Instantons in QM and in YM theories.

- Vainshtein, Zakharov, Novikov, Shifman,
  Soviet Physics Uspekhi,
  Vol. 25 (1982), 195
  "ABC of Instantons"
- Coleman, "Aspects of Symmetry"

Anomalies

- Jackiw,
  in "Current Algebra & Anomalies"

Green's fn in QM

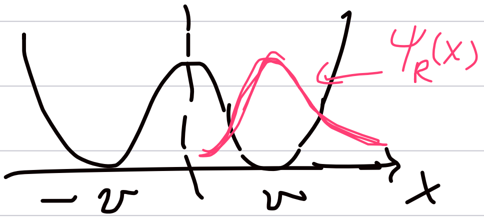
$$\langle x_f | e^{-iH(t_f-t_i)} | x_i \rangle = N \int DX \, e^{iS/\hbar} ; \quad S = \int dt \, L(x,\dot{x})$$

→ Feynman diagrams, pert. calculations, ...

→ Nonperturbative effects.

Double well potential

$$V(x) = \lambda (x^2 - v^2)^2$$



Schrödinger eq. ⇒ The ground state / 1st exc. st.

not $\psi_R(x)$ or $\psi_L(x)$, but $\psi_\pm(x) = \frac{1}{\sqrt{2}}(\psi_R(x) \pm \psi_L(x))$

and $\delta E_\pm \sim \text{const.} \, e^{-\frac{1}{\hbar}\int dx |p|}$ → tunnel effect.

In path-integral approach, we are interested in

$$\underset{\underset{A}{\parallel}}{\langle -v | e^{-HT} | v \rangle} , \quad \underset{\underset{B}{\parallel}}{\langle \underset{\uparrow x_f}{v} | e^{-HT} | \underset{\nwarrow x_i}{-v} \rangle} , \quad \text{etc.}$$

They are given by

$$A = \int_{x(-T/2)=-v}^{x(T/2)=-v} DX \, e^{-S_E} ; \quad B = \int_{x(-T/2)=-v}^{x(T/2)=+v} DX \, e^{-S_E} ,$$

where the Euclidean time $t = -i\underset{\substack{\uparrow \\ \text{real}}}{\tau}$ has been introduced.

$$iS = i\int dt\left(\frac{m}{2}\dot{x}^2 - V(x)\right) = -\int d\tau\left(\frac{m}{2}\underbrace{\dot{x}}_{dX/d\tau}^2 + V(x)\right)$$

$$\equiv -S_E$$

S_E describes motion in $-V(x)$.

To compute A and B, appeal to the functional saddle-point method.

$$X(\tau) = X_{cl}(\tau) + \delta X(\tau)$$

$$S_E = S_E(X_{cl} + \delta X) = S_E(X_{cl}) + \cancel{\frac{\delta S}{\delta X}\Big|_{cl}}\cdot\delta X + \frac{1}{2}\frac{\delta^2 S}{\delta X^2}\Big|_{cl}\cdot(\delta x)^2 + \cdots$$

X_{cl} : sol of

$$0 = \frac{\delta S}{\delta X} = \frac{\delta}{\delta X}\int d\tau\left(\frac{m}{2}\dot{x}^2(\tau) + V(\tau)\right)$$

$$= -m\ddot{x}(\tau) + \frac{dV}{dX}$$

$$\left(\frac{\delta X(\tau')}{\delta X(\tau)} = \delta(\tau' - \tau)\right)$$

$$\therefore\ m\ddot{X} = \frac{dV}{dX}$$

(motion in $-V$)

Soln

$$\frac{1}{2}m\dot{X}^2 - V = E = \text{const.}$$

(energy cons.)

$$= 0$$

then

$$\frac{dX}{d\tau} = \sqrt{\frac{2V(x)}{m}}$$

$$d\tau = \sqrt{\frac{m}{2V(x)}}\,dx \Rightarrow X_{cl} = X_{cl}(\tau)$$

Quantum fluctuation

$$S_E = S_{cl}{}^{//\,0} + \frac{1}{2}\int d\tau\, \delta x\left[-\frac{d^2}{d\tau^2} + V''(x)\right]\delta x + \cdots$$

Introduce the eigenmodes

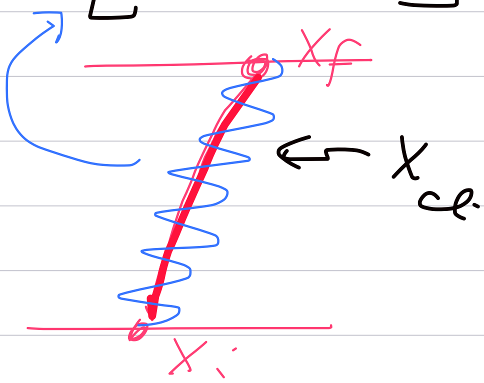
$$\left[-\frac{d^2}{d\tau^2} + V''(x_{cl}(\tau))\right] q_n = \epsilon_n q_n \,, \quad q_n\left(\pm\frac{T}{2}\right) = 0$$

$$\int d\tau\, q_n(\tau)\, q_m(\tau) = \delta_{n,m}$$

$$\int \mathcal{D}X \equiv \prod_n \int dc_n \qquad \text{then}$$

$$\int \mathcal{D}X\, e^{-S_E} = e^{-S_E^{(cl)}} \cdot \int \prod_n dc_n\, e^{-\frac{1}{2}\sum_n c_n^2 \epsilon_n}$$

$$= e^{-S_E^{(cl)}} \cdot \prod_n \epsilon_n^{-1/2} = e^{-S_E^{(cl)}} \cdot \left[\det\left(-\frac{d^2}{d\tau^2} + V''\right)\right]^{-1/2}$$



$$S_E = \int d\tau\left(\frac{m}{2}\dot{x}^2(\tau) + V(\tau)\right) = \int d\tau\, m\dot{x}^2 = \int dx\, m\dot{x}$$

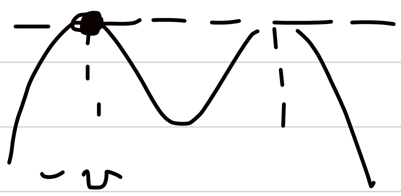
$$= \int dx\sqrt{2mV(x)}$$

Amplitude A

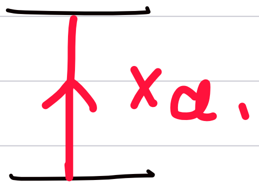
Soln of $m\ddot{x} = V$ w/ $x(\pm T/2) = -v$

by inspection :

$x_{cl}(\tau) = -v, \ \forall \tau$!



$S_E(x_{cl}) = 0$



$V''(x_{cl}) = V''(-v) = m\omega^2$, const.

$V(x) = \lambda(x^2 - v^2)^2$

$= \frac{1}{2}m\omega^2(x+v)^2 + \cdots$ $\Rightarrow$ $V''(x)\Big|_{-v} = 8\lambda v^2$

$\omega \equiv \sqrt{\frac{8\lambda v^2}{m}}$

But this is H.O. ! $(x+v \rightarrow x)$

$V = \frac{1}{2}m\omega^2 x^2 \rightarrow V'' = m\omega^2$

quadr fluc. $\left(-\frac{d^2}{d\tau^2} + \omega^2\right) q_n = \epsilon_n q_n$, $q_n(\pm T/2) = 0$.

$\epsilon_n - \omega^2 = \frac{\pi^2 n^2}{T^2} \equiv \lambda_n^2$, $\lambda_n = \frac{\pi n}{T}$

$q_n(\tau) = A_n \sin \lambda_n(\tau + T/2)$

Then

$$\langle 0| e^{-HT} |0\rangle = N \int DX e^{-S}$$

$$\Rightarrow N e^{-S_{cl}} \prod_n \epsilon_n^{-1/2} = N \prod_n \left(\omega^2 + \frac{\pi^2 n^2}{T^2}\right)^{-1/2}$$

$$= N \underbrace{\prod_n \left(\frac{\pi^2 n^2}{T^2}\right)^{-1/2}} \cdot \prod_n \left(1 + \frac{\omega^2 T^2}{\pi^2 n^2}\right)^{-1/2}$$

$$= \langle 0| e^{-H_0 T} |0\rangle \qquad ''$$

$$\left(\prod_{n=1}^{\infty} \left(1 + \frac{y^2}{n^2}\right) = \frac{\mathrm{sh}\, y}{\pi y}\right)$$

$$= \frac{1}{\sqrt{2\pi T}} \cdot \left(\frac{\mathrm{sh}\, \omega T}{\omega T}\right)^{-1/2}$$

$$\xrightarrow[T \to \infty]{} \left(\frac{\omega}{\pi}\right)^{1/2} e^{-\omega T/2} \qquad \therefore \quad E_0 = \omega/2, \quad |\psi_0(0)|^2 = \left(\frac{\omega}{\pi}\right)^{1/2}$$

O.K.

- This is all for Harmonic Oscillator.
- For amplitude A of the double well,

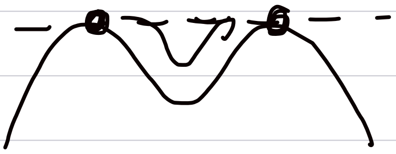
$$\langle -v| e^{-HT} |-v\rangle,$$

this is correct to lowest order, but there is something else to be taken into account.

$\Rightarrow$ Consider Amplitude B.

Amplitude B.

$$\frac{\delta S}{\delta x}\Big|_{x_{cl}} = 0 \qquad x(-T/2) = -v \qquad x(+T/2) = v$$



→ Nontrivial soln!

→ Obvious that such a soln exists.

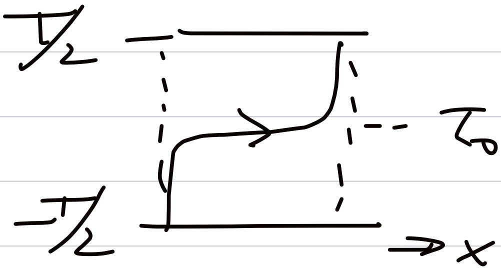
$$V = \lambda(x^2-v^2)^2$$

$$\frac{dx}{d\tau} = \sqrt{\frac{2V(x)}{m}} \Longrightarrow$$

$$\tau-\tau_0 = \int_0^x dx\sqrt{\frac{m}{2V}} = \sqrt{\frac{m}{2\lambda}}\int_0^x \frac{dx}{v^2-x^2}$$

$$\Longrightarrow x_{cl} = x_{inst} = v\,\mathrm{th}\,\omega_1(\tau-\tau_0) \quad \text{"Instanton"}$$

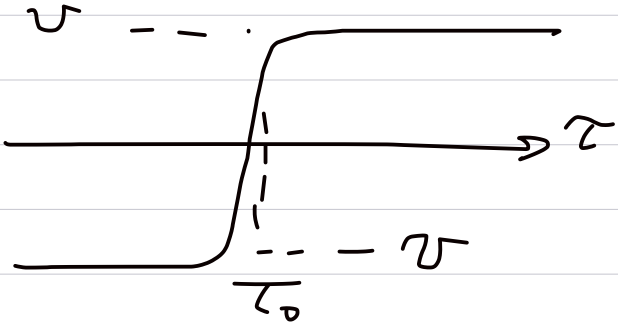
$$\omega_1 \equiv \sqrt{\frac{2\lambda v^2}{m}} = \frac{\omega}{2}.$$



$$\Rightarrow S_E^{inst} = \int d\tau\left(\frac{m\dot{x}^2}{2}+V\right)$$

$$\Rightarrow \int_{-v}^{v} dx\sqrt{2mV(x)}$$

$$= \int dx\,|p|$$



$$= \frac{4}{3}\sqrt{2m\lambda}\,v^3$$

$$= \omega^3/12\lambda.$$

Instanton is localized in time, of the order of the width $\Delta\tau \simeq \frac{1}{\omega_1}$

$\tau\to\infty$, $x^{inst}\sim v$

$V(x)\simeq\frac{m}{2}\omega_1^2(x-v)^2$

$\Rightarrow \frac{dx}{d\tau}\sim\omega_1(v-x)$

$\therefore v-x\simeq e^{-\omega_1\tau}$.

Quadratic fluctuation

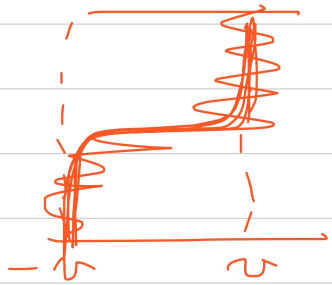
$$S_E = S^{inst} + \frac{1}{2}\int d\tau\, \delta x(\tau)\left[-m\frac{d^2}{d\tau^2} + V''(x^{inst}(\tau))\right]\delta x(\tau)$$

$$\int \mathcal{D}x\, e^{-S_E/\hbar} = e^{-S^{inst}/\hbar}\, N\cdot \det\left(-m\frac{d^2}{d\tau^2}+V''(x_{cl})\right)^{-1/2}$$

$$\overset{?}{=} \langle v|\, e^{-HT}\,|-v\rangle$$

Some more considerations needed.

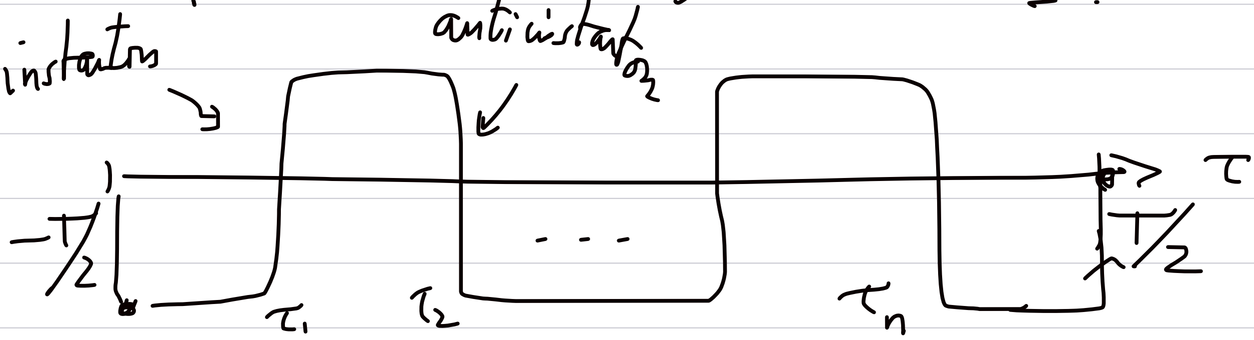
(i) $N\det\left(-m\frac{d^2}{d\tau^2}+V''(x_{cl})\right)^{-1/2} = \boxed{\sqrt{\frac{m\omega}{\pi\hbar}}\, e^{-\omega T/2}}\cdot K$



K = correction factor

$= \omega\cdot\sqrt{\frac{6S^{inst}}{\pi\hbar}}$

(ii) at large T there are many other solns.



they must be summed !

Will give:

$$\int_{-T/2}^{T/2} d\tau_n \int_{-T/2}^{\tau_n} d\tau_{n-1} \cdots \int_{-T/2}^{\tau_2} d\tau_1 = \frac{1}{n!}T^n \qquad !$$

Also, $S_E \longrightarrow n\, S^{inst}$

These other solns exist also for amplitude A!

Topology of the configurations $\Rightarrow$

(A): $$\langle -v| e^{-HT} |-v\rangle = \langle v| e^{-HT} |v\rangle = \left(\frac{m\omega}{\hbar}\right)^{1/2} e^{-\omega T/2} \sum_{n\ \text{even}} \frac{\left(K e^{-\frac{\omega T}{2}} \cdot T\right)^n}{n!}$$

$$= \left(\frac{m\omega}{\hbar}\right)^{1/2} e^{-\omega T/2} \cdot ch\left(K e^{-\omega T/2}\, T\right) ;$$

(B) $$\langle v| e^{-HT} |-v\rangle = \langle -v| e^{-HT} |v\rangle = \left(\frac{m\omega}{\hbar}\right)^{1/2} e^{-\omega T/2} \sum_{n=\text{odd}} \frac{\left(K e^{-\omega T/2}\, T\right)^n}{n!}$$

$$= \left(\frac{m\omega}{\hbar}\right)^{1/2} e^{-\omega T/2}\, sh\left(K e^{-S^{inst}/\hbar}\, T\right)$$

But then the eigenstates of the evolution operator e^{-HT} is $$\frac{|v\rangle \pm |-v\rangle}{\sqrt{2}} \equiv |\psi_\pm\rangle ;$$

$$\langle \psi_+ | e^{-HT} | \psi_- \rangle = 0 ;$$

$$\langle \psi_+ | e^{-HT} | \psi_+ \rangle = \left(\frac{m\omega}{\hbar}\right)^{1/2} e^{-\omega T/2}\, e^{K e^{-S_0/\hbar} \cdot T} ,$$

$$\langle \psi_- | \cdots | \psi_- \rangle = \cdots \quad e^{-K e^{-S_0/\hbar}\, T}$$

$$\Rightarrow \quad \Delta E = \pm 2 K \hbar\, e^{-S^{inst}/\hbar} = \pm\, 2\omega\hbar \sqrt{\frac{6 S^{inst}}{\pi\hbar}}\, e^{-S^{inst}/\hbar} .$$

∘ Dilute-gas approximation

O.K. if $T/n \gg \frac{1}{\omega}$

I size ↘

$\delta t \rightarrow e^{S_0/\hbar}$

check: $n \sim K e^{-S_0/\hbar} T \quad \therefore \quad \frac{T}{n} \sim \frac{1}{K e^{-S_0/\hbar}} \sim \frac{e^{S_0/\hbar}}{\omega} \gg \frac{1}{\omega}$

O.K.

- Sum over instanton center : DO IT BETTER !

$$X(\tau) = X_{cl}(\tau-\tau_1) + \delta X(\tau)$$

$$\delta X(\tau) = \sum_n c_n q_n(\tau)$$

$$S_E = S^{inst} + \frac{1}{2}\int d\tau\, \delta X(\tau)\underbrace{\left[-\frac{m\, d^2}{2\, d\tau^2} + \frac{1}{2}V''\right]}_{D}\delta X(\tau)$$

$$[\; D \;]\, q_n = \epsilon_n q_n$$

Problem : ∃ Soln w. $\epsilon_0 = 0$ (zeromode)

$(\det D)^{-1/2}$ does not make sense !

Origin : $S(x_{cl}(\tau-\tau_1)) = S(x_{cl}(\tau-\tau_1-\delta\tau_1))$

⇒ δX contains a zeromode (translation)

$$q_0 = c\,\frac{dX_{cl}}{d\tau}(\tau-\tau_1)\,, \quad \int d\tau\, |q_0|^2 = 1$$

$$\longrightarrow c \quad = \left(\frac{S^{inst}}{m}\right)^{-1/2}\frac{dX_{cl}}{d\tau}$$

$\delta X = \sum_n c_n q_n$

Or $\quad dX(\tau) = \frac{dX_{cl}}{d\tau_1}\, d\tau_1 = q_0\, dc_0$

$$\therefore \quad dc_0 = (2\pi\hbar)^{1/2}\left(\frac{S^{inst}}{m}\right)^{1/2} d\tau_1$$

Jacobian

Integ δX → collective coordinate

Now $\int \mathcal{D}x\, e^{-\frac{S_E}{\hbar}} = \int \prod_n{}' dc_n\, dc_0\; e^{-S^{(inst)} - \frac{1}{2}\sum_{n\neq 0} c_n^2 \epsilon_n}$

$$= (2\pi\hbar)^{1/2} \cdot \left(\frac{S^{inst}}{m}\right)^{1/2} \underset{\displaystyle T}{\underset{\downarrow}{d\tau_1}} \Bigg[\underset{\displaystyle \prod_{n\neq 0} \epsilon_n}{\underset{\shortparallel}{\det{}'(\qquad)}}\Bigg]^{-1/2}$$

- Similar for n instanton configuration.

Observations.

- $I-\bar{I}$ not an exact soln.

$$S^{I+\bar{I}} = \left(S^{I} + S^{\bar{I}}\right)\left(1 + O\left(e^{-2\sqrt{\lambda v^2}\, t_{12}/\sqrt{2}}\right)\right)$$

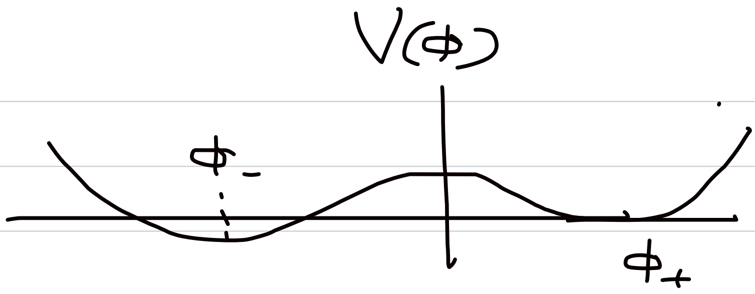
as $|t_{12}| \gg \frac{1}{\omega}$, $\left|\frac{\Delta S}{S}\right| = e^{-\omega t_{12}} \ll 1$.

- "$I-\bar{I}$" configuration $|t_{12}| \to 0$ badly defined (they attract each other.)
- Perturbative corrections around I solution are also badly behaved ($\sim c^n n! g^n$, $c>0$) Not Borel-summable !! (cfr. gx^4)
- These ambiguities exactly cancel (Bogomolny 1980)

✓ Zinn-Justin, et. al. ~1980

✓ Resurgent Series 2012-... (Dunne, Unsal, ...)

Metastable vacua.



$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)$$

$\langle\phi\rangle = \phi_-$ true vacuum

$\langle\phi\rangle = \phi_+$ metastable vac.

Decay of the false vacuum.

- analogous to the decay of superheated liquid.

a bubble of size $R \rightarrow \Delta E = \epsilon \cdot R^3$

Small bubble $\epsilon R^3 < 4\pi R^2 \cdot T \rightarrow$ shrinks

Larger bubble $\epsilon R^3 > 4\pi R^2 T \rightarrow$ grows

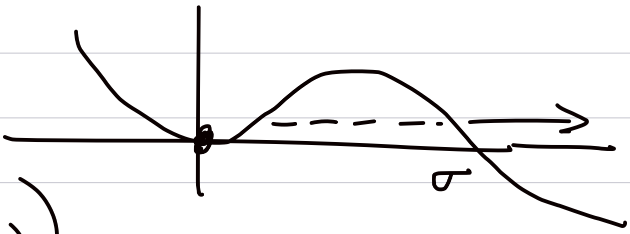
$\rightarrow$ fill the whole space (vapor phase)

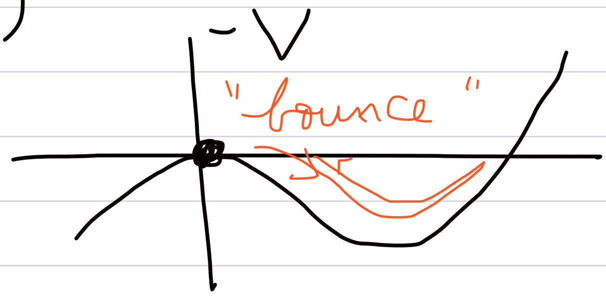
QM analogue. (Decay of the metastable b. st.)

$$\langle x=0|e^{-HT}|x=0\rangle$$

$$= \int \mathcal{D}X\, e^{-\frac{1}{\hbar}\int d\tau\left(\frac{\dot{x}^2}{2}+V(x)\right)}$$

$$X(\tau) = X_{cl}(\tau) + \delta X$$





multi-bounce solns.

$$S^{\text{bounce}}_{cl} = \int d\tau\left(\frac{\dot{x}^2}{2}+V\right) = \int d\tau\, \dot{x}^2 = 2\int_0^\sigma dx\sqrt{2V} = 2\int_0^\sigma dx\,|p|$$

Observations

(i) As in the double-well case, $\delta X = \sum_n c_n q_n$.

$$\delta X = c_1 \frac{dX_{cl}}{d\tau} + \dots \longrightarrow \text{zero mode}$$

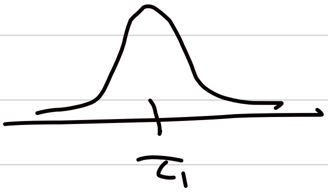
$$D_2 q_1 = \left(-\frac{d^2}{d\tau^2}+V''(x_{cl})\right) q_1(\tau) = 0.$$

$$\to dc_1 \propto d\tau_1 \to T.$$

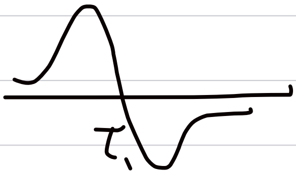
(ii) Summing over n bounce solns, $n=0,1,2,\dots$

$$\to \sum \frac{T^n}{n!}(\ )^n \to e^{K e^{-\frac{1}{\hbar}S^b_{cl}}\,T}$$

(ii) But $X_{cl}(\tau)$ looks like



$\frac{\partial X}{\partial \tau}$ looks like



→ has a node!

It is not the ground state of $\mathcal{D}_2 q_n = \epsilon_n q_n$

$\rightarrow \exists\, \epsilon_0$ such that $\epsilon_0 < 0$.

$$\left[\det\left(-\frac{d^2}{d\tau^2}+V''(x_{cl})\right)\right]^{-1/2} = i \det{}'[\quad]^{-1/2}$$

$$= \sqrt{\frac{m\omega}{\pi\hbar}}\, e^{-\omega T/2}\cdot K$$

$K = i|K|$

$$\delta E = K e^{-\frac{1}{\hbar}S^{bounce}}$$

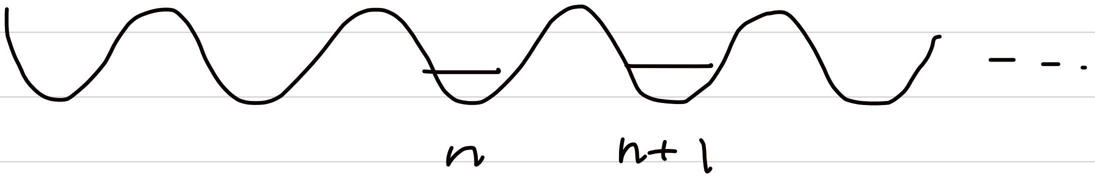
$$= i|K| e^{-\frac{1}{\hbar}S^{bounce}}$$

$$E \rightarrow \underset{=\omega/2}{E_{real}} + i\Gamma/2$$

$$\Gamma = |K|\, e^{-\frac{1}{\hbar}S^{bounce}}$$

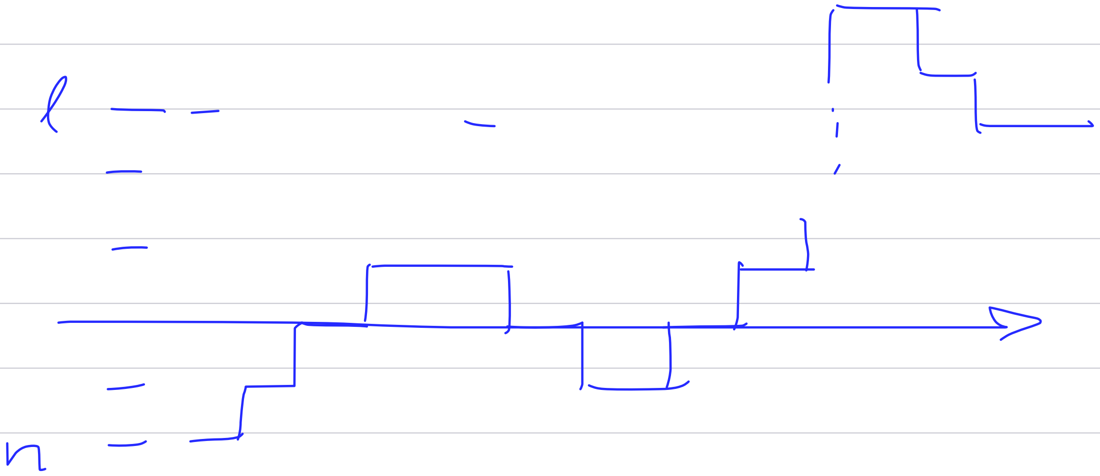
where $S^{bounce} = 2\int dx\sqrt{2V} = 2\int_0^{\sigma} dx\,|p|$

Periodic potential.



$$\langle n+1|\, e^{-HT}\, |n\rangle = K T e^{-S_0/\hbar}\, e^{-\omega T/2}\sqrt{\frac{M\omega}{\pi\hbar}}$$

$$\langle l|\, e^{-HT}\, |n\rangle = \sum_{\substack{N_1, N_2 \\ N_1-N_2 = l-n}} \frac{\left(K T e^{-S/\hbar}\right)^{N_1+N_2}}{N_1!\; N_2!}\; \underbrace{e^{-\omega T/2}\sqrt{\frac{m\omega}{\pi\hbar}}}_{A_0}$$



Eigenstates of the evolution operator e^{-HT}

$$|\theta\rangle = N \sum_n e^{in\theta} |n\rangle$$

$$\Rightarrow \langle\theta'|\, e^{-HT} |\theta\rangle = N^2 \sum_{n,\, l}\; \sum_{\substack{N_1, N_2 \\ N_1-N_2=l-n}} e^{in\theta}\, e^{-il\theta'}\, \frac{(\quad)^{N_1+N_2}}{N_1!\; N_2!}\, A_0$$

$$= N^2 \sum_{n, N_1, N_2} e^{in(\theta-\theta')}\, e^{-i(N_1-N_2)\theta'}\, \frac{(\quad)^{N_1+N_2}}{N_1!\; N_2!}\, A_0$$

$$= \delta(\theta-\theta') \cdot e^{e^{-i\theta}(KTe^{-S_0/\hbar})} \cdot e^{e^{i\theta}(\quad)} A_0$$

$$= \delta(\theta-\theta') \exp\left[2KTe^{-S_0/\hbar} \cos\theta - \frac{\omega T}{2} \right] \sqrt{\frac{M\omega}{\pi\hbar}}$$

$$\therefore \; E = \frac{\omega\hbar}{2} - 2Ke^{-S_0/\hbar} \cos\theta$$

(conduction band ; $\theta \sim ka$ = lattice momentum).

QCD : θ vacua (later)

Decay of a false vacuum.

Coleman, PRD15(77) 2929

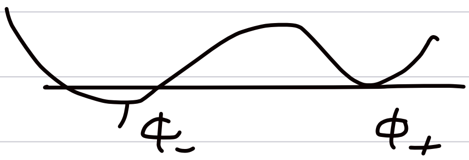
$$x(t) \to x_i(t) \to \phi(t,\mathbf{x}) \; ; \; \sum_i \to \int d^3x$$

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - U(\phi) \; ;$$

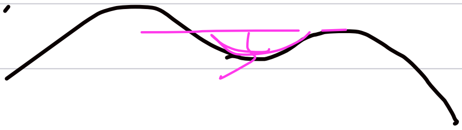
$$\mathcal{L}_E = \frac{1}{2}\left[(\partial_\tau\phi)^2 + (\nabla\phi)^2\right] + U(\phi).$$

Eq. motion

$$\left(\frac{\partial^2}{\partial\tau^2} + \nabla^2\right)\phi(\tau,\mathbf{x}) = U'(\phi)$$



Boundary cond. for bounce soln.



$$\begin{cases} \lim_{\tau\to\pm\infty} \phi(\tau,\mathbf{x}) = \phi_+ \\ \frac{\partial\phi}{\partial\tau}(0,\mathbf{x}) = 0 \end{cases}$$

$$\Gamma/V = A\, e^{-B/\hbar}$$

Observation:

If $\phi_0(\tau,\mathbf{x})$ is a soln, so is $\phi_0(\tau,\mathbf{x}+\mathbf{a})$

Γ contains $\int d^3x = V \Rightarrow \frac{\Gamma}{V} =$ decay prob. per volume

The coefficient B is given by

$$B = S_E = \int d\tau\, d^3x \left[\frac{1}{2}\left(\frac{\partial\phi}{\partial\tau}\right)^2 + \frac{1}{2}(\nabla\phi)^2 + U(\phi) \right]$$

For B to be finite, need

$$\phi(\tau, x) \underset{|x|\to\infty}{=} \phi_+ \quad ,$$

MINIMIZE NOW S_E

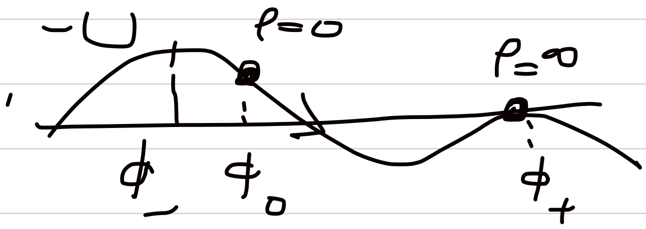
Ansatz

$$\phi = \phi(\rho), \quad \rho = \sqrt{\tau^2 + x^2} \Rightarrow \begin{array}{l} S_E = 2\pi^2 \cdot \\ \int_0^\infty d\rho\, \rho^3 \left[\frac{1}{2}\left(\frac{d\phi}{d\rho}\right)^2 + U\right] \end{array}$$

Eq. of motion: $$\frac{d^2\phi}{d\rho^2} + \frac{3}{\rho}\frac{d\phi}{d\rho} = U'(\phi)$$

B.c. $$\phi(\rho) \xrightarrow[\rho\to\infty]{} \phi_+ \quad ; \quad \frac{d\phi}{d\rho}\Big|_0 = 0 .$$

a 1D particle, in a force $+U'(\phi)$, with a viscous damping force

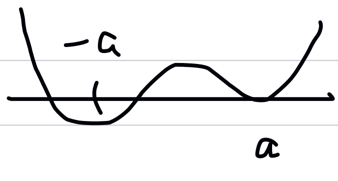


$\rightarrow \exists\, \phi_0 \quad \phi(0) = \phi_0$, such that $\phi(\infty) = \phi_+$.

Example $$U = \underbrace{\frac{\lambda}{8}\left(\phi^2 - \frac{\omega^2}{\lambda}\right)^2}_{U_0} + \frac{\epsilon}{2a}(\phi - a) \quad (*)$$

$a = \sqrt{\frac{\omega^2}{\lambda}}$, $\phi_+ = a$, $\phi_- = -a$

$\omega^2 = U''(a)$, $\epsilon = \mathcal{E}_+ - \mathcal{E}_-$



Compute B for small ϵ.

$\phi(0)$ must be close to ϕ_-; the particle stays until large "time", $\rho = R$., when it transverses the valley from ϕ_0 towards ϕ_+.

Near $\rho \simeq R$, one can neglect the damping force, and also ϵ-dependent term in U. Eq. of motion of 1D analogue mechanical problem:

$$\frac{d^2\phi}{dx^2} = U_0'(\phi) \Longrightarrow \quad x = \int_{\phi_0}^{\phi_+} \frac{d\phi}{\sqrt{2U_1(\phi)}}$$

$$\Rightarrow \quad S_1 = \int dx \left[\frac{1}{2}\left(\frac{d\phi}{dx}\right)^2 + U_0\right]$$

$$= \int_{-a}^{a} d\phi \sqrt{2U(\phi)}$$

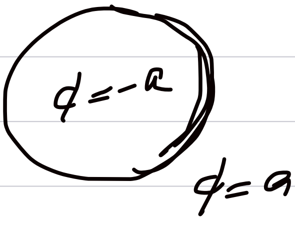
$$\Longrightarrow \quad \phi_1 = a\, th\left(\frac{\omega}{2}x\right); \quad U_0 = \frac{\lambda}{8}\left(\phi^2 - \frac{\omega^2}{\lambda}\right)^2$$

$$S_1 = \omega^3/3\lambda;$$

$$\phi_1(x) \underset{x\to\pm\infty}{\sim} \pm\left(a - K e^{-\omega|x|}\right)$$

By using this soln of the 1D problem,

$$\begin{aligned}\phi(\rho) &= -a, & \rho &\ll R\\ &\simeq \phi_1(\rho - R), & \rho &\simeq R\\ &\approx a & \rho &\gg R\end{aligned}$$



R can be fixed by the minimiz. of S_E

$$S_E = 2\pi^2 \int_0^\infty \rho^3 d\rho \left[\frac{1}{2}\left(\frac{d\phi}{d\rho}\right)^2 + U(\phi) \right]$$

$$= \underbrace{-\frac{1}{2}\pi^2 R^4}_{\uparrow}\epsilon + \underset{\uparrow}{\pi^2 R^3} S_1$$

inside the bubble   wall

$S_3 = 2\pi^2 R^3$

$V = \frac{2\pi^2}{4} R^4 = \frac{\pi^2 R^4}{2}$

$$\frac{dS_E}{dR} = 0 = -2\pi^2 R^3 \epsilon + 6\pi^2 R^2 S_1$$

$$\rightarrow \quad R = 3 S_1 / \epsilon$$

$\Rightarrow$ Resubst.

$$S_E = \frac{27\pi^2 S_1^4}{2\epsilon^3} \underset{\epsilon \to 0}{\gg} 1$$

and $\frac{\Gamma}{V} \sim e^{-S_E/\hbar}$.

For the concrete model,

$$B = \frac{\pi^2 \omega^{12}}{6\epsilon^3 \lambda^4}, \quad \left(\text{Validity } \; \omega^4/\epsilon\lambda \gg 1 \right)$$

Fate of the false vacuum.

$\phi_f \Rightarrow \phi_-$: quantum transition.

The classical fields make a jump, say, at $t=0$ to $\phi(t=0, x) = \phi(\tau=0, x)$, $\frac{\partial\phi}{\partial t} = 0$.

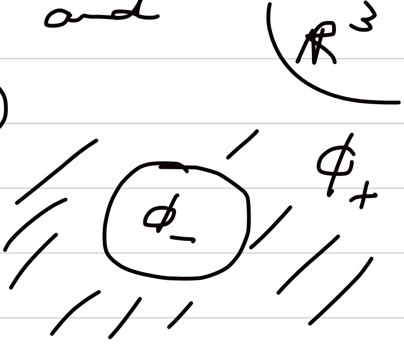
after that, it evolves accordly to.

$$-\frac{\partial^2\phi}{\partial t^2} + \nabla^2\phi = U'(\phi).$$

The sol. is also an analytic continuation of the Euclidean soln.

$$\phi(t, x) = \phi\left(\rho = \sqrt{x^2 - t^2}\right)$$

$\rho \cong R \Rightarrow |x| = R$ at $t = 0$ and
(a spherical bubble)

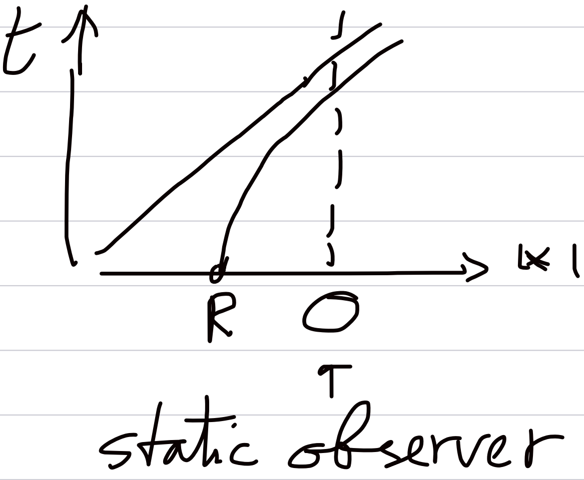


and grows at

$$|x|^2 - t^2 = R^2$$

↳ $R \sim$ microscopic size
$\sim$ fm.

- grows at the speed of light.



- Energy of the expanding bubble

$$\approx \frac{4\pi|x|^3}{3}\epsilon \leftarrow$$ just the energy difference between $\phi_\pm$

Final observations :

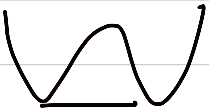
(i) $\epsilon \to 0$ $B = \infty$: <u>No decay</u>

$$S_E = \frac{27\pi^2 S_1^4}{2\epsilon^3} \leftarrow D = 3 \text{ (space dim)}$$

<u>Spontaneous symmetry breaking</u>

$(\phi \to -\phi)$.

cfr QM (zerodim F.T.)

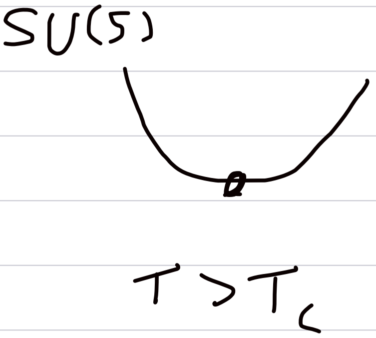


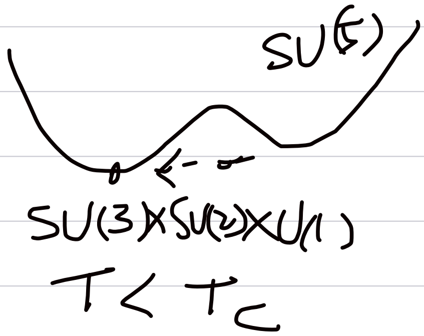
the tunnelling not <u>suppressed</u>.

(ii) S_1 : microscopic information.

(fundamental, particle physics ⟺ cosmology)

(iii) <u>GUT</u>





Instantons in SU(2) (YM) theories.

$$e^{-W} = \int \mathcal{D}A_\mu \mathcal{D}\bar\psi \mathcal{D}\psi \, e^{-S}$$

$$S = S_{gauge} + S_{matter}$$

$$S_{gauge} = \frac{1}{2}\int d^4x \, \mathrm{Tr}(G_{\mu\nu}G_{\mu\nu}) \, ; \quad G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]$$

$$S_{matter} = \sum_{quarks} i\bar\psi\gamma^\mu(\partial_\mu - igA_\mu)\psi$$

- Finite action fields

$$A_\mu \to iU^\dagger \partial_\mu U \quad \text{f} \ |x| \to \infty$$

$$\mathrm{Tr}\, G_{\mu\nu}G_{\mu\nu} \to 0$$

Classified by: $\swarrow$ $|x| = R \to \infty$

$$A : S^3 \to SU(2) \sim S^3 \, , \quad \pi_3(SU(2)) = \mathbb{Z}$$

- "Winding" # = Pontryagin number

$$q = \frac{g^2}{32\pi^2}\int d^4x \, G^a_{\mu\nu}\tilde{G}^a_{\mu\nu} \in \mathbb{Z} \, , \quad \boxed{\tilde{G}^a_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}G^a_{\mu\nu}}$$

It's a surface term

$$= \int_{R^4} \partial_\mu K^\mu d^4x \, ; \quad K^\mu = \frac{1}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}\,\mathrm{Tr}\left(G_{\alpha\beta}A_\gamma - \frac{2}{3}A_\alpha A_\beta A_\gamma\right)$$

Instanton

$$S = \int \frac{1}{2} \mathrm{Tr}\, G_{\mu\nu} G^{\mu\nu} = \frac{1}{4}\int \mathrm{Tr}\left[(G_{\mu\nu} - \tilde{G}_{\mu\nu})^2 + 2\, G_{\mu\nu}\tilde{G}^{\mu\nu}\right]$$

$$\Longrightarrow \frac{1}{4}\int d^4x\, \mathrm{Tr}(G_{\mu} - \tilde{G}_{\mu\nu})^2 + \frac{1}{2}\int d^4x\, \mathrm{Tr}\, G_{\mu\nu}\tilde{G}_{\mu\nu}$$

$$\geq \frac{8\pi^2}{g^2} \cdot n \qquad n = 0, 1, 2, \cdots$$

Search for solns of (Belavin Polyakov Shwartz Tyupkin)

$$G_{\mu\nu} = \pm \tilde{G}_{\mu\nu} \quad \left(\begin{matrix}\text{self-}\\ \text{dual}\end{matrix}\right); \quad S^{IJ} = \frac{8\pi}{g^2}$$

$$A^{inst}_{\mu} = \frac{\tau^a}{2} A^a_{\mu} = \frac{2i}{g}\, \frac{\tau_{\mu\nu}(x - x_0)^{\nu}}{(x - x_0)^2 + \rho^2}; \quad \left(\begin{matrix}\tau_\mu = \\ (1, -i\sigma_i)\end{matrix}\right)$$