December, 2018

Notes on the (two-)Weyl Fermi arcs

1 Notations

Let us consider the Bloch hamiltonian H,

$$H = v \begin{pmatrix} -i\partial_3 & h - \partial_2 \\ h + \partial_2 & i\partial_3 \end{pmatrix} , \tag{1.1}$$

where

$$h = \frac{\left[(-i\partial_1)^2 - k_0^2 \right]}{2k_0} . \tag{1.2}$$

The parameters v and k_0 have respectively the dimensions of a velocity and of a momentum, so that the eigenvalues of H represent energy values. (In our notation, $\hbar = 1 = c$.) In $\mathbb{R}^3$, the hamiltonian commutes with the components of the momentum, so one can put

$$\psi_{\boldsymbol{p}}(\boldsymbol{x}) = e^{i\boldsymbol{p}\boldsymbol{x}} \, \chi(\boldsymbol{p}) . \tag{1.3}$$

One finds

$$H \, \chi(\boldsymbol{p}) = v \begin{pmatrix} p_3 & h(p_1) - ip_2 \\ h(p_1) + ip_2 & -p_3 \end{pmatrix} \chi(\boldsymbol{p}) , \tag{1.4}$$

with

$$h(p_1) = \frac{p_1^2 - k_0^2}{2k_0} = \frac{(p_1 + k_0)(p_1 - k_0)}{2k_0} . \tag{1.5}$$

Therefore the eigenvalues of the hamiltonian are given by

$$H \, u(\boldsymbol{p}) = E(\boldsymbol{p}) \, u(\boldsymbol{p}) \quad , \quad H \, v(\boldsymbol{p}) = -E(\boldsymbol{p}) \, v(\boldsymbol{p}) \tag{1.6}$$

with

$$E(\boldsymbol{p}) = v \, \varepsilon(\boldsymbol{p}) = v \sqrt{h^2(p_1) + p_2^2 + p_3^2} \, , \tag{1.7}$$

and the corresponding normalised “eigenspinors” take the form

$$u(\boldsymbol{p}) = n(\boldsymbol{p}) \begin{pmatrix} \varepsilon(\boldsymbol{p}) + p_3 \\ h(p_1) + ip_2 \end{pmatrix} \quad , \quad v(\boldsymbol{p}) = n(\boldsymbol{p}) \begin{pmatrix} -h(p_1) + ip_2 \\ \varepsilon(\boldsymbol{p}) + p_3 \end{pmatrix} , \tag{1.8}$$

where the normalization factor $n(\boldsymbol{n})$ is given by

$$n(\boldsymbol{p}) = \frac{1}{\sqrt{2\varepsilon(\boldsymbol{p})(\varepsilon(\boldsymbol{p}) + p_3)}} \ . \tag{1.9}$$

A direct computation shows that

$$\begin{aligned} u_1^*(\boldsymbol{p})u_1(\boldsymbol{p}) &= \frac{\varepsilon + p_3}{2\varepsilon} \quad , \quad u_1^*(\boldsymbol{p})u_2(\boldsymbol{p}) = \frac{h + ip_2}{2\varepsilon} \\ u_2^*(\boldsymbol{p})u_1(\boldsymbol{p}) &= \frac{h - ip_2}{2\varepsilon} \quad , \quad u_2^*(\boldsymbol{p})u_2(\boldsymbol{p}) = \frac{\varepsilon - p_3}{2\varepsilon} \ ; \end{aligned} \tag{1.10}$$

and

$$\begin{aligned} v_1^*(\boldsymbol{p})v_1(\boldsymbol{p}) &= \frac{\varepsilon - p_3}{2\varepsilon} \quad , \quad v_1^*(\boldsymbol{p})v_2(\boldsymbol{p}) = -\frac{h + ip_2}{2\varepsilon} \\ v_2^*(\boldsymbol{p})v_1(\boldsymbol{p}) &= -\frac{h - ip_2}{2\varepsilon} \quad , \quad v_2^*(\boldsymbol{p})v_2(\boldsymbol{p}) = \frac{\varepsilon + p_3}{2\varepsilon} \ . \end{aligned} \tag{1.11}$$

So one finds

$$u_\alpha^*(\boldsymbol{p})u_\beta(\boldsymbol{p}) + v_\alpha^*(\boldsymbol{p})v_\beta(\boldsymbol{p}) = \delta_{\alpha\beta} \ . \tag{1.12}$$

2 Boundary conditions in half space

We consider the manifold

$$M = \mathbb{R}^2 \times \mathbb{R}_+ \quad , \quad (x^1, x^2, x^3) \quad \text{with} \quad x^3 \geq 0 \ . \tag{2.1}$$

The boundary conditions so that H is hermitian,

$$\langle H\varphi \,|\, \psi\rangle = \langle \varphi \,|\, H\psi\rangle \tag{2.2}$$

for any state $|\,\psi\rangle$ and $\langle\varphi|$, imply

$$\varphi^\dagger(\boldsymbol{x})\,\sigma^3\,\psi(\boldsymbol{x})\,\big|_{x^3=0} = 0 \tag{2.3}$$

Equation (2.3) is satisfied when each wave function $\psi(\boldsymbol{x})$ verifies

$$\left(\sigma^1\cos\gamma + \sigma^2\sin\gamma\right)\ \psi(\boldsymbol{x})\,\big|_{x^3=0} \ = \ \psi(\boldsymbol{x})\,\big|_{x^3=0} \ , \tag{2.4}$$

where the parameter $0 \leq \gamma \leq 2\pi$ characterises the boundary conditions. Since

$$\sigma(\gamma) \equiv \left(\sigma^1\cos\gamma + \sigma^2\sin\gamma\right) = \begin{pmatrix} 0 & e^{-i\gamma} \\ e^{i\gamma} & 0 \end{pmatrix} \ , \tag{2.5}$$

one finds

$$\sigma(\gamma)\,w(\gamma) = w(\gamma) \quad , \quad \text{with} \quad w(\gamma) = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\gamma/2} \\ e^{i\gamma/2} \end{pmatrix} \ . \tag{2.6}$$

The complete set of solutions of the constraint (2.4) is composed of two parts: the so-called scattering states and the edge states.

2.1 Scattering states

The wave functions of the scattering states have the form

$$\phi_{\boldsymbol{p}}^{(+)}(\boldsymbol{x}) = e^{i(p_1x^1+p_2x^2-p_3x^3)}\, u(p_1,p_2,-p_3) + S_+(\boldsymbol{p}) e^{i\boldsymbol{px}}\, u(\boldsymbol{p}) \ , \tag{2.7}$$

and

$$\phi_{\boldsymbol{p}}^{(-)}(\boldsymbol{x}) = e^{i(p_1x^1+p_2x^2-p_3x^3)}\, v(p_1,p_2,-p_3) + S_-(\boldsymbol{p}) e^{i\boldsymbol{px}}\, v(\boldsymbol{p}) \ , \tag{2.8}$$

where $\boldsymbol{px} = p_1x^1 + p_2x^2 + p_2x^3$, with $p_3 \geq 0$. They verify

$$H\, \phi_{\boldsymbol{p}}^{(+)}(\boldsymbol{x}) = E(\boldsymbol{p})\, \phi_{\boldsymbol{p}}^{(+)}(\boldsymbol{x}) \quad , \quad H\, \phi_{\boldsymbol{p}}^{(-)}(\boldsymbol{x}) = -E(\boldsymbol{p})\, \phi_{\boldsymbol{p}}^{(-)}(\boldsymbol{x}) \ . \tag{2.9}$$

One can imagine that $S_\pm(\boldsymbol{p})$ are the scattering transition amplitudes (or the element of the scattering matrix) for particles, with incoming momentum $(p_1, p_2, -p_3)$, which are scattered on the plane $x^3 = 0$ and have final momentum $\boldsymbol{p} = (p_1, p_2, p_3)$. The boundary condition (2.4) means

$$u(p_1,p_2,-p_3) + S_+(\boldsymbol{p})\, u(\boldsymbol{p}) \propto w(\gamma) \quad , \quad v(p_1,p_2,-p_3) + S_-(\boldsymbol{p})\, v(\boldsymbol{p}) \propto w(\gamma) \ . \tag{2.10}$$

One finds

$$S_+(\boldsymbol{p}) = \left[\frac{\varepsilon + p_3}{\varepsilon - p_3}\right]^{1/2} \frac{-\varepsilon + p_3 + e^{-i\gamma}(h + ip_2)}{\varepsilon + p_3 - e^{-i\gamma}(h+ip_2)} \ , \tag{2.11}$$

$$S_-(\boldsymbol{p}) = -\left[\frac{\varepsilon + p_3}{\varepsilon - p_3}\right]^{1/2} \frac{\varepsilon - p_3 + e^{i\gamma}(h - ip_2)}{\varepsilon + p_3 + e^{i\gamma}(h-ip_2)} \ . \tag{2.12}$$

Note that

$$|\, S_\pm(\boldsymbol{p})\, | = 1 \ , \tag{2.13}$$

and

$$S_\pm^*(\boldsymbol{p}) = S_\pm(p_1, p_2, -p_3) \ . \tag{2.14}$$

2.2 Complex poles of the scattering matrix

The amplitudes $S_\pm$ shown in equations (2.11) and(2.12) have poles in the complex p_3-plane. In particular $S_+(\boldsymbol{p})$ has a pole when

$$p_3 = \widetilde{p}_3 = i\ [p_2\, \cos\gamma - h(p_1) \sin\gamma] \ , \tag{2.15}$$

and

$$h(p_1)\ \cos\gamma + p_2\ \sin\gamma \geq 0 \ . \tag{2.16}$$

In this case, one has

$$\varepsilon(p_1, p_2, \widetilde{p}_3) = h(p_1)\ \cos\gamma + p_2\ \sin\gamma \ . \tag{2.17}$$

On the other hand, $S_-(\boldsymbol{p})$ has a pole when

$$p_3 = \widetilde{p}_3 = i\ [p_2\, \cos\gamma - h(p_1) \sin\gamma] \ , \tag{2.18}$$

and

$$[\, h(p_1)\ \cos\gamma + p_2\ \sin\gamma\,] \leq 0 \ . \tag{2.19}$$

In this case, one has

$$\varepsilon(p_1, p_2, \widetilde{p}_3) = - \ [h(p_1) \ \cos\gamma + p_2 \ \sin\gamma] \ . \tag{2.20}$$

In order to determine the residues of the amplitudes which are associated to the previous poles, let us put

$$p_3 = \widetilde{p}_3 + \epsilon \ . \tag{2.21}$$

Then

$$\mathcal{R}_+ = \lim_{\epsilon\to 0} \epsilon\, S_+(\boldsymbol{p}) = \frac{h(p_1) \ \cos\gamma + p_2 \ \sin\gamma}{\sqrt{h^2+p_2^2}} \, 2\,\widetilde{p}_3 \ . \tag{2.22}$$

Similarly, in the case of S_- one gets

$$\mathcal{R}_- = \lim_{\epsilon\to 0} \epsilon\, S_-(\boldsymbol{p}) = \frac{h(p_1) \ \cos\gamma + p_2 \ \sin\gamma}{\sqrt{h^2+p_2^2}} \, 2\,\widetilde{p}_3 \tag{2.23}$$

It is useful to compute the value of the spinors $u(\boldsymbol{p})$ and $v(\boldsymbol{p})$ when $p_3 = \widetilde{p}_3$.

At the pole of $S_+(\boldsymbol{p})$, one finds

$$u(\boldsymbol{p})\Big|_{p_3=\widetilde{p}_3} = \sqrt{\frac{h+ip_2}{\varepsilon(p_1,p_2,\widetilde{p}_3)}}\, w(\gamma) = \sqrt{\frac{h+ip_2}{h\cos\gamma + p_2\sin\gamma}}\, w(\gamma) \ , \tag{2.24}$$

and at the pole of $S_-(\boldsymbol{p})$ one gets

$$v(\boldsymbol{p})\Big|_{p_3=\widetilde{p}_3} = \sqrt{\frac{-h+ip_2}{\varepsilon(p_1,p_2,\widetilde{p}_3)}}\, w(\gamma) = \sqrt{\frac{h-ip_2}{h\cos\gamma + p_2\sin\gamma}}\, w(\gamma) \ . \tag{2.25}$$

3 Completeness relations

In order to recover the completeness relations, let us consider the integral

$$I^{(+)} = \int' \frac{d^3p}{(2\pi)^3} \, \left(\phi^{(+)}_{\boldsymbol{p}}(\boldsymbol{y})\right)^*_\beta \left(\phi^{(+)}_{\boldsymbol{p}}(\boldsymbol{x})\right)_\alpha \ , \tag{3.1}$$

where the integral on the momenta is restricted by $p_3 \geq 0$. Let us define

$$\widehat{p} = (p_1, p_2, -p_3) \ . \tag{3.2}$$

Then

$$\begin{aligned} I^{(+)} &= \int' \frac{d^3p}{(2\pi)^3} \Big[e^{-i\widehat{\boldsymbol{p}}\boldsymbol{y}} u^*_\beta(\widehat{\boldsymbol{p}}) + S^*_+(\boldsymbol{p}) e^{-i\boldsymbol{p}\boldsymbol{y}} u^*_\beta(\boldsymbol{p}) \Big] \Big[e^{i\widehat{\boldsymbol{p}}\boldsymbol{x}} u_\alpha(\widehat{\boldsymbol{p}}) + S_+(\boldsymbol{p}) e^{i\boldsymbol{p}\boldsymbol{x}} u_\alpha(\boldsymbol{p}) \Big] \\ &= \int' \frac{d^3p}{(2\pi)^3} \Big\{ e^{-i\widehat{\boldsymbol{p}}\boldsymbol{y}} e^{i\widehat{\boldsymbol{p}}\boldsymbol{x}} u^*_\beta(\widehat{\boldsymbol{p}}) u_\alpha(\widehat{\boldsymbol{p}}) + S^*_+(\boldsymbol{p}) S_+(\boldsymbol{p}) e^{-i\boldsymbol{p}\boldsymbol{y}} e^{i\boldsymbol{p}\boldsymbol{x}} u^*_\beta(\boldsymbol{p}) u_\alpha(\boldsymbol{p}) \\ &\quad + S_+(\boldsymbol{p}) e^{-i\widehat{\boldsymbol{p}}\boldsymbol{y}} e^{i\boldsymbol{p}\boldsymbol{x}} u^*_\beta(\widehat{\boldsymbol{p}}) u_\alpha(\boldsymbol{p}) + S^*_+(\boldsymbol{p}) e^{-i\boldsymbol{p}\boldsymbol{y}} e^{i\widehat{\boldsymbol{p}}\boldsymbol{x}} u^*_\beta(\boldsymbol{p}) u_\alpha(\widehat{\boldsymbol{p}}) \Big\} \ . \end{aligned} \tag{3.3}$$

Since $S_+^*(\boldsymbol{p})S_+(\boldsymbol{p}) = 1$, one finds

$$\int' \frac{d^3p}{(2\pi)^3}\Big\{e^{-i\widehat{\boldsymbol{p}}\boldsymbol{y}}e^{i\widehat{\boldsymbol{p}}\boldsymbol{x}}u_\beta^*(\widehat{\boldsymbol{p}})u_\alpha(\widehat{\boldsymbol{p}}) + S_+^*(\boldsymbol{p})S_+(\boldsymbol{p})e^{-i\boldsymbol{p}\boldsymbol{y}}e^{i\boldsymbol{p}\boldsymbol{x}}u_\beta^*(\boldsymbol{p})u_\alpha(\boldsymbol{p})\Big\} =$$
$$= \int \frac{d^3p}{(2\pi)^3}\, e^{i\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{y})}u_\beta^*(\boldsymbol{p})u_\alpha(\boldsymbol{p})\ , \tag{3.4}$$

where the last integral in d^3p is performed in all space, without constraints. Similarly, because of the identity (2.14),

$$S_+^*(\boldsymbol{p}) = S_+^*(p_1,p_2,p_3) = S_+(p_1,p_2,-p_3)\ , \tag{3.5}$$

one has

$$\int' \frac{d^3p}{(2\pi)^3}\Big\{S_+(\boldsymbol{p})e^{-i\widehat{\boldsymbol{p}}\boldsymbol{y}}e^{i\boldsymbol{p}\boldsymbol{x}}u_\beta^*(\widehat{\boldsymbol{p}})u_\alpha(\boldsymbol{p}) + S_+^*(\boldsymbol{p})e^{-i\boldsymbol{p}\boldsymbol{y}}e^{i\widehat{\boldsymbol{p}}\boldsymbol{x}}u_\beta^*(\boldsymbol{p})u_\alpha(\widehat{\boldsymbol{p}})\Big\} =$$
$$= \int \frac{d^3p}{(2\pi)^3}\, e^{ip_1(x^1-y^1)+ip_2(x^2-y^2)}e^{ip_3(x^3+y^3)}S_+(\boldsymbol{p})\, u_\beta^*(\widehat{\boldsymbol{p}})u_\alpha(\boldsymbol{p})\ . \tag{3.6}$$

In the computation of the integral $\int_{-\infty}^{\infty} dp_3$, one has to note that since $x^3 \geq 0$ and $y^3 \geq 0$, one can integrate on a large semi-circle on the upper half-plane of the complex p_3-plane. By means of the residue theorem, one then gets

$$\int \frac{d^3p}{(2\pi)^3}\, e^{ip_1(x^1-y^1)+ip_2(x^2-y^2)}e^{ip_3(x^3+y^3)}S_+(\boldsymbol{p})\, u_\beta^*(\widehat{\boldsymbol{p}})u_\alpha(\boldsymbol{p}) = \tag{3.7}$$
$$= -\int \frac{d^2p}{(2\pi)^2}\, e^{ip_1(x^1-y^1)+ip_2(x^2-y^2)}e^{-[p_2\cos\gamma - h\sin\gamma](x^3+y^3)}2[p_2\cos\gamma - h\sin\gamma] \times$$
$$\times w_\beta^*(\gamma)\, w_\alpha(\gamma)\, \Theta(p_2\cos\gamma - h\sin\gamma \geq 0)\, \Theta(h\cos\gamma + p_2\sin\gamma \geq 0)\ .$$

Let the wave function of the edge states with positive energy be

$$\zeta_{\boldsymbol{p}}^{(+)}(\boldsymbol{x}) = \sqrt{2[p_2\cos\gamma - h\sin\gamma]}\, e^{i(p_1x^1+p_2x^2)}\, e^{-x^3[p_2\cos\gamma - h\sin\gamma]}\, w(\gamma) \times$$
$$\times\, \Theta(p_2\cos\gamma - h\sin\gamma \geq 0)\, \Theta(h\cos\gamma + p_2\sin\gamma \geq 0)\ . \tag{3.8}$$

Then,

$$I^{(+)} = \int \frac{d^3p}{(2\pi)^3}\, e^{i\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{y})}u_\beta^*(\boldsymbol{p})u_\alpha(\boldsymbol{p}) - \int \frac{d^2p}{(2\pi)^2}\, \left[\zeta_{\boldsymbol{p}}^{(+)}(\boldsymbol{y})\right]_\beta^* \left[\zeta_{\boldsymbol{p}}^{(+)}(\boldsymbol{x})\right]_\alpha\ . \tag{3.9}$$

Let us now consider

$$I^{(-)} = \int' \frac{d^3p}{(2\pi)^3}\Big[e^{-i\widehat{\boldsymbol{p}}\boldsymbol{y}}v_\beta^*(\widehat{\boldsymbol{p}}) + S_-^*(\boldsymbol{p})e^{-i\boldsymbol{p}\boldsymbol{y}}v_\beta^*(\boldsymbol{p})\Big]\Big[e^{i\widehat{\boldsymbol{p}}\boldsymbol{x}}v_\alpha(\widehat{\boldsymbol{p}}) + S_-(\boldsymbol{p})e^{i\boldsymbol{p}\boldsymbol{x}}v_\alpha(\boldsymbol{p})\Big]$$
$$= \int' \frac{d^3p}{(2\pi)^3}\Big\{e^{-i\widehat{\boldsymbol{p}}\boldsymbol{y}}e^{i\widehat{\boldsymbol{p}}\boldsymbol{x}}v_\beta^*(\widehat{\boldsymbol{p}})v_\alpha(\widehat{\boldsymbol{p}}) + S_-^*(\boldsymbol{p})S_-(\boldsymbol{p})e^{-i\boldsymbol{p}\boldsymbol{y}}e^{i\boldsymbol{p}\boldsymbol{x}}v_\beta^*(\boldsymbol{p})v_\alpha(\boldsymbol{p})$$
$$+S_-(\boldsymbol{p})e^{-i\widehat{\boldsymbol{p}}\boldsymbol{y}}e^{i\boldsymbol{p}\boldsymbol{x}}v_\beta^*(\widehat{\boldsymbol{p}})v_\alpha(\boldsymbol{p}) + S_-^*(\boldsymbol{p})e^{-i\boldsymbol{p}\boldsymbol{y}}e^{i\widehat{\boldsymbol{p}}\boldsymbol{x}}v_\beta^*(\boldsymbol{p})v_\alpha(\widehat{\boldsymbol{p}})\Big\}\ . \tag{3.10}$$

As before, one gets

$$\int' \frac{d^3p}{(2\pi)^3}\Big\{e^{-i\widehat{\boldsymbol{p}}\boldsymbol{y}}e^{i\widehat{\boldsymbol{p}}\boldsymbol{x}}v^*_\beta(\widehat{\boldsymbol{p}})v_\alpha(\widehat{\boldsymbol{p}}) + S^*_-(\boldsymbol{p})S_-(\boldsymbol{p})e^{-i\boldsymbol{p}\boldsymbol{y}}e^{i\widehat{\boldsymbol{p}}\boldsymbol{x}}v^*_\beta(\boldsymbol{p})v_\alpha(\boldsymbol{p})\Big\} = \\ = \int \frac{d^3p}{(2\pi)^3}\, e^{i\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{y})}v^*_\beta(\boldsymbol{p})v_\alpha(\boldsymbol{p}) \ . \tag{3.11}$$

One also has

$$\int' \frac{d^3p}{(2\pi)^3}\Big\{S_-(\boldsymbol{p})e^{-i\widehat{\boldsymbol{p}}\boldsymbol{y}}e^{i\boldsymbol{p}\boldsymbol{x}}v^*_\beta(\widehat{\boldsymbol{p}})v_\alpha(\boldsymbol{p}) + S^*_-(\boldsymbol{p})e^{-i\boldsymbol{p}\boldsymbol{y}}e^{i\widehat{\boldsymbol{p}}\boldsymbol{x}}v^*_\beta(\boldsymbol{p})v_\alpha(\widehat{\boldsymbol{p}})\Big\} = \\ = \int \frac{d^3p}{(2\pi)^3}\, e^{ip_1(x^1-y^1)+ip_2(x^2-y^2)}e^{ip_3(x^3+y^3)}S_-(\boldsymbol{p})\, v^*_\beta(\widehat{\boldsymbol{p}})v_\alpha(\boldsymbol{p}) \ . \tag{3.12}$$

By integrating on a large semi-circle in the complex p_3-plane, one finds

$$\int \frac{d^3p}{(2\pi)^3}\, e^{ip_1(x^1-y^1)+ip_2(x^2-y^2)}e^{ip_3(x^3+y^3)}S_-(\boldsymbol{p})\, v^*_\beta(\widehat{\boldsymbol{p}})v_\alpha(\boldsymbol{p}) = \tag{3.13}$$

$$= -\int \frac{d^2p}{(2\pi)^2}\, e^{ip_1(x^1-y^1)+ip_2(x^2-y^2)}e^{-[p_2\cos\gamma - h\sin\gamma](x^3+y^3)}2[p_2\cos\gamma - h\sin\gamma]\times \\ \times w^*_\beta(\gamma)\, w_\alpha(\gamma)\,\Theta(p_2\cos\gamma - h\sin\gamma\geq 0)\,\Theta(h\cos\gamma + p_2\sin\gamma \leq 0) \ .$$

Let the wave function of the edge states with negative energy be

$$\zeta^{(-)}_{\boldsymbol{p}}(\boldsymbol{x}) \;=\; \sqrt{2[p_2\cos\gamma - h\sin\gamma]}\, e^{i(p_1x^1+p_2x^2)}\, e^{-x^3[p_2\cos\gamma - h\sin\gamma]}\, w(\gamma)\ \times \\ \times\,\Theta(p_2\cos\gamma - h\sin\gamma\geq 0)\,\Theta(h\cos\gamma + p_2\sin\gamma \leq 0) \ . \tag{3.14}$$

Then,

$$I^{(-)} = \int \frac{d^3p}{(2\pi)^3}\, e^{i\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{y})}v^*_\beta(\boldsymbol{p})v_\alpha(\boldsymbol{p}) - \int \frac{d^2p}{(2\pi)^2}\,\left[\zeta^{(-)}_{\boldsymbol{p}}(\boldsymbol{y})\right]^*_\beta\left[\zeta^{(-)}_{\boldsymbol{p}}(\boldsymbol{x})\right]_\alpha \ . \tag{3.15}$$

Finally, by taking the sum of equations (3.9) and (3.15) one derives

$$\int' \frac{d^3p}{(2\pi)^3}\,\Big\{\left(\phi^{(+)}_{\boldsymbol{p}}(\boldsymbol{y})\right)^*_\beta\left(\phi^{(+)}_{\boldsymbol{p}}(\boldsymbol{x})\right)_\alpha + \left(\phi^{(-)}_{\boldsymbol{p}}(\boldsymbol{y})\right)^*_\beta\left(\phi^{(-)}_{\boldsymbol{p}}(\boldsymbol{x})\right)_\alpha\Big\} \\ + \int \frac{d^2p}{(2\pi)^2}\,\Big\{\left[\zeta^{(+)}_{\boldsymbol{p}}(\boldsymbol{y})\right]^*_\beta\left[\zeta^{(+)}_{\boldsymbol{p}}(\boldsymbol{x})\right]_\alpha + \left[\zeta^{(-)}_{\boldsymbol{p}}(\boldsymbol{y})\right]^*_\beta\left[\zeta^{(-)}_{\boldsymbol{p}}(\boldsymbol{x})\right]_\alpha\Big\} = \delta_{\alpha\beta}\,\delta^3(\boldsymbol{x}-\boldsymbol{y}) \ . \tag{3.16}$$

This equation shows that the wave functions represent a complete set of states.

4 Summary of the one-particle wave functions

Scattering states:

$$\phi^{(+)}_{\boldsymbol{p}}(\boldsymbol{x}) = e^{i(p_1x^1+p_2x^2-p_3x^3)}\, u(p_1,p_2,-p_3) + S_+(\boldsymbol{p})e^{i\boldsymbol{p}\boldsymbol{x}}\, u(\boldsymbol{p}) \ , \tag{4.1}$$

and

$$\phi_{\boldsymbol{p}}^{(-)}(\boldsymbol{x}) = e^{i(p_1 x^1 + p_2 x^2 - p_3 x^3)}\, v(p_1, p_2, -p_3) + S_-(\boldsymbol{p}) e^{i\boldsymbol{p}\boldsymbol{x}}\, v(\boldsymbol{p}) \;, \tag{4.2}$$

with

$$H\, \phi_{\boldsymbol{p}}^{(+)}(\boldsymbol{x}) = v\sqrt{h^2(p_1) + p_2^2 + p_3^2}\, \phi_{\boldsymbol{p}}^{(+)}(\boldsymbol{x}) \tag{4.3}$$

$$H\, \phi_{\boldsymbol{p}}^{(-)}(\boldsymbol{x}) = -v\sqrt{h^2(p_1) + p_2^2 + p_3^2}\, \phi_{\boldsymbol{p}}^{(-)}(\boldsymbol{x}) \;. \tag{4.4}$$

Edge states:

$$\begin{aligned} \zeta_{\boldsymbol{p}}(\boldsymbol{x}) \quad &= \quad \sqrt{2[p_2 \cos\gamma - h \sin\gamma]}\, e^{i(p_1 x^1 + p_2 x^2)}\, e^{-x^3 [p_2 \cos\gamma - h \sin\gamma]}\, w(\gamma) \;\times \\ &\times \Theta(p_2 \cos\gamma - h \sin\gamma \geq 0) \;, \end{aligned} \tag{4.5}$$

and

$$H\, \zeta_{\boldsymbol{p}}(\boldsymbol{x}) = v(h \cos\gamma + p_2 \sin\gamma)\, \zeta_{\boldsymbol{p}}(\boldsymbol{x}) \;. \tag{4.6}$$

5 Regions

The function

$$h = h(p_1) = \frac{p_1^2 - k_0^2}{2k_0}$$

is shown in Figure 1, and the value of h satisfies

$$-\frac{k_0}{2} \leq h \leq +\infty \;.$$

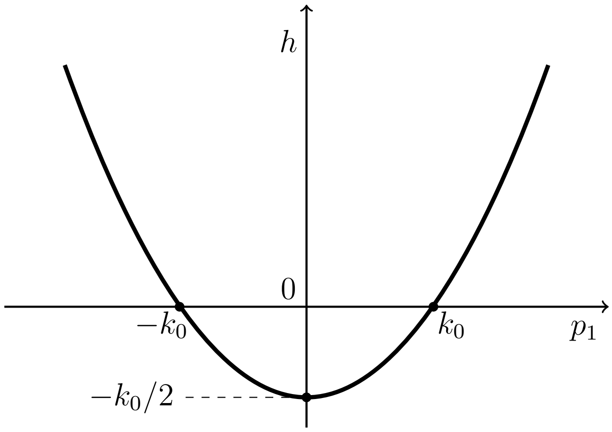

Figure 1

The region of the (p_2, h)-plane which is defined by the condition (existence of the bound edge states)

$$q_2 = q_2(p_1, p_2) = p_2 \cos\gamma - h(p_1) \sin\gamma \geq 0 \tag{5.1}$$

corresponds to the union of the two shaded regions which are shown in Figure 2, where it is assumed that $0 < \gamma < \pi/2$. The dotted (unlimited) region containing a plus sign $(+)$ is defined by the constraint (5.1) and by the additional constraint (positive energy)

$$q_1 = q_1(p_1, p_2) = h(p_1)\cos\gamma + p_2 \sin\gamma \geq 0 \; . \tag{5.2}$$

Whereas the (limited) region with pattern vertical lines, and congaing a minus sign $(-)$, is defined by the constraint (5.1) and by the additional constraint (negative energy)

$$q_1 = h\cos\gamma + p_2 \sin\gamma \leq 0 \; . \tag{5.3}$$

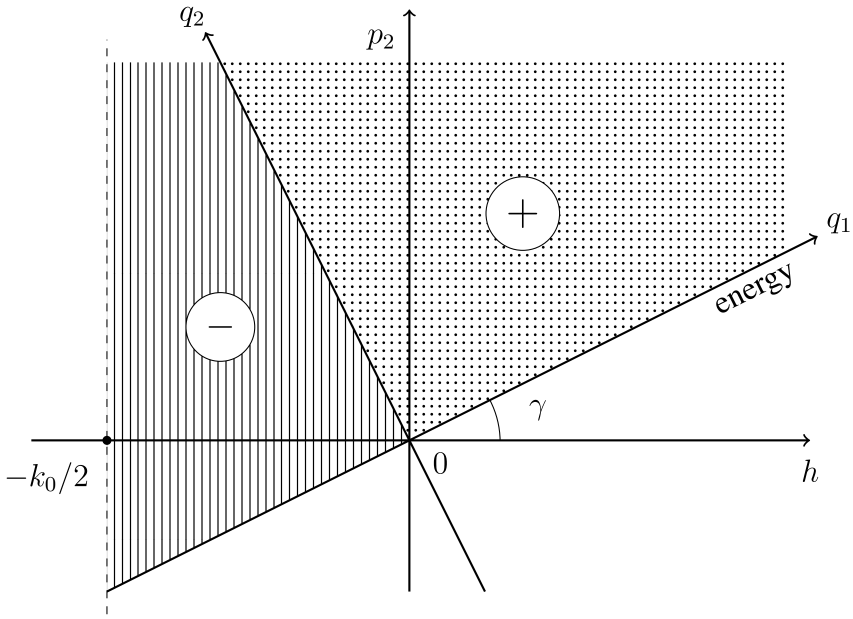

Figure 2

6 Fermi arcs

Let us concentrate on the edge states with vanishing energy.

6.1 $(\gamma = 0)$

When $\gamma = 0$, the vanishing value of the energy implies $h(p_1) = 0$ and then

$$p_1 = \pm k_0 \; . \tag{6.1}$$

The value of p_2 must be positive, but it can assume arbitrary values, as shown in Figure 3.

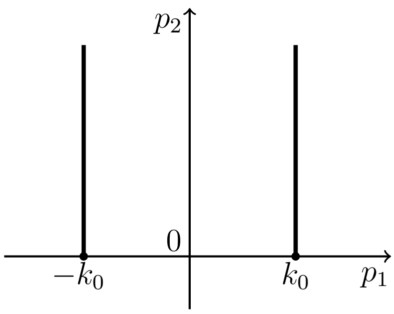

Figure 3. ($\gamma = 0$)

6.2 $(0 < \gamma < \pi/2)$

When $0 < \gamma < \pi/2$, the zero energy constraint gives

$$h(p_1) = -p_2 \tan\gamma \ , \tag{6.2}$$

which implies

$$p_2 = \frac{k_0^2 - p_1^2}{2k_0} \cot\gamma \ . \tag{6.3}$$

The existence of the boundary states condition reads

$$-h \cot\gamma \, \cos\gamma - h \sin\gamma = -h \frac{1}{\sin\gamma} \geq 0 \ , \tag{6.4}$$

which implies $h < 0$. Therefore, it must be $-k_0 \leq p_1 \leq k_0$, and the corresponding Fermi arc is shown in Figure 4.

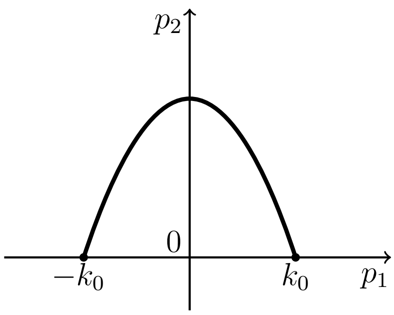

Figure 4. ($0 < \gamma < \pi/2$)

6.3 $(\gamma = \pi/2)$

When $\gamma = \pi/2$, the existence of edge states implies $h_(p_1) \leq 0$ and then $-k_0 \leq p_1 \leq k_0$. The vanishing condition for the energy requires $p_2 = 0$. The Fermi arc is shown in Figure 5.

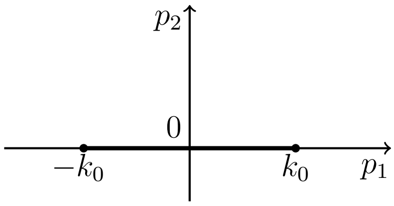

Figure 5. ($\gamma = \pi/2$)

6.4 ($\pi/2 < \gamma < \pi$)

When $\pi/2 < \gamma < \pi$, one has

$$\cos\gamma = -\cos(\pi - \gamma) < 0 \quad , \quad \sin\gamma = \sin(\pi - \gamma) > 0 \; .$$

The existence of edge states gives the constraint

$$p_2 \leq -h(p_1)\,\frac{\sin(\pi - \gamma)}{\cos(\pi - \gamma)} = \frac{k_0^2 - p_1^2}{2k_0}\,\tan(\pi - \gamma) \; . \tag{6.5}$$

On the other hand, the zero energy condition reads

$$p_2 = h(p_1)\,\frac{\cos(\pi - \gamma)}{\sin(\pi - \gamma)} = \frac{p_1^2 - k_0^2}{2k_0}\,\cot(\pi - \gamma) \; . \tag{6.6}$$

Therefore, the resulting Fermi arc is shown in Figure 6.

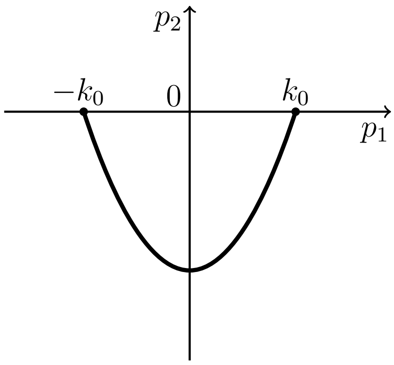

Figure 6. ($\pi/2 < \gamma < \pi$)

6.5 ($\gamma = \pi$)

When $\gamma = \pi$, the condition for the existence of edge states is given by

$$-p_2 \geq 0 \; , \tag{6.7}$$

and the zero energy constraint gives

$$-h(p_1) = 0 \;\longrightarrow\; p_1 = \pm k_0 \; . \tag{6.8}$$

The associated Fermi arc is shown in Figure 7.

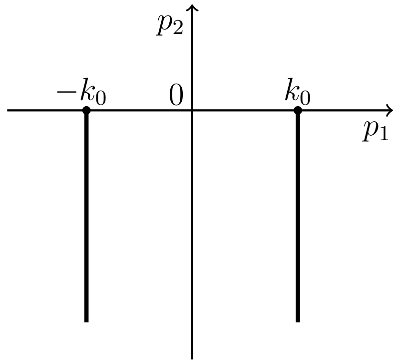

Figure 7. ($\gamma = \pi$)

6.6 ($-\pi < \gamma < -\pi/2$)

When $-\pi < \gamma < -\pi/2$, both $\cos\gamma$ and $\sin\gamma$ assume negative values. The zero energy condition gives

$$p_2 = -h(p_1)\frac{\cos\gamma}{\sin\gamma} = \frac{k_0^2 - p_1^2}{2k_0}\frac{\cos\gamma}{\sin\gamma} \ . \tag{6.9}$$

On the other hand, the existence of the edge states requires

$$p_2\cos\gamma - h\sin\gamma = -h\frac{\cos^2\gamma}{\sin\gamma} - h\sin\gamma \geq 0 \ , \tag{6.10}$$

which implies

$$h(p_1) \geq 0 \ , \tag{6.11}$$

and then $p_1 \leq -k_0$ or $p_1 \geq k_0$. The resulting Fermi arc is shown in Figure 8.

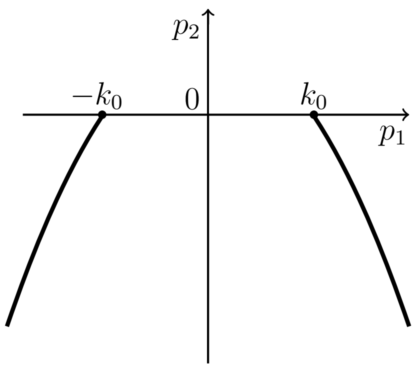

Figure 8. ($-\pi < \gamma < -\pi/2$)

6.7 ($\gamma = -\pi/2$)

When $\gamma = -\pi/2$, the zero energy condition gives

$$p_2 = 0 \ , \tag{6.12}$$

whereas the constraint due to the existence of edge states reads

$$h(p_1) \geq 0 \ , \tag{6.13}$$

which implies that $p_1 \leq -k_0$ or $p_1 \geq k_0$. In this case, the Fermi arc is shown in Figure 9.

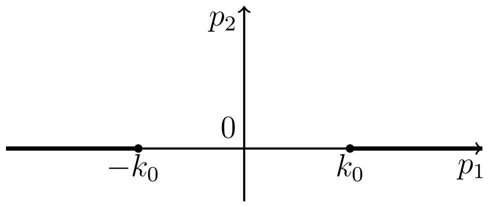

Figure 9. ($\gamma = -\pi/2$)

6.8 ($-\pi/2 < \gamma < 0$)

When $-\pi/2 < \gamma < 0$, one has $\cos\gamma > 0$ and $\sin\gamma = -|\sin\gamma| < 0$. The zero energy condition leads to

$$p_2 = h(p_1)\frac{\cos\gamma}{|\sin\gamma|} \ . \tag{6.14}$$

The existence of edge states condition states

$$h\frac{\cos^2\gamma}{|\sin\gamma|} + h\,|\sin\gamma| = \frac{h}{|\sin\gamma|} \geq 0 \ , \tag{6.15}$$

which implies $p_1 \geq k_0$ and $p_1 \leq -k_0$. So the corresponding Fermi arc is shown in Figure 10.

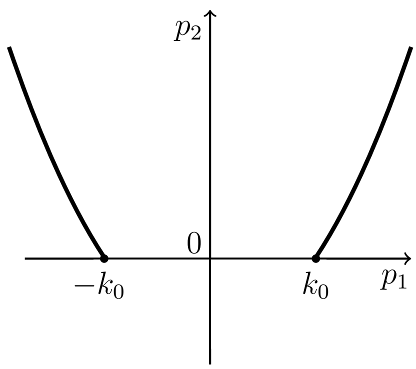

Figure 10. ($-\pi/2 < \gamma < 0$)

In what follows we shall consider the case in which

$$0 \leq \gamma \leq \pi \ .$$