Superconductivity for Particular Theorists[*)]

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No one did more than Nambu to bring the idea of spontaneously broken symmetries to the attention of elementary particle physicists. And, as he acknowledged in his ground-breaking 1960 article "Axial Current Conservation in Weak Interactions", Nambu was guided in this work by an analogy with the theory of superconductivity, to which Nambu himself had made important contributions. It therefore seems appropriate to honor Nambu on his birthday with a little pedagogical essay on superconductivity, whose inspiration comes from experience with broken symmetries in particle theory. I doubt if anything in this article will be new to the experts, least of all to Nambu, but perhaps it may help others, who like myself are more at home at high energy than at low temperature, to appreciate the lessons of superconductivity.

§ 1. Introduction

There is something peculiar about standard textbook treatments of superconductivity. On one hand, the reader learns that superconductors exhibit startling phenomena that can be predicted with extraordinary accuracy. For instance, electrical resistance is so low that currents can circulate for years without perceptible decay; the magnetic flux through a loop of such currents or through a vortex line in a superconductor is quantized to high precision; and a junction between two superconductors at different voltage produces an alternating current with a frequency that is so accurately predicted that it provides the best experimental value of the fundamental constant $e/\hbar$. But in deriving these results, textbooks generally use models that — despite their great historical importance — are surely no better than reasonably good approximations. There are macroscopic models, like that of Ginzburg and Landau, in which cooperative states of electrons are represented with a complex scalar field. And there is the microscopic model of Bardeen, Cooper and Schrieffer, from which the Ginzburg-Landau theory can be derived, and in which electrons appear explicitly, but are assumed to interact only by single-phonon exchange. How can one possibly use such approximations to derive predictions about superconducting phenomena that are of essentially unlimited accuracy?

The answer to this question is well understood by experts in superconductivity. When pressed, they will explain that the high-precision predictions about superconductors actually follow not only from the models themselves, but more generally from the fact that these models exhibit a spontaneous breakdown of electromagnetic gauge invariance in a superconductor. The importance of broken symmetry in superconductivity has been

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especially emphasized by Anderson. One needs detailed models like that of Bardeen, Cooper and Schrieffer to explain the mechanism for the spontaneous symmetry breakdown, and as a basis for approximate quantitative calculations, but not to derive the most important exact consequences of this breakdown. But if the general features of superconductivity are in fact model independent consequences of the spontaneous breakdown of electromagnetic gauge invariance, then why not derive them in this way in textbooks?

This article offers such a derivation. No new results are obtained, and no attempt is made to derive numerical results for quantities like critical temperatures, coherence lengths, etc., that cannot be predicted with very high accuracy anyway. The one goal of the derivations below is to show explicitly how the fundamental properties of superconductors may be explained without introducing any unnecessary approximations.

§ 2. Broken gauge invariance

We start with the assumption that, for whatever reason, electromagnetic gauge invariance is spontaneously broken in a superconductor.*) Nothing here will depend on the specific mechanism by which the symmetry breakdown occurs.

The electromagnetic gauge group is $U(1)$, the group of multiplication of fields $\psi(x)$ of charge q with the phases

$$\psi(x) \to \exp(i\Lambda q/\hbar)\psi(x). \tag{1}$$

(For the moment, we will ignore the interactions of the electromagnetic field, so Λ is for now a position-independent phase. Local gauge invariance will be introduced a little later.) We assume that all charges q are integer multiples of the electron charge $-e$, so phases Λ and $\Lambda+2\pi\hbar/e$ are to be regarded as identical.

This $U(1)$ is spontaneously broken to Z_2, the subgroup consisting of $U(1)$ transformations with $\Lambda=0$ and $\Lambda=\pi\hbar/e$. The assumption that Z_2 is unbroken arises of course from the physical picture that, while pairs of electron operators can have non-vanishing expectation value, individual electron operators do not. But for us it only is important that Z_2 is unbroken.

According to our general understanding of spontaneously broken symmetries, any system described by a Lagrangian with symmetry group G, when in a phase in which G is spontaneously broken to a subgroup H, will possess a set of Nambu-Goldstone excitations, described by fields that transform under G like the coordinates of the coset space G/H. (That is, the Nambu-Goldstone fields transform under G like the coordinates used to label the elements of G itself, but with two field values identified if they are in the same coset; i.e., if one can be transformed into the other by multiplication with an element of the unbroken subgroup H.) In relativistic quantum field theories this transformation property requires that Nambu-Goldstone bosons have zero mass; a more general statement is that their energy vanishes in the limit of zero momentum.

In our case, there will be a single Nambu-Goldstone excitation, described by a field $\phi(x)$ that transforms under $G=U(1)$ like the phase Λ itself. The group $U(1)$ has the multiplication rule $g(\Lambda_1)g(\Lambda_2)=g(\Lambda_1+\Lambda_2)$, so under a gauge transformation with parameter Λ, the field $\phi(x)$ will undergo the transformations

$$\phi(x) \to \phi(x)+\Lambda. \tag{2}$$

*) This is itself an approximation, valid in the limit of large numbers of atoms.

But $\phi(x)$ parameterizes $U(1)/Z_2$, not $U(1)$, so $\phi(x)$ and $\phi(x)+\pi\hbar/e$ are taken to be equivalent field values:

$$\phi(x)\equiv\phi(x)+\pi\hbar/e. \tag{3}$$

To see that the theory must involve such a Nambu-Goldstone field, start with a $U(1)$-invariant Lagrangian involving only ordinary fields $\psi(x)$ (like one-electron operators) that have conventional $U(1)$ transformation properties (1), and write all such fields as

$$\psi(x)\equiv\exp(iq\phi(x)/\hbar)\,\tilde{\psi}(x) \tag{4}$$

with gauge-invariant fields $\tilde{\psi}(x)$ subject to some convenient constraint, as for instance that some particular bilinear combination of the one-electron operators is real. This is of course a purely formal construction, that could be carried out whether or not electromagnetic gauge invariance were spontaneously broken. The characteristic property of a system with broken symmetry is that the quantity $\phi(x)$ behaves as a propagating field; the second variational derivative of the Lagrangian with respect to $\phi(x)$ has non-vanishing expectation value. We will return to this point briefly at the end of this section, and more explicitly in § 3.

Now we turn on the interaction of the superconductor with the electromagnetic fields $\boldsymbol{B}$ and $\boldsymbol{E}$. These are written as usual in terms of vector and scalar potentials, as

$$\boldsymbol{B}=\nabla\times\boldsymbol{A},\quad \boldsymbol{E}=-\nabla A^0-\frac{\partial\boldsymbol{A}}{\partial t}. \tag{5}$$

Their interaction is governed by the principle of local gauge invariance, under which the Nambu-Goldstone field $\phi(x)$ transforms as before under $U(1)$, but now with space-time-dependent phase:

$$\phi(x)\rightarrow\phi(x)+\Lambda(x). \tag{6}$$

The potentials themselves transform as usual, as

$$\boldsymbol{A}(x)\rightarrow\boldsymbol{A}(x)+\nabla\Lambda(x),$$
$$A^0(x)\rightarrow A^0(x)-\frac{\partial\Lambda(x)}{\partial t}. \tag{7}$$

However, it is important to note that in this formalism all other field operators are gauge-invariant. This is because we introduced the Goldstone bosons by writing the ordinary fields $\psi(x)$ according to Eq. (4) in terms of gauge-invariant fields $\tilde{\psi}(x)$.

It follows that the Lagrangian[*)] for the superconductor plus electromagnetic field may be written as

$$L=\frac{1}{2}\int d^3x(\boldsymbol{E}^2-\boldsymbol{B}^2)+L_m[\nabla\phi-\boldsymbol{A},\ \dot{\phi}+A^0,\ \tilde{\psi}]. \tag{8}$$

Here the matter Lagrangian L_m is a more-or-less unknown functional of the gauge-invariant combinations of $\partial_\mu\phi$ and A_μ, as well as of unspecified gauge-invariants $\tilde{\psi}$ representing the other excitations of the system.

The reader may recognize this as a highly generalized version of the Ginzburg-

*) By "Lagrangian" I mean here something like the effective action of quantum field theory, including effects of quantum fluctuations. I must admit that I am not sure how to justify the use of this formalism in dealing with dissipative systems, like real metals.

Landau theory, with $2e\phi/\hbar$ playing the role of the phase of the wave function in that theory. Many of the arguments below are adapted from standard treatments of the Ginzburg-Landau theory.*) I will have a little more to say about the Ginzburg-Landau theory in the final section.

From the matter Lagrangian, we can obtain the electric current and charge density as variational derivatives

$$J(x)=\frac{\delta L_m}{\delta A(x)}, \tag{9}$$

$$\varepsilon(x)=-\frac{\delta L_m}{\delta A^0(x)}=-\frac{\delta L_m}{\delta \dot{\phi}(x)}. \tag{10}$$

The Lagrangian equations of motion for $\phi(x)$ then yield

$$\frac{\partial}{\partial t}\varepsilon(x)=-\frac{\partial}{\partial t}\frac{\delta L_m}{\delta \dot{\phi}(x)}=-\frac{\delta L_m}{\delta \phi(x)}=-\nabla\cdot\frac{\delta L_m}{\delta A(x)},$$

which we recognize as the equation of charge conservation

$$\frac{\partial}{\partial t}\varepsilon(x)=-\nabla\cdot J(x). \tag{11}$$

The fact that Eq. (11) follows from the equation of motion for $\phi(x)$ alone is a consequence of the formalism we are using, in which $\phi(x)$ is the only gauge-noninvariant matter field.

We will not need to specify the structure of the functional L_m, but we will need to assume that in the absence of external electromagnetic fields, the superconductor has a stable equilibrium configuration with vanishing fields

$$\nabla\phi-A=\dot{\phi}+A^0=0. \tag{12}$$

The existence of such an equilibrium configuration is equivalent to the requirement that, for small values of $\nabla\phi-A$ and $\dot{\phi}+A^0$, the leading terms in the matter Lagrangian L_m are at least of second order in these quantities.**) Furthermore, the assumption that electromagnetic gauge invariance is spontaneously broken is equivalent to the statement that the coefficients of the terms in the L_m of second order in $\nabla\phi-A$ and $\dot{\phi}+A^0$ have non-vanishing expectation values, so that ϕ behaves like an ordinary physical excitation. (The formalism discussed in § 7 shows that this would not be the case for unbroken gauge invariance.)

We now turn to the consequences of these general assumptions.

§ 3. Meissner effect

One of the most important aspects of superconductivity is that, in the static case with $\dot{\phi}=A^0=0$, the quantity $\nabla\phi-A$ must vanish deep in a large superconductor. It follows

*) I am not sufficiently familiar with the history of the theory of superconductivity to know whom to credit with the derivations presented here, or to judge what might be original in these derivations. I will therefore not attempt to give references to the original literature in this article.

**) The absence of first-order terms in $\nabla\phi-A$ can also be inferred from reflection and rotation invariance, but we need a special assumption about equilibrium at (12) to deal with real superconductors, wher rotation symmetry is broken by the crystal structure. (Also, as I learned from Nambu, it has been pointed out by Anderson that terms linear in $\nabla\phi-A$ are forbidden if there are no perturbations that violate time-reversal invariance.)

that the magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$ must also vanish deep in the superconductor.

To derive these results, note first that, when $\dot{\phi} = A^0 = 0$, the matter Lagrangian for sufficiently small values of $\mathbf{A} - \nabla \phi$ may according to our general assumptions be written as a quadratic functional of these quantities

$$L_m = L_{m0} - \frac{1}{2}\int C_{ij}(\mathbf{x}, \mathbf{x}')(A_i(\mathbf{x}) - \nabla_i\phi(\mathbf{x})) \times (A_j(\mathbf{x}') - \nabla_j'\phi(\mathbf{x}'))d^3x d^3x' + \cdots\cdots \tag{13}$$

with L_{m0} independent of $\mathbf{A}$ and ϕ. The kernel C is clearly symmetric

$$C_{ij}(\mathbf{x}, \mathbf{x}') = C_{ji}(\mathbf{x}', \mathbf{x}). \tag{14}$$

Also, since in these circumstances $-L_m$ is the energy of the system, and since we assume that there is a *stable* equilibrium with $\mathbf{A} - \nabla \phi = 0$, the kernel C must be also positive, in the sense that

$$\int C_{ij}(\mathbf{x}, \mathbf{x}')a_i(\mathbf{x})a_j(\mathbf{x}')d^3x d^3x' \geq 0 \tag{15}$$

for all $a_i(\mathbf{x})$. The characteristic property of a superconductor is that C_{ij} does not vanish. From Eqs. (13) and (9), we obtain the Pippard formula for the electric current density:

$$J_i(\mathbf{x}) = -\int C_{ij}(\mathbf{x}, \mathbf{x}')(A_j(\mathbf{x}') - \nabla_j'\phi(\mathbf{x}'))d^3x'. \tag{16}$$

In the absence of other long-range forces, the function $C_{ij}(\mathbf{x}, \mathbf{x}')$ will be only moderately non-local, vanishing when $|\mathbf{x} - \mathbf{x}'|$ is greater than some characteristic distance ξ, so (16) can be assumed to hold when $\mathbf{A} - \nabla \phi$ is small near $\mathbf{x}$, whether or not it is small everywhere.

Now consider a large superconductor placed in a small steady magnetic field. Suppose for a moment that all or part of the field could penetrate the superconductor, with $\mathbf{A} - \nabla \phi$ taking values typically of some order A. The energy of the superconductor would then be increased by an amount of order $C\xi^3L^3A^2$, where C is a typical magnitude of the kernel $C_{ij}(\mathbf{x}, \mathbf{x}')$ in Eq. (13), ξ is a typical value of its range, and L is a typical length scale of the superconductor. The magnetic field B will be of order A/L, so this energy can be written as $C\xi^3L^5B^2$. But it costs only an energy of order B^2L^3 to expel the field from the superconductor entirely, so for sufficiently large superconductors, with $C\xi^3L^2 \gg 1$, it will be energetically favorable for $\mathbf{A} - \nabla \phi$ to be zero almost everywhere in the superconductor. This result does not apply in an ordinary non-superconducting metal, because there $C_{ij} = 0$, and the energy density there does not depend on $\mathbf{A}$, but only on $\mathbf{B}$.

For purposes of illustration, it is convenient to make the simplifying assumption that the material of the superconductor is invariant under translations, rotations, and reflections. The kernel in Eq. (16) then takes the form

$$C_{ij}(\mathbf{x}, \mathbf{x}') = \delta_{ij}c(|\mathbf{x} - \mathbf{x}'|^2) + \nabla_i\nabla_j d(|\mathbf{x} - \mathbf{x}'|^2) \tag{17}$$

with unknown (but finite-range) functions c and d. The field equation then reads

$$\nabla \times \boldsymbol{B}(\boldsymbol{x}) = -\int d^3x' c(|\boldsymbol{x}-\boldsymbol{x}'|^2)[\boldsymbol{A}(\boldsymbol{x}') - \nabla'\phi(\boldsymbol{x}')]$$

$$- \nabla \nabla \cdot \int d^3x' d(|\boldsymbol{x}-\boldsymbol{x}'|^2)[\boldsymbol{A}(\boldsymbol{x}') - \nabla'\phi(\boldsymbol{x}')]. \tag{18}$$

To eliminate ϕ we take the curl, which gives

$$\nabla^2 \boldsymbol{B}(\bar{x}) = \int d^3\boldsymbol{x}' c(|\boldsymbol{x}-\boldsymbol{x}'|^2)\boldsymbol{B}(\boldsymbol{x}'). \tag{19}$$

For a superconductor that fills the half-space $y>0$, with an external field in the x-direction, the solution of Eq. (19) [and $\nabla \cdot \boldsymbol{B}=0$] is

$$B_x = B_0 e^{-y/\lambda}, \quad B_y = B_z = 0, \tag{20}$$

where λ is the penetration depth, given by the solution of

$$1/\lambda^2 = \int dxdydz\; c(x^2+y^2+z^2)e^{-y/\lambda}. \tag{21}$$

(Note that λ is real, because the positivity condition (15) makes $c(r^2)$ positive. As λ increases from 0 to ∞, the left-side decreases steadily from ∞ to 0, while the right-side increases steadily from 0 to some finite value, so (21) always has just one solution.)

The current, given by $\nabla \times \boldsymbol{B}$, is then

$$J_z = \frac{1}{\lambda} B_0 e^{-y/\lambda}, \quad J_x = J_y = 0. \tag{22}$$

The solution of Eq. (18) is

$$(\nabla\phi - \boldsymbol{A})_z = \lambda B_0 e^{-y/\lambda}, \quad (\nabla\phi - \boldsymbol{A})_x = (\nabla\phi - \boldsymbol{A})_y = 0. \tag{23}$$

This solution is strictly valid only deep in the superconductor, i.e., for $y \gg \lambda$, both because we ignored edge effects in solving Eq. (19), and also because, unless B_0 itself is very small, it is only deep in the superconductor that the fields are small enough to allow the use of the quadratic approximation (13) to the Lagrangian. Thus B_0 in Eq. (20) should be regarded as an integration constant, which gives the value of the magnetic field *extrapolated* to the surface, rather than the actual surface field.

§ 4. Flux quantization

In a simply-connected time-independent superconductor the Nambu-Goldstone field ϕ is irrelevant, because we can eliminate it by a gauge transformation with $\Lambda(x)$ equal to $-\phi(x)$. However, in a multiply connected superconductor, $\phi(x)$ can jump by multiples of $\pi\hbar/e$, while Λ must be continuous, so this gauge transformation may not be possible.

Consider for example a superconducting ring, with thickness much greater than the penetration depth λ. Draw a closed contour $\mathscr{S}$ that follows the ring, deep inside it. On $\mathscr{S}$, $\nabla\phi - \boldsymbol{A}$ vanishes, so the change in ϕ around $\mathscr{S}$ is

$$\Delta\phi = \oint_{\mathscr{S}} \nabla\phi \cdot d\boldsymbol{l} = \oint_{\mathscr{S}} \boldsymbol{A} \cdot d\boldsymbol{l} = \Phi, \tag{24}$$

where Φ is the magnetic flux through the area $\mathscr{A}$ bounded by $\mathscr{S}$

$$\Phi = \int_{\mathscr{A}} (\nabla \times \boldsymbol{A}) \cdot d\boldsymbol{S}. \tag{25}$$

Since ϕ can only change around the ring by multiples of $\pi\hbar/e$, we must have quantized flux through the ring

$$\Phi = n\pi\hbar/e. \quad (n=0, \pm1, \pm2, \cdots) \tag{26}$$

§ 5. Infinite conductivity

The result of the previous section has the consequence that a current flowing through a superconducting loop cannot decay smoothly, but only in jumps at which the flux $|\Phi|$ drops by multiples of $\pi\hbar/e$. This shows that the supercurrent is not affected by ordinary electrical resistance, but this derivation was limited to *thick closed* rings. In order to understand the phenomenon of infinite conductivity in a more general context, it is necessary to say a little about time-dependent effects in superconductors. (This will also be needed in the next section, where we calculate the Josephson frequency.)

We recall that, according to Eq. (10), the charge density is given by

$$-\varepsilon(x) = \frac{\delta L_m}{\delta\dot{\phi}(x)},$$

so $-\varepsilon(x)$ is the dynamical variable canonically conjugate to $\phi(x)$. In the Hamiltonian formalism, the matter Hamiltonian H_m is then to be regarded as a functional of $\phi(x)$ and $\varepsilon(x)$ rather than of $\phi(x)$ and $\dot{\phi}(x)$, with the time-dependence of ϕ given by

$$\dot{\phi}(x) = \frac{\delta H_m}{\delta(-\varepsilon(x))}. \tag{27}$$

The "voltage" at any point can be regarded as the change in the energy density per change in the charge density at that point, or

$$V(x) \equiv \frac{\delta H_m}{\delta\varepsilon(x)}. \tag{28}$$

Hence the time-dependence of the Nambu-Goldstone field at any point is simply given by the voltage

$$\dot{\phi}(x) = -V(x). \tag{29}$$

One immediate consequence is that a piece of superconducting wire which carries a steady current, with time-independent fields, must have zero voltage difference between its ends. If the voltage difference were not zero, then according to Eq. (29) the gradient $\nabla\phi(x)$ would have to be time-dependent, leading (as in Eq. (16)) to time-dependent currents or fields. A zero voltage difference at finite current is just what we mean by infinite conductivity.

Some textbooks relate the infinite conductivity of superconductors directly to the existence of an energy gap separating a Fermi sea of paired electrons from their excited unpaired states. This seems to me misleading. The arguments given here show that infinite conductivity depends only on the spontaneous breakdown of electromagnetic gauge invariance, and would presumably occur even if the charged particles, whose pairing produced the symmetry breakdown, were bosons rather than fermions. In any case, these are known examples of superconductors without gaps. The one respect in which Fermi statistics plays a crucial role in superconductors is that the existence of a Fermi surface enhances the long range effects of the phonons whose exchange is responsible for the electron pairing. But this is only important in showing that electromagnetic gauge invariance can be spontaneously broken, not in deriving the general consequences of this breakdown discussed here, such as infinite conductivity.

§ 6. The AC Josephson current

Consider a gap between two superconductors. Assuming no gradients along the surface of the gap, and no vector potential, gauge invariance will allow the contribution of this junction to the matter action to depend only on the difference $\Delta\phi$ between the Nambu-Goldstone field in the two superconductors. Furthermore, we can shift ϕ in *either* superconductor by a multiple of $\pi\hbar/e$ without changing its physical significance, so the contributions of the junction to the matter action must be a *periodic* function of $\Delta\phi$

$$L_{\text{junction}} = \mathcal{A} F(\Delta\phi), \tag{30}$$

$$F(\Delta\phi) = F(\Delta\phi \pm \pi\hbar/e), \tag{31}$$

where $\mathcal{A}$ is the junction area. (Josephson calculated this function, and in effect found it to be proportional to $\cos(2e\Delta\phi/\hbar)$, but this specific result was based on an approximate model, and cannot be regarded as having the same general validity as the periodicity itself.)

In the presence of a vector potential, the difference $\Delta\phi$ must be replaced with the gauge invariant quantity

$$\Delta\phi = \int d\boldsymbol{l}\cdot(\nabla\phi - \boldsymbol{A}) \tag{32}$$

the integral being taken over a line joining the superconductors. We can then derive the current per unit area across the junction as

$$\boldsymbol{J} = \frac{\delta L_{\text{junction}}}{\delta \boldsymbol{A}} = -F'(\Delta\phi)\,\hat{n}, \tag{33}$$

where $\hat{n}$ is the unit normal to the gap.

Now suppose that the two superconductors are maintained at uniform but different voltages, with a voltage difference ΔV. According to Eq. (29), the Nambu-Goldstone field ϕ in each superconductor will decrease at a rate equal to the voltage in that superconductor, so the difference in these fields will have time dependence (now taking $\boldsymbol{A}=0$)

$$\Delta\phi = -t\Delta V + \text{constant}. \tag{34}$$

Combining (31), (33) and (34), we see that the current oscillates, with a frequency

$$\nu = \frac{e|V|}{\pi\hbar}. \tag{35}$$

However the oscillation is not in general a pure sine wave — in principle all harmonics of the frequency (35) will be present.

§ 7. Order parameter and vortex lines

Up to this point, we have not had to say anything about the other excitations (symbolized by $\tilde{\psi}$) in the matter Lagrangian L_m. This is just as well, because in general we have no precise information about them. There is one situation, however, in which we can make model-independent statements about other excitations. It is when the material is brought, by whatever means, close to a state in which it loses its superconductivity.

When electromagnetic gauge invariance is unbroken, the excitations of the system must fall into ordinary linear representations of the gauge group. Since the group is $U(1)$, these representations are just complex singlet fields transforming as in Eq. (1). Each can be decomposed into a modulus and a phase. By continuity then, when the material is close to losing its superconductivity, the Nambu-Goldstone field $\phi(x)$ must be accompanied with a modulus field $\rho(x)$, so that there is a linearly transforming field, known as the order parameter

$$\chi(x)=\rho(x)\exp(2ie\phi(x)/\hbar). \tag{36}$$

The argument of the exponential is taken to be $2ie\phi/\hbar$ in order that $\chi(x)$ should transform as in Eq. (1), with $q=2e$, which is necessary in order that a non-zero expectation value of $\chi(x)$ will break $U(1)$ down to Z_2, as assumed in § 2.

The modular field $\rho(x)$ plays an important role in the dynamics of vortex lines. These arise when a superconductor is placed in a magnetic field that is strong enough so that tubes of magnetic flux penetrate the material. We can show in this case that each such flux tube must contain a vortex line (or perhaps a finite vortex tube), along which electromagnetic gauge invariance is *not* broken. The behaviour of the modular field $\rho(x)$ near a vortex line turns out to be uniquely determined by the total magnetic flux through the flux tube.

To see this, begin by drawing a contour that circles the flux tube, but far outside it, deep in the superconducting material. Just as in § 4, the quantity $\nabla\phi - A$ vanishes on this contour, so the change in ϕ around the contour equals the line integral of A around it, and hence equals the magnetic flux Φ_{TOT} through it. The change in ϕ is an integer multiple n of $\pi\hbar/e$, so

$$\Delta\phi=\Phi_{\mathrm{TOT}}=n\pi\hbar/e. \tag{37}$$

Now consider shrinking the contour, always keeping it within superconducting material, where everything is a smooth function of ϕ. As we shrink it into the region of high field, it is no longer true that $A=\nabla\phi$, so the magnetic flux Φ through the contour can decrease from the value Φ_{TOT}, but $\Delta\phi$ must always be an integer multiple of $\pi\hbar/e$, and hence by continuity will continue to be given by Eq. (37).

Since ϕ changes by a finite fixed amount around the contour, it is not possible to shrink this contour down to zero area without encountering a region in which the field ϕ is irrelevant, which requires that in this region $\rho=0$ and the $U(1)$ symmetry is restored. We will here consider only the case where this region of unbroken symmetry is one-dimensional, a vortex line along which $\rho=0$.

The "order parameter" (36) must be a smooth function of position, so if it vanishes along some vortex line, then it must do so as a power series in the distance from the line. Suppose we label some point on the vortex line as $x=y=z=0$, and let the z-axis run locally along the line. Then for small x, y and z, the order parameter may be written as a sum of terms of the form

$$(x\pm iy)^l,\quad (x\pm iy)^l(x^2+y^2),\quad (x\pm iy)^l(x^2+y^2)^2,\ \cdots$$
$$l=0,\ 1,\ 2,\ \cdots. \tag{38}$$

In cylindrical coordinates z, r, θ, the Nambu-Goldstone field then behaves as

$$\phi\rightarrow\pm\frac{\hbar l}{2e}\theta \tag{39}$$

so comparing with (37), we see that l is related to the flux quantum number n by

$$n=\pm l. \tag{40}$$

Also, the leading term in the modulus is

$$\rho\propto r^l=r^{|n|}. \tag{41}$$

The results (39) ~ (41) apply very close to the vortex lines. Very far from the vortex line, we know in general only that $\nabla\phi-A$ vanishes exponentially, while ρ approaches a constant. At intermediate distances, there is not much (apart from (37)) that can be said without relying on specific models. However, for a *straight* isolated vortex line, symmetry arguments allow us to go a little farther. The invariances under rotations around the z-axis and under electromagnetic gauge invariance are spontaneously broken here, but Eq. (39) exhibits an unbroken symmetry under combined rotations by angles α and electromagnetic gauge transformations with $\Lambda=\hbar n\alpha/2e$. We can expect that this will then be a symmetry of the whole system, not just for $r\rightarrow 0$, in which case for all r the Nambu-Goldstone field must be given by

$$\phi=\frac{\hbar n}{2e}\theta+\phi_0(r). \tag{42}$$

With the introduction of the modulus field ρ we have come very close to the Ginzburg-Landau theory (and to the similar treatment by Feynman). In that theory, the Lagrangian is written as a sum of a few local terms of limited dimensionality involving a complex field $\chi(x)$ and its gauge-covariant derivative. Of course, we have made no such assumptions about the form of the Lagrangian, and hence we obtain less detailed predictions. Beyond this there is a difference of point of view: Instead of regarding $\phi(x)$ as the phase of a complex wave function used in an approximate treatment of electron pairs, we regard it here as a Nambu-Goldstone field, which inevitably accompanies the breakdown of electromagnetic gauge invariance. Planck's constant $\hbar$ does not appear in the differential equations governing ϕ, but only in certain topological conditions, which arise solely because of the incidental fact of charge quantization,*) and then only in the

*) If charge were not quantized then the Meissner effect and vanishing resistivity would still follow from broken electromagnetic gauge invariance just as in §§ 3 and 5. However in place of flux quantization we would have the result that the magnetic flux through a thick superconducting ring vanishes. Also, the time-dependent current produced by a voltage difference across a Josephson junction would be aperiodic. Planck's constant would appear nowhere in the macroscopic description of superconductivity.

combination $e/\hbar$. From this point of view, superconductivity is not macroscopic quantum mechanics; it is the classical field theory of a Nambu-Goldstone field.

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