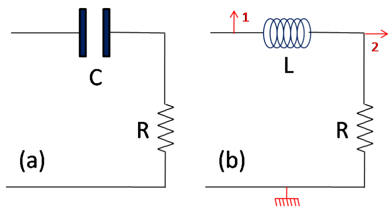
A method to solve the temporal behavior of RC and RL circuits

fuso@df.unipi.it; http://www.df.unipi.it/∼fuso/dida

(Dated: version 1.1 - FF, November 10, 2012)

This short text discusses a simple and rather straightforward method to solve the relevant equations in circuits comprising resistors and capacitors, or resistors and inductances. Transient (i.e., time-dependent) solutions are considered.



h!

FIG. 1. Schemes of the simple circuits considered in the text, made of a resistor-capacitor (a) and resistor-inductance series (b).

I. INTRODUCTION

We will consider here the simple circuits illustrated in Fig. 1, consisting in the series of either a resistor and a capacitor or a resistor and an inductance . Such components will be considered as ideal, i.e., the total (equivalent) resistance of the circuits is R, the capacity C, the self-inductance L. In other words, we won't consider internal or spurious factors such as, the resistance of the coil forming the inductance, and the internal resistance of the generators, if any. We will analyze as first the behavior of the RC circuit. We will then see that the relevant equations for the RL are formally similar and we will convert the solution of the RC into that of the RL circuit. Note that the method presented here, based exclusively on real quantities, has nothing to do with the (more powerful and effective) approach based on complex quantities and functions, which you will study later on during the course.

II. THE RC CIRCUIT

We will first assume that an (ideal) voltage generator V_0 is connected at time $t_0 = 0$ to the circuit of Fig. 1(a). The initial condition is that the capacitor has no net charge on its plates. Therefore, we will investigate the capacitor charging, which is well known to you all. Its analysis is however useful for the next steps, where we will assume an alternate voltage source. Said $Q(t)$ the time dependent charge at the capacitor plates [the initial condition is $Q(t = t_0 = 0) = 0$], the equation mastering the circuit is:

$$V_0 = \frac{Q(t)}{C} + RI(t) \ , \tag{1}$$

as derived from the definition of capacitance and the Ohm's law. The current $I(t)$ flowing into the resistor is given by the charge leaving the capacitor plate (a minus sign must be used). Note that, if we assume $V_0 > 0$ (the positive pole of the generator is connected to the capacitor), the sign of such current is negative, leading to $I(t) = -(-dQ(t)/dt) = dQ(t)/dt$. Hence, the equation reads:

$$V_0 = \frac{Q(t)}{C} + R\frac{dQ(t)}{dt} \tag{2}$$

that is, after algebraic manipulations:

$$\frac{dQ(t)}{dt} = \frac{1}{RC}(CV_0 - Q(t)) \ . \tag{3}$$

Note that, for dimensional reasons, RC must be a time: we will place hereafter $RC = \tau$. As you know, this is a first order differential equation which can be easily solved. A very rough method (for physicists, not mathematicians!) consists in rewriting the equation above as

$$-\frac{1}{RC}dt = -\frac{dQ(t)}{CV_0 - Q(t)} = \frac{d\xi}{\xi} \ , \tag{4}$$

where in the last term we have introduced the variable $\xi = CV_0 - Q(t)$, with $dQ(t) = -d\xi$. The equation can be integrated in both sides for the time range $t_0 = (0, t)$, corresponding to $\xi_0 = CV_0$ and $\xi(t) = CV_0 - Q(t)$ (we have used the initial condition on the charge mentioned above), respectively. Integration leads to

$$\ln(\frac{\xi(t)}{\xi_0}) = -\frac{t}{\tau} \ , \tag{5}$$

that, after coming back to the explicit variable $Q(t)$, reads:

$$Q(t) = CV_0(1 - \exp(-t/\tau)) \ . \tag{6}$$

The above equation dictates the temporal evolution of the charge at the capacitor plates. The other relevant quantities can then be obtained. For instance, the current flowing into the resistor is $I(t) = dQ(t)/dt = (CV_0/\tau)exp(-t/\tau) = (V_0/R)exp(-t/\tau)$ (the negative sign coming out from the derivative is canceled because of the previously mentioned considerations on charge and current signs), whereas the voltage drop across the resistor is $\Delta V_R(t) = RI(t) = V_0 \exp(-t/\tau)$. Note that the expression for the voltage measured between the capacitor plates is different: $\Delta V_C(t) = Q(t)/C = V_0(1 - \exp(-t/\tau))$.

A. RC with an alternate generator

Let's now consider an alternate voltage generator in the place of the continuous generator assumed before. In particular, we will put $V(t) = V_0 \cos(\omega t)$, which describes an AC generator (ideal, with no internal resistance and purely "monochromatic"). The relevant equation is now

$$V_0 \cos(\omega t) = \frac{Q(t)}{C} + R\frac{dQ(t)}{dt} \ . \quad (7)$$

From a qualitative point of view, the behavior of the circuit depends on the angular frequency of the generator. For instance, in the case of small ω, or, better, $\omega\tau << 1$, the capacitor will have time to charge and discharge (the time constant of both processes is $\tau = RC$), hence reaching the steady state conditions where no current flow from or to the capacitor plates. In the steady state, no current flows through the resistor, hence no voltage will be read across the resistor. In fact, this is what happens, for instance, in the AC filter at the input stage of an oscilloscope, which is used to cut (remove) the continuous (DC) component of a signal. In the opposite case, $\omega\tau >> 1$, the capacitor will not have enough time to complete the charge/discharge process, and the current will flow through the resistor like the capacitor were a short-circuit. Very nice! We have a system which responds to the frequency.

Let's look for a more quantitative solution. In particular, we want to see if a trial solution of the kind $Q(t) = Q_0 \cos(\omega t + \phi)$, with Q_0 and ϕ parameters to be determined, satisfies the equality stated in Eq. 7. We remind that $dQ(t)/dt = -\omega Q_0 \sin(\omega t + \phi)$ and also that $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$ and $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$.

By placing the derivative of the trial function $Q(t)$ into the right side of Eq. 7 and using the above mentioned trivial trigonometric relations, we get:

$$V_0 \cos(\omega t) = \frac{Q_0}{C}(\cos(\omega t)\cos\phi - \sin(\omega t)\sin\phi) - \quad (8)$$
$$- \omega Q_0 R(\sin(\omega t)\cos\phi + \cos(\omega t)\sin\phi) \ . \quad (9)$$

If we want the trial function $Q(t)$ to be a solution, i.e., to ensure the equality between left and right sides, we must separately consider the terms oscillating as $\sin(\omega t)$ and $\cos(\omega t)$. For the first terms, we get

$$0 = -\frac{Q_0}{C}\sin\phi - \omega Q_0 R\cos\phi \ , \quad (10)$$

that is

$$\tan\phi = -\omega\tau \ , \quad (11)$$

where $\tau = RC$. For the equality of the terms containing $\cos(\omega t)$ we need

$$V_0 = \frac{Q_0}{C}\cos\phi - \omega Q_0 R\sin\phi \ , \quad (12)$$

that is

$$Q_0 = \frac{V_0}{\cos\phi}\frac{1}{\frac{1}{C} - \omega R\tan\phi} = CV_0\frac{1}{\sqrt{1+\omega^2\tau^2}} \ , \quad (13)$$

where for the last passage we have used $\tan\phi = -\omega\tau$, as found above, and the trigonometric relation $\cos\phi = 1/\sqrt{1+\tan^2\phi}$.

Therefore, the charge at the capacitor plate oscillates at the angular frequency ω with an amplitude Q_0 proportional to $1/\sqrt{1+\omega^2\tau^2}$ and a dephasing $\phi = \arctan(-\omega\tau)$ with respect to the "forcing" oscillation. The limiting cases lead to $Q_0 \to CV_0$, $\phi \to 0$, and $Q_0 \to 0$, $\phi \to -\pi/2$ for $\omega\tau << 1$ ($\omega\tau \to 0$) and $\omega\tau >> 1$ ($\omega\tau \to \infty$), respectively.

The amplitude of the voltage read between the capacitor plates will be

$$\Delta V_{C,max} = \frac{Q_0}{C} = V_0\frac{1}{\sqrt{1+\omega^2\tau^2}} \ , \quad (14)$$

that drops to zero for increasing ω.

The amplitude of the voltage read across the resistor, which is proportional to $I(t) = dQ(t)/d(t)$, is instead given by

$$\Delta V_{R,max} = RI_{max} = R\frac{dQ}{dt}|_{max} = V_0\frac{\omega\tau}{\sqrt{1+\omega^2\tau^2}} \quad (15)$$

which is null for $\omega\tau \to 0$, but tends to V_0 for increasing ω (the dephasing tends to π, as you can easily check).

Very, very nice! We have again a behavior which depends on the angular frequency! Try to find out the mathematical and physical implications for that, by noticing, among other aspects, that the considered circuit can act, for specific frequencies and choices of the voltage to be taken as the output (the one read across either the capacitor or the resistor) as a "low-pass" or "high-pass" frequency filter. This is well illustrated by Fig. 2, where the amplitude of the signal read across the capacitor and the resistor [panels (a) and (b), respectively] are numerically calculated as a function of the (dimensionless) product $\omega\tau$.

III. THE RL CIRCUIT

Let's now switch to the RL circuit depicted in Fig. 1(b), and use it in the same configuration as in Sec. II. The equation ruling the temporal behavior of the circuit will now read

$$V_0 = L\frac{dI(t)}{dt} + RI(t) \ , \quad (16)$$

where we have used the Ohm's law to express the voltage drop across the resistor and the equation describing the voltage between the inductor ends ($\Delta V_{ind} = LdI/dt$). Note that the choice of sign is correct. In fact, the minus sign appearing in the Faraday's law at the basis of the inductor equation (see, e.g., the situation established when

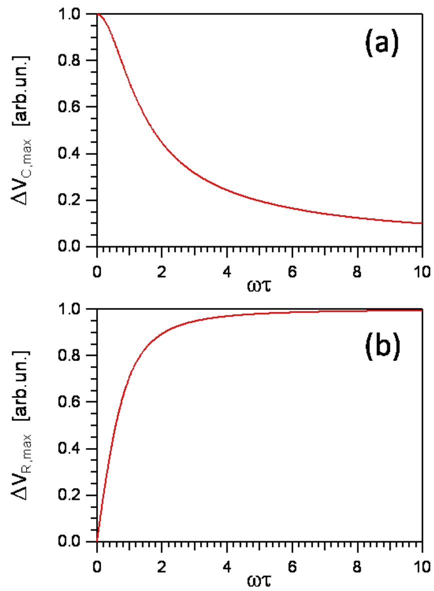


FIG. 2. Numerical calculations of the maximum amplitude $\Delta V_{C,max}$ and $\Delta V_{R,max}$ defined in the text [plot (a) and (b), respectively] as a function of $\omega\tau$. Bear in mind that $\omega = 2\pi f$, with f frequency of the alternate generator. High and low frequencies components of the "input" signal $V(t)$ are practically cut (heavily suppressed) depending on whether the "output" signal is measured across the capacitor or the resistor.

an ideal solenoid is used as a model inductor) states that the output lead of the inductor stays at a voltage lower than its input, where input and output are defined according to the direction of the current flowing through the solenoid (do your best to convince yourselves on that!).

Formally, Eq. 16, which replaces Eq. 1, is similar to Eq. 1, but for the different physical quantities it contains. The solution, assuming $t_0 = 0$ and $I_0 = 0$, will then be:

$$I(t) = \frac{V_0}{R}(1 - \exp(-t/\tau)) \tag{17}$$

with $\tau = L/R$ (note the difference with respect to the time constant of the RC circuit!).

The process is somehow similar to the one experienced by a capacitor during its charging: now, it is the current which increases as a function of time approaching the limit level V_0/R. One option to appreciate the intimate motivations for such a similar behavior is based on considering that during the charging process a capacitor gets an energy $CV_0^2/2$ (an equal amount of energy is "dissipated" by the Joule effect of the resistor - please, demonstrate it!). In the process involving the inductor, it gets an energy $LI^2/2$ (an equal amount of energy is "dissipated" by the resistor, as well). So both processes entail energy storage "somewhere in the space" (in the capacitor or the inductor, respectively), energy which is obviously provided by the generator.

A. RL with an alternate generator

Consider the RL circuit connected to to the alternate generator, $V(t) = V_0 \cos(\omega t)$. The differential equation mastering the system is

$$V_0 \cos(\omega t) = L\frac{dI(t)}{dt} + RI(t) , \tag{18}$$

very similar, formally, to Eq. 7.

The solution, easily obtained by following the above mentioned steps, is $I(t) = I_0 \cos(\omega t + \phi)$ with

$$\tan\phi = -\omega\tau \tag{19}$$

$$I_0 = \frac{V_0}{R}\frac{1}{\sqrt{1+\omega^2\tau^2}} . \tag{20}$$

The voltage read across the resistor will now have an amplitude

$$\Delta V_{R,max} = RI_{max} = V_0\frac{1}{\sqrt{1+\omega^2\tau^2}} , \tag{21}$$

whose behavior as a function of $\omega\tau$ is the same of the previously calculated $\Delta V_{C,max}$ [see Fig. 2(a)]. Therefore, if the "output" is taken across the resistor, the RL circuit acts as a "low-pass" frequency filter. Note in particular that the (angular) frequency $\omega_{1/2}$ corresponding to an output signal amplitude equal to one half of the input signal amplitude (this is required in some practical exercise!), can be obtained by solving the algebraic equation

$$V_0\frac{1}{2} = V_0\frac{1}{\sqrt{1+\omega_{1/2}^2\tau^2}} , \tag{22}$$

that leads, after a few basic manipulations, to $\omega_{1/2} = \sqrt{3}/\tau = \sqrt{3}R/L$ (remember that we are here assuming all internal resistances negligible, that must be carefully verified in practical experiments). The corresponding frequency $f_{1/2}$ of the signal generator is $f_{1/2} = 2\pi\omega_{1/2} = 2\pi\sqrt{3}R/L \approx 65R/L$.

On the contrary, the voltage across the inductor will be $\Delta V_L = LdI(t)/dt$. Its amplitude is

$$\Delta V_{L,max} = L\frac{dI(t)}{dt}|_{max} = V_0\frac{\omega\tau}{\sqrt{1+\omega_{1/2}^2\tau^2}} , \tag{23}$$

which behaves similarly to the amplitude of the voltage read across the resistor in the RC circuit. In other words, if the "output" is read across the inductor, a "high-pass" frequency filter will be attained. Please note, once more, that the behavior of the circuit depends on what you are measuring.

IV. THE DEPHASING AND ITS VISUALIZATION

As already mentioned, one gets in both circuits (but with different values and expressions of the characteristic

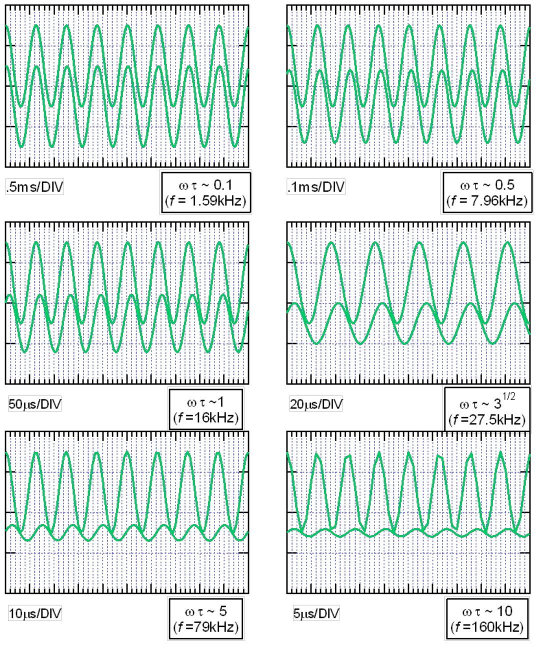


FIG. 3. Numerical calculations of what should be observed in an ideal experiment carried out according to the recipes described in the text. The oscilloscope screen is simulated: note that the sweep speed (TIME/DIV) is adjusted according to the measured signals, as indicated in the bottom left of the plots. Note also that, to make the simulation a little bit more realistic, a characteristic time $\tau = 100\mu s$ has been considered and that the simulated traces corresponding to the two channels have been vertically shifted each other for the sake of clarity.

time τ) $\phi = \arctan(-\omega\tau)$. In order to predict something which can be visualized in practical experiments, we will concentrate on the RL circuit, assuming that $V(t)$ and $V_R(t) = RI(t)$ are simultaneously measured.[1] This can be easily accomplished by using a two channel oscilloscope, with, say, the probe of channel 1 is connected to the signal generator and that of channel 2 to the resistor [the relevant nodes are indicated in red with 1 and 2 in Fig. 1(b)]. Obviously, as also shown in the figure, the "bottom" line of the circuit must be connected to ground.

The term ϕ plays the role of dephasing between the periodical signals visualized by the two channel traces. In order to get a stable and reliable visualization, you must obviously take care of the oscilloscope trigger, for instance by triggering on channel 1 (trigger source CH1, trigger mode "normal", trigger level and slope appropriate to the situation). Figure 3 shows what you should expect to observe on the screen for a few selected choices of the generator frequency $f = 2\pi\omega$ (the plots are calculated numerically, the real experimental conditions will possibly lead to much less neat traces!). The graphs account also for the amplitude decrease as a function of $\omega\tau$ discussed in the previous section. The change in the dephasing (whether negative or positive is impossible to understand from the plots, consider this point!) is clearly evident!

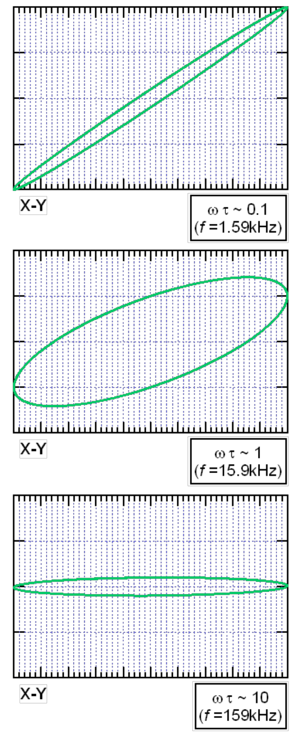


FIG. 4. Same of Fig. 3, but simulating the oscilloscope in the X-Y operating mode (Y is channel 2, X is channel 1). The same V/DIV setting is used for all plots (for both channels).

The previous simulation refers to the use of the oscilloscope in the Y-t mode, as required to visualize waveforms (voltages which vary as a function of time). We can also simulate the oscilloscope readout seen when the X-Y operating mode is selected. This is shown in Fig. 4 for a few selected choices of $\omega\tau$. Elliptical traces are obtained, and you must do your best to understand why!

[1] In the practice, one would avoid to take the reference signal connecting the oscilloscope to the point 1 in Fig. 1 in order to prevent altering the circuit behavior. As a matter of fact, the oscilloscope, as any instrumentation, has its own input impedance (typically modeled by an RC circuit with R = 1 Mohm and C = 25 pF). Even though this leads quite often to practically negligible effects, a safer way to get the reference signal is to use the TRIG OUTPUT or SYNC output of the signal generator, which consists in a TTL signal having the same frequency and phase of the generated signal.