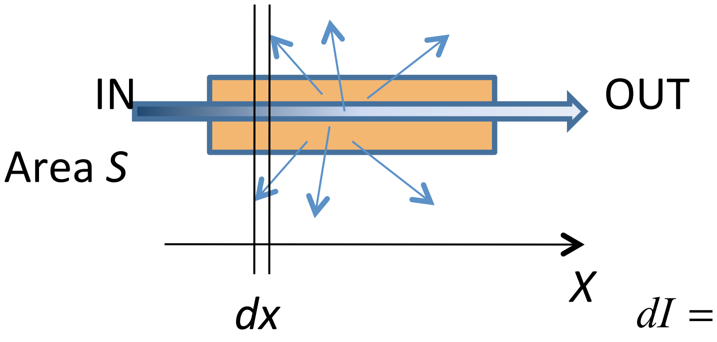
Setting $\Delta N = (N_2 - N_1)$, if $B_{12}=B_{21}$
and neglecting spontaneous em:

$$\frac{dF}{dt} = B_{21}Fh\nu(N_2 - N_1) \propto \Delta N$$

The sample behavior depends on the population difference ΔN

ABSORPTION COEFFICIENT

Let’s now find an explicit expression for the absorption coefficient *α*



Power within the medium : $P = Scu_n$
Intensity within the medium : $I = P/S = cu_n$
Intensity variation in $dV = Sdx$:

$$dI = \frac{dP}{S} = \frac{dU/dt}{S} = \frac{h\nu \frac{dF}{dt} Sdx}{S} = (h\nu)^2 F B_{21} \Delta N dx = u_\nu h\nu B_{21} \Delta N dx =$$

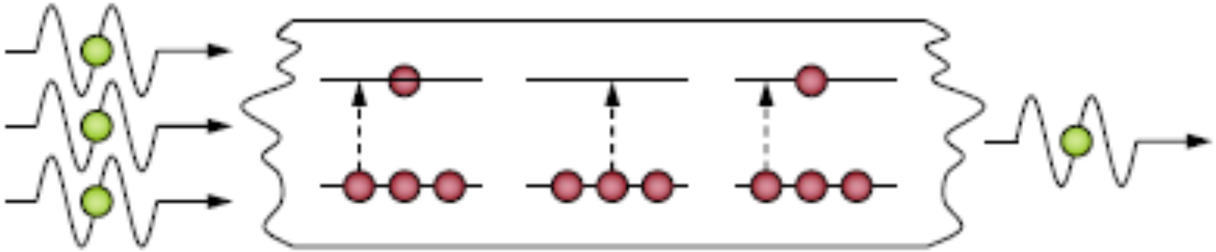
$$= \frac{I}{c} h\nu B_{21} \Delta N dx = I \alpha dx \rightarrow I(x) = I_0 \exp(\alpha x)$$

The sign of α (absorption coefficient) depends on ΔN:
$\Delta N < 0$ → $\alpha < 0$ (absorption, actually)
$\Delta N > 0$ → $\alpha > 0$ (amplification!)

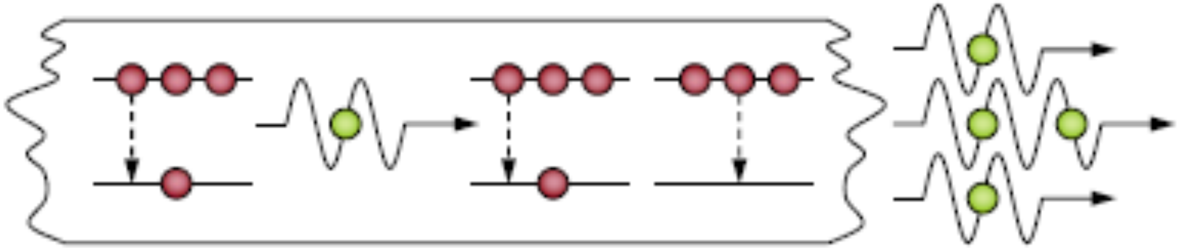
Amplification in a medium (quantum matter) interacting with radiation depends on the occurrence of population inversion

ABSORPTION vs AMPLIFICATION

Without inversion of population (i.e., in ordinary conditions) the incoming radiation is absorbed



But in the presence of inversion of population the incoming radiation is amplified (output number of photons larger than input)



RELAXATION

In order to proceed with rate equations describing practical problems, we have to recall that, due to spontaneous emission and <u>other possible (non radiative) processes,</u> in the absence of radiation (u_ν=0) the system must evolve towards the ground state:

$$N_1 \rightarrow N_1^0 \; ; \; N_2 \rightarrow N_2^0 \quad \text{that is } \Delta N \rightarrow \Delta N^0$$

Such a **relaxation** of the system must be explicitly put into the equations:

$$dN_2/dt|_{\text{relax}} = -\gamma_2(N_2-N_2^0) \qquad dN_1/dt|_{\text{relax}} = -\gamma_1(N_1-N_1^0)$$

(with γ_j relaxation rate: for the sake of simplicity we put $\gamma_2 = \gamma_1$

New form of the rate equations including relaxation:

$$\frac{dN_2}{dt} = -B_{21}u_\nu N_2 + B_{12}u_\nu N_1 - \gamma(N_2 - N_2^0)$$

$$\frac{dN_1}{dt} = B_{21}u_\nu N_2 - B_{12}u_\nu N_1 - \gamma(N_1 - N_1^0)$$

STEADY STATE SOLUTIONS AND SATURATION

$$\frac{dN_2}{dt} = -B_{21}u_\nu N_2 + B_{12}u_\nu N_1 - \gamma(N_2 - N_2^0)$$

$$\frac{dN_1}{dt} = B_{21}u_\nu N_2 - B_{12}u_\nu N_1 - \gamma(N_1 - N_1^0)$$

$$\frac{d(\Delta N)}{dt} = \frac{dN_2}{dt} - \frac{dN_1}{dt} = -2B_{21}u_\nu \Delta N - \gamma(\Delta N - \Delta N^0) = 0$$

Let's look for the population difference Δ*N* in stationary conditions

$$\Delta N = \frac{\gamma \Delta N^0}{2B_{21}u_\nu + \gamma}$$

Alternative expression:

$$\Delta N = \frac{\Delta N^0}{2B_{21}u_\nu / \gamma + 1} = \frac{\Delta N^0}{I / I_S + 1}$$

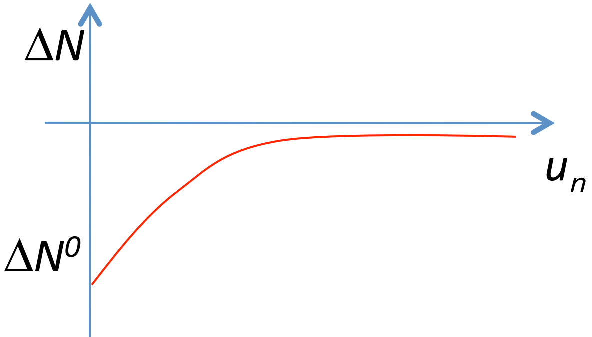
with I_S saturation intensity (depending only on the quantum system)

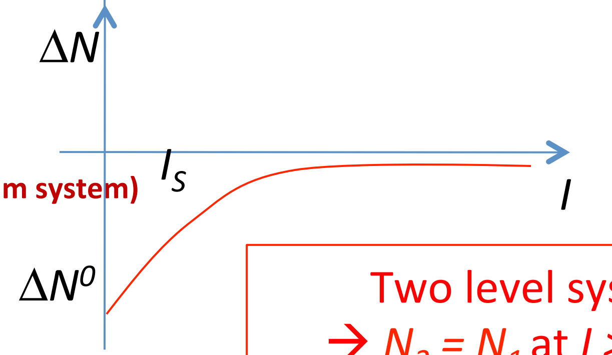
$$I_S = \frac{2B_{21}u_\nu}{\gamma}$$

that is, introducing the "lineshape" $\gamma(\nu-\nu_0)$:

$$I_S = \frac{2B_{21}u_\nu g(\nu-\nu_0)}{\gamma}$$

(we will see more on linshapes later on)





Two level system (ideal)
→ $N_2 = N_1$ at $I >> I_S$ → $\Delta N \leq 0$

No population inversion possible!

CONCLUSIONS

Considering the matter in quantum terms (quantum systems, with a set of discrete energy levels) enables recovering the classical behavior (namely, absorption) and makes possible to envision two distinct emission processes

Emission can in fact be either spontaneous or stimulated, with strong differences in the properties of the emitted photons

Stimulated emission can lead to amplification of light; moreover, the stimulated emitted photons are similar each other (→ coherence)

QM (starting from the simple Einstein's procedure) allows us to evaluate the relevant rates → rate equations can be written down

Amplification turns possible only in the case of inversion of population

At the steady state, an ideal two level quantum system cannot lead to inversion of population → only absorption is expected!

What to do, then, to get amplification? (We'll see in the following…)

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