Supersymmetric gauge theories.

[1]

Two-component spinors.

$$\psi_\alpha \sim (\tfrac{1}{2},0) \; ; \; \bar{\psi}_{\dot\alpha} \sim (0,\tfrac{1}{2}) \text{ of } L_+^\uparrow \sim SL(2,\mathbb{C})/\mathbb{Z}_2$$

They transform as $(\alpha,\beta=1,2 \; ; \; \dot\alpha,\dot\beta=1,2)$

$$\psi_\alpha \to \psi'_\alpha = M_\alpha{}^\beta \psi_\beta \; ; \; \bar\psi_{\dot\alpha} \to \bar\psi'_{\dot\alpha} = (M^*)_{\dot\alpha}{}^{\dot\beta} \bar\psi_{\dot\beta}$$

$$\det M = 1 .$$

Define also

$$\psi^\alpha \equiv \epsilon^{\alpha\beta}\psi_\beta \; ; \; \bar\psi^{\dot\alpha} \equiv \epsilon^{\dot\alpha\dot\beta}\bar\psi_{\dot\beta} \; ; \qquad \epsilon^{12} = -\epsilon^{21} = 1$$

$$(\bar\psi_{\dot\alpha} \equiv (\psi_\alpha)^*)$$

$$\epsilon_{12} = -\epsilon_{21} = -1$$

$$\sigma^\mu = (-1, \sigma^i)$$
$$\bar\sigma^\mu = (-1, -\sigma^i)$$

Define now $P \equiv p_\mu \sigma^\mu \qquad (L \Leftrightarrow \pm M)$

a vector p_μ (P) transforms as

$$P \to P' = M P M^\dagger \;, \Rightarrow p^\mu \sim (\tfrac{1}{2},\tfrac{1}{2})$$

also $\gamma^\mu = \begin{pmatrix} & \sigma^\mu \\ \bar\sigma^\mu & \end{pmatrix}$, $\psi_D = \begin{pmatrix} \chi_\alpha \\ \bar\psi^{\dot\alpha} \end{pmatrix}$; $\psi_M = \begin{pmatrix} \chi_\alpha \\ \bar\chi^{\dot\alpha} \end{pmatrix}$

and $Q_\alpha = \begin{pmatrix} Q_\alpha \\ \bar Q^{\dot\alpha} \end{pmatrix}$ Susy generators

$\Longrightarrow$

Note on the Lorentz group.

$$J_k = \frac{1}{2}\epsilon^{klm} M_{lm} \; ; \qquad (k, l, m = 1,2,3)$$
$$K_m \equiv M_{m0}$$

Df: $M_m = (J_m + iK_m)/2$
$N_m = (J_m - iK_m)/2$

$\Rightarrow$ $[M_m, M_n] = i\epsilon^{mnk} M_k$
$[N_m, N_n] = i\epsilon^{mnk} N_k$
$[M_m, N_n] = 0$

$\therefore$ Alg.$(L) \sim SU(2)\times SU(2)$

As a group L is non compact, $e^{i\omega^{\mu\nu}M_{\mu\nu}}$

$SU(2)\times SU(2)$ is compact $e^{i\omega^m M_m + i\, 2^n N_n}$

$\Rightarrow$ Finite repr. of the Lorentz group cannot be unitary. !

More on two-component spinors.

$$\psi_\alpha \to \psi'_\alpha = M_\alpha{}^\beta \psi_\beta \; ; \quad \bar\psi_{\dot\alpha} \to \bar\psi'_{\dot\alpha} = (M^*)_{\dot\alpha}{}^{\dot\beta} \bar\psi_{\dot\beta}$$

$$\psi^\alpha \to \psi^{\alpha\prime} = \psi^\beta (M^{-1})_\beta{}^\alpha \; ; \quad \bar\psi^{\dot\alpha} \to \bar\psi^{\dot\alpha\prime} = (M^*)^{-1}{}_{\dot\beta}{}^{\dot\alpha} \bar\psi^{\dot\beta}$$

$$\left.\begin{array}{l} \sigma^\mu \equiv (-1, \sigma^i) \\ \bar\sigma^\mu \equiv (-1, -\sigma^i) \end{array}\right\} \quad \mathrm{Tr}\, \bar\sigma^\mu \sigma^\nu = 2 g^{\mu\nu}$$

$$g^{\mu\nu} = (1 \;\; -1 \;\; -1 \;\; -1)$$

Vector p^μ $\left(\Leftrightarrow P = p_\mu \sigma^\mu ; \;\; p^\mu = \frac{1}{2} \mathrm{Tr} (\bar\sigma^\mu P) \right)$

Lorentz tr:

$$P \to P' = M P M^\dagger$$

Exercise show $M = e^{\frac{1}{2} i \phi \sigma_3}$ $\to$ rot. ; $M = e^{\frac{1}{2} w \sigma_3}$ $\to$ boost

N.B. $\det P = \det \begin{pmatrix} p^0 + p^3 & p^1 - i p^2 \\ p^1 + i p^2 & -p^0 p^3 \end{pmatrix} = p_0^2 - \mathbf{p}^2$ $\quad SL(2, \mathbb{C})$

is invariant!

$\therefore \quad p^\mu \sim (1/2, 1/2)$

N.B. $\quad \psi^\alpha \psi^\beta \sim (1,0) + (0,0)$ $\qquad$ Df $\psi\psi \equiv \psi^\alpha \psi_\alpha$, $\quad \bar\psi\bar\psi = \bar\psi_{\dot\alpha} \bar\psi^{\dot\alpha}$

$$\psi^\alpha \psi_\alpha \to \psi M^{-1} M \psi = \psi^\alpha \psi_\alpha \quad \text{i.e. inv.}$$

$$\psi^\alpha \chi_\alpha \quad \text{i.v.}$$

$$\bar\psi_{\dot\alpha} \bar\psi^{\dot\alpha} \to \bar\psi . M^* (M^*)^{-1} \bar\psi = \bar\psi \bar\psi \quad \text{inv.}$$

$$\psi^\alpha \sigma^\mu_{\alpha\dot\alpha} \partial_\mu \bar\psi^{\dot\alpha} \longrightarrow \psi \sigma^\mu \partial_\mu \bar\psi \quad \text{inv !}$$

$$\psi_\alpha \bar\psi_{\dot\beta} \sim p^\mu (\sigma_\mu)_{\alpha\dot\beta} \sim (1/2, 1/2)$$

2 compt vs 4 compt spinors → chiral repr.

$$\gamma^\mu = \begin{pmatrix} & \sigma^\mu \\ \bar\sigma^\mu & \end{pmatrix} \qquad \{\gamma^\mu, \gamma^\nu\} = 2 g^{\mu\nu}$$

Dirac $\psi_D = \begin{pmatrix} \chi_\alpha \\ \bar\psi^{\dot\alpha} \end{pmatrix} \Leftrightarrow$ 2 Weyl.

Majorana $\psi_M = \begin{pmatrix} \chi_\alpha \\ \bar\chi^{\dot\alpha} \end{pmatrix} \Leftrightarrow$ 1 Weyl (neutrinos?)

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} \mathbb{1} & \\ & -\mathbb{1} \end{pmatrix} ; \quad \gamma^0 = \begin{pmatrix} & -\mathbb{1} \\ -\mathbb{1} & \end{pmatrix} ,$$

$$\bar\psi_D \equiv \psi_D^\dagger \gamma^0 = (\bar\chi_{\dot\alpha}, \psi^\alpha)\begin{pmatrix} & -1 \\ -1 & \end{pmatrix} = (-\psi^\alpha, -\bar\chi_{\dot\alpha})$$

$$\bar\psi_M = (-\chi^\alpha, -\bar\chi_{\dot\alpha}) = C\psi_M^T ; \quad C = -i\gamma^0\gamma^2 = -\begin{pmatrix} -1 & & & \\ & 1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}$$

$$= -\begin{pmatrix} \epsilon^{\alpha\beta} & \\ & \epsilon_{\dot\alpha\dot\beta} \end{pmatrix} \quad \text{charge conj.}$$

$$\bar\psi_D\psi_D = (-\psi^\alpha, -\bar\chi_{\dot\alpha})\begin{pmatrix} \chi_\alpha \\ \bar\psi^{\dot\alpha} \end{pmatrix} = -\psi\chi - \bar\chi\bar\psi$$

N.B. $\psi\chi = \psi^\alpha\chi_\alpha = -\chi_\alpha\psi^\alpha = \chi^\alpha\psi_\alpha = \chi\psi$

↳ grassmanian (operator)

$$(\psi\chi)^\dagger = (\psi^\alpha\chi_\alpha)^\dagger = \bar\chi_{\dot\alpha}\bar\psi^{\dot\alpha} = \bar\chi\bar\psi = \bar\psi\bar\chi$$

$$(\psi\sigma^\mu\bar\chi)^\dagger = \chi\sigma^\mu\bar\psi$$

So $$\bar\psi i\gamma^\mu\partial_\mu\psi_D = (-\psi^\alpha, -\bar\chi_{\dot\alpha})\begin{pmatrix} 0 & i\sigma^\mu_{\alpha\dot\alpha}\partial_\mu \\ i\bar\sigma^{\mu\dot\alpha\alpha}\partial_\mu & 0 \end{pmatrix}\begin{pmatrix} \chi_\alpha \\ \bar\psi^{\dot\alpha} \end{pmatrix}$$

$$= -i\psi\sigma^\mu\partial_\mu\bar\psi - i\bar\chi\bar\sigma^\mu\partial_\mu\chi = (\quad)^\dagger \quad \text{o.k.}$$

also, $(\bar\sigma^\mu)^{\dot\alpha\alpha} \equiv \epsilon^{\dot\alpha\dot\beta}\epsilon^{\alpha\beta}(\sigma^\mu)_{\beta\dot\beta}$ ✓

chirality & helicity.

$$i\gamma^\mu\partial_\mu \psi_D = 0 \longrightarrow \begin{cases} i\sigma^\mu\partial_\mu \bar\psi = 0 \\ i\bar\sigma^\mu\partial_\mu \chi = 0 \end{cases}$$

$$(i\partial_0 + i\sigma^i\partial_i)\bar\psi = 0$$

$$E\bar\psi \equiv i\partial_0\bar\psi = -(\sigma\cdot p)\bar\psi \quad \therefore \sigma\cdot p < 0$$

$\therefore$ $\bar\psi$ ha la chiralità negativa.

ψ " " positiva.

[2]. Susy algebra

$$\{Q^i_\alpha, \bar{Q}^j_{\dot\beta}\} = -2\,\sigma^\mu_{\alpha\dot\beta} P_\mu \delta^{ij} \quad \begin{pmatrix} i=1,2\cdots N \\ N\text{-supersym} \end{pmatrix}$$

$$\{Q^i_\alpha, Q^j_\beta\} = 0$$

$$[P^\mu, Q] = 0$$

$$i[M^{\mu\nu}, Q_\alpha] = (\sigma^{\mu\nu} Q)_\alpha$$

$$i[M^{\mu\nu}, \bar{Q}^{\dot\alpha}] = (\bar\sigma^{\mu\nu})^{\dot\alpha}{}_{\dot\beta}\, \bar{Q}^{\dot\beta}$$

$$[P^\mu, P^\nu] = 0$$

$$i[M^{\mu\nu}, P^\lambda] = g^{\mu\lambda}P^\nu - g^{\nu\lambda}P^\mu$$

$$i[M^{\mu\nu}, M^{\lambda\kappa}] = g^{\mu\lambda}M^{\nu\kappa} - \cdots$$

$$\sigma^{\mu\nu} \equiv \frac{\sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu}{4}$$

gen. Lorentz tr. in the spinor basis

$$\Sigma^{\mu\nu} = \frac{1}{4}[\gamma^\mu, \gamma^\nu] = \begin{pmatrix} \sigma^{\mu\nu} & 0 \\ 0 & \bar\sigma^{\mu\nu} \end{pmatrix};$$

N.B. $\{Q^i_\alpha, Q^j_\beta\} = \epsilon_{\alpha\beta}\epsilon^{ij}(c_1 Z_1 + i c_2 Z_2)$ $(N=2)$

$(c_1, c_2$: central charges $)$

$N=1 \qquad P^\mu = (E \quad , 0)$ rest syst of everything

$\to \{Q_\alpha, \bar{Q}_{\dot\beta}\} = 2E\delta_{\alpha\dot\beta}$

$\langle\{Q, \bar{Q}\}\rangle \geq 0$

$\Longrightarrow \quad E \geq 0 \qquad \therefore \quad \Lambda_{cont} = 0.$

$Q_\alpha|0\rangle = 0 \longrightarrow \langle 0|E|0\rangle = 0.$

(One of the original motivation)

(i) Haag-Sohnius-Lopszanski theorem.

Supersymmetry algebra is the only algebra which contains Poincaré algebra, which generalizes it by additional Grassmann algebra, and that is consistent with nontrivial S matrix of QFT.

This evades the famous Coleman-Mandula theorem: the only symmetry algebra which generalizes the Poincaré algebra to include internal symmetry algebra, and that is consistent with nontrivial S-matrix is

$$g_{\text{Poincaré}} \otimes g_{\text{internal symmetry}}.$$

(ii) Jacobi's identity reads

$$[A, [B, C]] \pm [B, [C, A]] \pm [C, [A, B]] = 0$$

where the anticomm. when both are Grassmanian, and the signs are determined by whether in that term the Grassmanian operators appear in an even or odd perm. w.r.t. the 1st term

(iii) For $N = 2$

the supersymmetry algebra can be generalized with so-called central extension:

$$\{Q^i_\alpha, Q^j_\beta\} = \epsilon_{\alpha\beta}\, \epsilon^{ij} (U + iV)$$

where U and V are the central charges.
(they commute with all generators)

It has been shown (Olive-Witten) that in $N=2$ non Abelian gauge theories U, V do appear, and correspond to electric and magnetic charges.

[3].

1 particle repr. of SUSY algebra.

(i) $N=1$ SUSY; massive particle.

$P^\mu = (M, \vec{0})$

$\{Q_\alpha, \bar{Q}_{\dot\alpha}\} = \delta_{\alpha\dot\alpha}\, 2M \Rightarrow$ Def. $\begin{cases} b_a^\dagger = \frac{1}{\sqrt{2M}} Q_\alpha \\ b_{\dot\alpha} = \frac{1}{\sqrt{2M}} \bar{Q}_{\dot\alpha} \end{cases}$

then $\{b_\alpha, b_\beta^\dagger\} = \delta_{\alpha\beta}$

$\Rightarrow$ $b^\dagger$ ~ creation operator; b ~ annihil.

Vac. $|0\rangle$ defined by $b_i|0\rangle = 0$ $(i=1,2)$

$|0\rangle$, $b_1^\dagger|0\rangle$, $b_2^\dagger|0\rangle$, $b_1^\dagger b_2^\dagger|0\rangle$

N.B. $\begin{cases} b_1^{\dagger 2}=0 \\ b_2^{\dagger 2}=0 \end{cases}$

$\Rightarrow$ 4 digenerate states $\left(\begin{array}{l} 2 \text{ bosons} \\ 2 \text{ fermions} \end{array}\right)$

Extended (N) SUSY

Q_α^i ; $\bar{Q}_{\dot\alpha}^i$ $\Rightarrow$ $2N$ copies of creation/ann. operators.

$$\Rightarrow d = \sum_{n=0}^{2N} \binom{2N}{n} = 2^{2N} \left\langle \begin{array}{l} 2^{2N-1} \text{ bosons} \\ 2^{2N-1} \text{ fermions.} \end{array}\right.$$

(ii) $N=1$ SUSY, $M=0 \Rightarrow P^\mu = (E,0,0,E)$

$$\{Q_\alpha, \bar{Q}_{\dot\alpha}\} = \begin{pmatrix} 2E & 0 \\ 0 & 0 \end{pmatrix}_{\alpha\dot\alpha}$$

$b_2^\dagger|0\rangle$ has 0-norm. $\Rightarrow$ NO.

$|0\rangle$, $b_1^\dagger|0\rangle$ only $\Rightarrow$ 2 states

N-SUSY Q_i $\bar{Q}_j$ only $i=1..N$

$$\Rightarrow d = \sum \binom{N}{n} = 2^N \text{ states}$$

(iii) $N=2$ susy, $M\neq 0$ particles, $P^\mu = (M, \vec{0})$ but with central charge.

$Q_1^1, Q_1^2, Q_2^1, Q_2^2 \rightarrow$ 4 + 4 charges (4, 4) Susy

$$\begin{cases} \{Q_\alpha^i, \bar{Q}_{\dot\alpha}^j\} = \delta^{ij}\delta_{\alpha\dot\alpha} 2M \\ \{Q_\alpha^i, Q_\beta^j\} = \epsilon_{\alpha\beta}\epsilon^{ij}(U + iV) \end{cases}$$

$U \to Q_e$, $V \to Q_m$ (Olive + Witten)

Def. $\frac{Q_1^1}{\sqrt{2M}} = b_1$; $\frac{Q_2^1}{\sqrt{2M}} = b_2$; $\frac{Q_1^2}{\sqrt{2M}} = b_3$; $\frac{Q_2^2}{\sqrt{2M}} = b_4$

(By $Q_\alpha^1 \to e^{i\gamma} Q_\alpha$, $Q_\alpha^2 \to Q_\alpha^2$)

$\{b_1, b_4\} = \alpha$; $\{b_2, b_3\} = -\alpha$

$\{b_1^\dagger, b_4^\dagger\} = \alpha$; $\{b_2^\dagger, b_3^\dagger\} = -\alpha$

$\{b_i, b_i^\dagger\} = 1$

$$\alpha = \frac{\sqrt{U^2+V^2}}{2M}$$

$\hookrightarrow \{b_1, b_4, b_1^\dagger, b_4^\dagger\} \Longleftrightarrow \{A, B, A^\dagger, B^\dagger\}$

such that A & B decouple ?

<u>Ans</u>

$$A \equiv b_1 \cos\theta + b_4^\dagger \sin\theta$$
$$B \equiv -b_1 \sin\theta + b_4^\dagger \cos\theta$$

(Similarly for b_2, b_3)

then $\{A, B\} = 0$; $\{A, B^\dagger\} = (\cos^2\theta - \sin^2\theta)\alpha = \cos 2\theta\, \alpha$

$\therefore$ choose $\theta = \pi/4 \Rightarrow$

$$A = (b_1 + b_4^\dagger)/\sqrt{2} \; ; \; B = (-b_1 + b_4^\dagger)/\sqrt{2}$$

$$\Rightarrow \{A, A^\dagger\} = \tfrac{1}{2}(1 + 1 + 2\alpha) = 1 + \alpha$$
$$\{B, B^\dagger\} = \tfrac{1}{2}(1 + 1 - 2\alpha) = 1 - \alpha$$

$$\frac{Q_1^1}{\sqrt{2M}} = b_1 \quad \text{etc.}$$

and $-\frac{U}{2M} = u$, $-\frac{V}{2M} = v$

$$\{b_i, b_j^\dagger\} = \delta_{ij}$$

$$\{b_1, b_4\} = u + iv, \quad \{b_2, b_3\} = -u - iv$$

$$\{b_1^\dagger, b_4^\dagger\} = u - iv.$$

$$Q_\alpha^1 \to e^{i\gamma} Q_\alpha^1 \qquad Q_\alpha^2 \to Q_\alpha^2$$

$$\left(\begin{array}{ll} b_1 \to e^{i\gamma} b_1, & b_2 \to e^{i\gamma} b_2 \\ b_4 \to b_4, & b_3 \to b_3 \end{array} \right.$$

$$\{b_1, b_4\} \to \cancel{e^{i\gamma}} \{b_1, b_4\} = \cancel{e^{i\delta}} \sqrt{u^2+v^2} \qquad \left(\delta = \arg \frac{v}{u}\right)$$

$$= \frac{\sqrt{U^2+V^2}}{2M}$$

$$\equiv \alpha$$

So

$1 > |\alpha|$: $A^\dagger, B^\dagger$ both create pos. norm states

$1 = |\alpha|$: One of $A^\dagger$ (or $B^\dagger$) creates a pos. norm state
the other $\Rightarrow$ zero norm. states $\times$

$1 < |\alpha|$ One of $A^\dagger, B^\dagger$ creates neg. norm states. NOT allowed.

Conclusion

$2M > \sqrt{U^2 + V^2} \Rightarrow$ 4 creation operators. $\Rightarrow 2^{2N} \underset{\mathcal{N}=2}{=} 16$ states (*)

"BPS - saturated" $2M = \sqrt{U^2+V^2} \Rightarrow$ 2 creation operators $\Rightarrow 2^N = 4$ states (**)

$2M < \sqrt{U^2+V^2} \Rightarrow$ NOT consistent.

(*), (**) known as a long and short $\mathcal{N}=2$ multiplets. (representation).

[4] Field Transformation

$$\{Q_\alpha, \bar{Q}_{\dot\alpha}\} = -2(\sigma^\mu)_{\alpha\dot\alpha} P_\mu , \quad P_\mu \equiv i\partial_\mu$$

Superfields and superspace.

$$F(x,\theta,\bar\theta) = f(x) + \theta\psi(x) + \bar\theta\bar\chi + \cdots + \theta^2\bar\theta^2 D$$

θ^α, $\bar\theta_{\dot\alpha}$ are Grassmannian coordinates.

$f(x)$, $\psi(x)$, ... are component fields.

$$\{\theta^1, \theta^2\} = 0, \quad (\theta^1)^2 = 0, \quad \theta^\alpha\theta_\alpha = \theta^1\theta_1 + \theta^2\theta_2 = 2\theta^2\theta^1$$

$\epsilon^{12} = 1$; $\epsilon_{12} = -1$ etc

Also, $\int d\theta = \frac{\partial}{\partial\theta}$.

$$\int d\theta^1\theta^1 = 1, \quad \int d^2\theta\ \theta\theta = 1.$$

$$\int d^2\theta\ \theta = \int d^2\theta . 1 = 0.$$

$$d^2\theta \equiv -\frac{1}{4}\epsilon_{\alpha\beta}\, d\theta^\alpha d\theta^\beta = -\frac{1}{2} d\theta^2 d\theta^1$$

$$\theta\theta = \delta^2(\theta)$$

Susy transformation $\equiv$ translation $\theta \to \theta + \xi$, $\bar\theta \to \bar\theta + \bar\xi$

$$\Rightarrow \quad Q_\alpha = \frac{\partial}{\partial\theta^\alpha} - i\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu \; ; \; \bar{Q}_{\dot\alpha} = \frac{\partial}{\partial\bar\theta^{\dot\alpha}} - i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\partial_\mu$$

$$\{Q_\alpha, \bar{Q}_{\dot\alpha}\} = -2i\sigma^\mu_{\alpha\dot\alpha}\partial_\mu \qquad \text{OK}.$$

In general, $F(x,\theta,\bar\theta)$ is reducible $\Rightarrow$ irreducible multiplets.

Susy transf. of the component fields defined by

$$F(x,\theta,\bar\theta) \Rightarrow \delta_\xi F = [\xi Q + \bar\xi \bar Q, F(x,\theta,\bar\theta)]$$
$$= \delta f(x) + \theta\delta\psi + \cdots$$

Introduce certain constraints (which however must not constrain the functional forms of the component fields), so as to reduce the # of compnts

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Introduce also the covariant derivatives

$$\bar D_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} - i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\partial_\mu \; ; \; D_\alpha = -\frac{\partial}{\partial\theta^\alpha} + i(\sigma^\mu\bar\theta)_\alpha\partial_\mu$$

which satisfy

- $\{D_\alpha, \bar D_{\dot\alpha}\} = -2i\,\sigma^\mu_{\alpha\dot\alpha}\partial_\mu$
- $\{Q, D\} = \{Q, \bar D\} = \{\bar Q, D\} = \{\bar Q, \bar D\} = 0$
- $\bar D^3 = D^3 = 0$.

N.B.
$$D^\alpha D_\alpha = \epsilon^{\alpha\beta} D_\beta D_\alpha = \epsilon^{\alpha\beta}\Big[\frac{\partial^2}{\partial\theta^\alpha\partial\theta^\beta} + i(\sigma^\mu\bar\theta)_\alpha\frac{\partial}{\partial\theta^\beta}\partial_\mu$$
$$- i(\sigma^\mu\bar\theta)_\beta\frac{\partial}{\partial\theta^\alpha}\partial_\mu + i(\sigma^\mu\bar\theta)_\alpha(i\sigma^\nu\bar\theta)_\beta\,\partial_\mu\partial_\nu\Big]$$
$$= 2\frac{\partial}{\partial\theta^1}\frac{\partial}{\partial\theta^2} + 2i(\sigma\bar\theta)_1\partial\frac{\partial}{\partial\theta^2} - 2i(\sigma\bar\theta)_2\partial\frac{\partial}{\partial\theta^1} + 2(i\sigma^\mu\bar\theta)_1(i\sigma^\nu\bar\theta)_2\,\partial_\mu\partial_\nu$$

$$D_1(DD) = \left(\frac{\partial}{\partial\theta^1} + i(\sigma^\mu\bar\theta)_1\partial_\mu\right)\cdot DD$$

$$= -2(i\sigma\bar\theta)_1\partial\frac{\partial^2}{\partial\theta^1\partial\theta^2} + 2(i\sigma^\mu\bar\theta)_1(i\sigma^\nu\bar\theta)_2\partial_\mu\partial_\nu\frac{\partial}{\partial\theta^1}$$

$$+ 2i(\sigma^\mu\bar\theta)_1\partial_\mu\frac{\partial^2}{\partial\theta^1\partial\theta^2} + i(\sigma^\mu\bar\theta)_1\partial_\mu(-2i\sigma\bar\theta)_2\partial\frac{\partial}{\partial\theta^1}$$

$$= 0 \,, \qquad \text{o.k.}$$

[5] Chiral superfields (scalar) and Wess-Zumino model.

Def. $\bar{D}\Phi = 0$. $\left(\begin{array}{c}\text{v.v. anti-chiral f.}\\ D\Phi^\dagger = 0\end{array}\right)$

↳ Solu: w. $y^\mu \equiv x^\mu + i\theta\sigma^\mu\bar\theta$

$$\Phi(x,\theta,\bar\theta) = A(y) + \sqrt{2}\,\theta\psi(y) + \theta^2 F(y)$$

($\theta^2 \to \theta\theta$)

$$= A(x) + i(\theta\sigma^\mu\bar\theta)\partial_\mu A - \frac{1}{4}\theta^2\bar\theta^2\Box A(x) + \sqrt{2}\,\theta\psi(x) + \frac{i}{\sqrt{2}}\theta^2(\partial_\mu\psi)\sigma^\mu\bar\theta + \theta\theta F(x)$$

where $(\theta\sigma^\mu\bar\theta)(\theta\sigma^\nu\bar\theta) = \frac{1}{2}(\theta\theta)(\bar\theta\bar\theta)g^{\mu\nu}$;

$(\theta\psi)(\theta\chi) = -\frac{1}{2}(\psi\chi)(\theta\theta)$, etc

N.B. $\bar{D}_{\dot\alpha} = -\left.\frac{\partial}{\partial\bar\theta^{\dot\alpha}}\right|_{y,\theta}$ $\quad(\bar{D}_{\dot\alpha}\, y = 0)$

N.B. $\bar{D}\Phi_1 = \bar{D}\Phi_2 = 0 \Rightarrow \bar{D}(\Phi_1\Phi_2) = 0$

$\bar{D}(\Phi_1 + \Phi_2) = 0$ $\quad\to$ Form a ring.

$\Rightarrow \bar{D}P(\Phi_1, \Phi_2 \cdots) = 0$ if $P(\Phi_i)$ holomorphic (no $\Phi_i^\dagger$)

$$\Phi^\dagger\Phi = A^\dagger A + \cdots + \theta\theta\bar\theta\bar\theta\Big[|F|^2 - \frac{1}{4}A^\dagger\Box A - \frac{1}{4}\Box A^\dagger A + \frac{1}{2}\partial_\mu A^\dagger\partial^\mu A + \frac{i}{2}\partial_\mu\bar\psi\bar\sigma^\mu\psi - \frac{i}{2}\bar\psi\bar\sigma^\mu\partial_\mu\psi\Big]$$

↳ the highest components.

Susy tr. of chiral fields.

$$\Phi = A(y) + \sqrt{2}\theta\psi + \theta\theta F \quad \Rightarrow$$

$$\begin{cases} [Q_\alpha, A] = \sqrt{2}\psi_\alpha ; \quad \{Q_\alpha, \psi_\beta\} = \sqrt{2}\epsilon_{\alpha\beta} F ; \\ [Q_\alpha, F] = 0 ; \\ [\bar{Q}_{\dot\alpha}, A] = 0 ; \quad [\bar{Q}_{\dot\alpha}, \psi_\alpha] = -i\sqrt{2}(\sigma^\mu)_{\alpha\dot\alpha}\partial_\mu A \\ [\bar{Q}_{\dot\alpha}, F] = -\sqrt{2}i(\sigma^\mu)_{\alpha\dot\alpha}\partial_\mu \psi^\alpha \end{cases}$$

Similarly for $\Phi^\dagger$:

$$\begin{cases} [\bar{Q}_{\dot\alpha}, \bar{A}] = \sqrt{2}\bar{\psi}_{\dot\alpha} ; \{\bar{Q}_{\dot\alpha}, \bar{\psi}^{\dot\beta}\} = -2\bar{F} ; [\bar{Q}, \bar{F}] = 0 ; \\ [Q_\alpha, \bar{A}] = 0 ; \{Q_\alpha, \bar{\psi}_{\dot\alpha}\} = \sqrt{2}i\sigma^\mu_{\alpha\dot\alpha}\partial_\mu \bar{A} ; [Q_\alpha, \bar{F}] = \sqrt{2}i\sigma^\mu_{\alpha\dot\alpha}\partial_\mu \bar{\psi}^{\dot\alpha} \end{cases}$$

$$\Rightarrow \quad \Phi^\dagger\Phi\big|_D \to \partial(\ ) ; \quad P(\Phi)\big|_F \to \partial(\ )$$

a supersymm. Lagrangian can be made by

$$\mathcal{L} = \Phi^\dagger\Phi\big|_{\theta\theta\bar\theta\bar\theta} + P(\Phi)\big|_{\theta\theta} + h.c$$

$$= \int d^4\theta\, \Phi^\dagger\Phi + \int d^2\theta\, P(\Phi) + h.c.$$

where $\int d^4\theta\, \theta\theta\bar\theta\bar\theta = 1$; $\int d^2\theta\, \theta\theta = 1$.

From $Q_\alpha = \frac{\partial}{\partial\theta^\alpha} - i\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu$ $\left(\frac{\text{similar}}{\bar Q}\right)$

the highest components

$P(\Phi_i)\big|_{\theta\theta}$ or $\Phi^\dagger\Phi\big|_{\theta\theta\bar\theta\bar\theta}$

transform into $\partial_\mu(\ )$ $\Longrightarrow$

The Lagrangian containing only chiral superfields (Wess-Zumino model)

is $\mathcal{L} = \sum_i \Phi_i^\dagger\Phi_i\big|_D + P(\Phi)\big|_F$

$\hookrightarrow \frac{m_{ij}}{2}\Phi_i\Phi_j + \frac{1}{3}g\Phi\Phi\Phi + \lambda\Phi\big|_{\theta\theta}$

<u>The simplest model</u>

$\mathcal{L} = -i\bar\psi\bar\sigma^\mu\partial_\mu\psi + \partial_\mu A^\dagger\partial^\mu A + F^\dagger F + m(AF - \frac{\psi\psi}{2})$

$+ m(AF - \frac{1}{2}\psi\psi) + g(A^2F - \psi\psi A) + \lambda F + \text{h.c.}$

Eq. of motion for F (auxiliary field) is

$F^\dagger + mA + gA^2 + \lambda = 0$

$\Rightarrow \mathcal{L} = -i\bar\psi\bar\sigma\partial_\mu\psi + \partial A^\dagger\partial A - \frac{m}{2}\psi\psi - \text{h.c.}$

$-g\psi\psi A - \text{h.c.} - V(A, A^\dagger)$,

$V(A, A^\dagger) = F^\dagger F = |\lambda + mA + gA^2|^2$

- In general W-Z model,

$$\mathcal{L} = \partial_\mu A^\dagger \partial^\mu A + i\,\partial_\mu\bar{\psi}\bar{\sigma}^\mu \psi - |\mathcal{P}'(A)|^2 - \frac{1}{2}\mathcal{P}''\psi\psi - h.c.$$

- Super current conservation

$$\delta\mathcal{L} = \delta\phi^a \cdot (\partial_\mu \Lambda^{a,\mu}) \quad (= 0 \text{ for internal symmetry})$$

then Def: $\dfrac{\delta\mathcal{L}}{\delta(\partial_\mu\phi^a)} - \Lambda^{a,\mu} \equiv J^{a,\mu}$

$$\Rightarrow \partial_\mu J^{a,\mu} = \partial_\mu \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi^a)} - \partial_\mu\Lambda^{a,\mu} =$$

$$\overset{\text{eq. mot.}}{=} \frac{\delta\mathcal{L}}{\delta\phi^a} - \partial_\mu\Lambda^{a,\mu} = 0 \qquad \text{o.k.}$$

In the W-Z model,

$$S^\mu_\alpha = -\sqrt{2}\,\partial_\nu A^\dagger (\sigma^\nu\bar{\sigma}^\mu\psi)_\alpha + i\sqrt{2}\,(\bar{\psi}\bar{\sigma}^\mu)^\beta\, \bar{\mathcal{P}}'(A^\dagger)\,\epsilon_{\alpha\beta}.$$

$\Rightarrow \partial_\mu S^\mu{}_\alpha = 0$ by use of eq. motion for $A, \psi, \bar{\psi}$. o.k.

[6] Vector (real) superfield and gauge theories

$$V = V^\dagger$$

$$V = C(x) + i\theta\chi - i\bar\theta\bar\chi + \frac{i}{2}\theta\theta[M+iN] - \frac{i}{2}\bar\theta\bar\theta[M-iN]$$
$$+ \theta\sigma^\mu\bar\theta v_\mu + i\theta\theta\bar\theta\left(\bar\lambda + \frac{i}{2}\bar\sigma^\mu\partial_\mu\chi\right) - i\bar\theta\bar\theta\theta\left(\lambda + \frac{i}{2}\sigma^\mu\partial_\mu\bar\chi\right)$$
$$+ \frac{1}{2}\theta\theta\bar\theta\bar\theta\left[D(x) - \frac{1}{2}\Box C\right],$$

C, D, M, N, v_μ all real.

$$\Longleftrightarrow \quad \Phi + \Phi^\dagger = A + A^* + \sqrt{2}(\theta\psi + \bar\theta\bar\psi) + \theta\theta F + \bar\theta\bar\theta F^\dagger$$
$$+ i(\theta\sigma^\mu\bar\theta)\partial_\mu(A - A^*) + \frac{i}{\sqrt{2}}\theta\theta\bar\theta\bar\sigma^\mu\partial_\mu\psi + \frac{i}{\sqrt{2}}\bar\theta\bar\theta\theta\sigma^\mu\partial_\mu\bar\psi$$
$$- \frac{1}{4}\theta\theta\bar\theta\bar\theta\,\Box(A + A^*)$$

Supersymmetric gauge transformation:

$$V \to V + \Phi + \Phi^\dagger$$

$$C \to C + A + A^*$$
$$\chi \to \chi - i\sqrt{2}\psi$$
$$M + iN \to M + iN - 2iF$$
$$v_\mu \to v_\mu - i\partial_\mu(A - A^*)$$
$$\lambda \to \lambda$$
$$D \to D$$

Eliminate C, χ, M, N !

In such a gauge,

$$V = \theta\sigma^\mu\bar\theta\, v_\mu + i\theta\theta\bar\theta\bar\lambda - i\bar\theta\bar\theta\theta\lambda + \frac{1}{2}\theta\theta\bar\theta\bar\theta D \;;$$

$$V^2 = \frac{1}{2}\theta\theta\,\bar\theta\bar\theta\, v^\mu v_\mu \;;$$

$$V^3 = 0 \quad \Longleftarrow \quad \text{Wess-Zumino gauge}$$

This still leaves the usual gauge transformation freedom,

$$v_\mu \to v_\mu + i\partial_\mu (A - A^*)$$

SQED

Def. $\boxed{W_\alpha = -\frac{1}{4}\bar D^2 D_\alpha V}$

$$= -i\lambda_\alpha(y) + \theta_\alpha D + \frac{i}{2}(\sigma^\mu\bar\sigma^\nu\theta)_\alpha F_{\mu\nu}(y) + \theta\theta\left(\sigma^\mu\partial_\mu\bar\lambda(y)\right)_\alpha$$

$$\bar W_{\dot\alpha} = i\bar\lambda_{\dot\alpha}(y^\dagger) + \bar\theta_{\dot\alpha} D(y^\dagger) - \frac{i}{2}\epsilon_{\dot\alpha\dot\gamma}(\bar\sigma^\mu\sigma^\nu\bar\theta)^{\dot\gamma} F_{\mu\nu}(y^\dagger) - \epsilon_{\dot\alpha\dot\beta}\bar\theta\bar\theta(\bar\sigma^\mu\partial_\mu\lambda)^{\dot\beta}$$

N.B. $\bar D W = 0 \Rightarrow W_\alpha$ is a chiral superfield

$$W^\alpha W_\alpha\big|_{\theta\theta} = -2i\lambda\sigma^\mu\partial_\mu\bar\lambda - \frac{1}{2}\left(F_{\mu\nu}^2 + iF_{\mu\nu}\tilde F^{\mu\nu}\right) + D^2$$

$$W^\alpha W_\alpha = -\frac{1}{4}\left(\bar D^2 W^\alpha D_\alpha V\right)$$

$$\Rightarrow \mathcal{L}_{SQED} = \frac{1}{4}\left\{ W^2\Big|_{\theta\theta} + \bar{W}^2\Big|_{\bar{\theta}\bar{\theta}} \right\} + \gamma V_D$$

$\gamma V_D \to \frac{\gamma}{2} D$

$V_D \to \theta\theta\bar{\theta}\bar{\theta}$

$$S = \frac{1}{4}\int d^4x\, d^2\theta\, W^2 + h.c.$$

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 - i\lambda\sigma^\mu\partial_\mu\bar{\lambda} + \frac{1}{2}D^2$$

Fayet-Iliopoulos term

→ Inclusion of matter fields (electrons)

→ NonAbelian gauge fields.

We do it all at once.

Introduce

$$V \equiv V^a t^a \ ; \quad t^a : \text{generators of e.g. } SU(N)$$

with normalization

$$\mathrm{Tr}\, t^a t^b = \frac{1}{2}\delta^{ab}.$$

Def. $W_\alpha \equiv -\frac{1}{4}\bar{D}^2 e^{-V} D_\alpha e^{V}$ $\left(\begin{array}{c}\bar{D}W_\alpha = 0 : \\ \text{a chiral superfield}\end{array}\right)$

Gauge tr:

$$e^{V} \longrightarrow e^{-i\Lambda^\dagger} e^{V} e^{i\Lambda} \qquad \left(\begin{array}{c}\bar{D}\Lambda = 0 \\ D\Lambda^\dagger = 0\end{array}\right) \quad (*)$$

$$\Rightarrow \quad W_\alpha \longrightarrow e^{-i\Lambda} W_\alpha e^{i\Lambda}$$

and the matter (chiral) superfield Φ,

$$\Phi \longrightarrow e^{-i\Lambda}\Phi \qquad (**)$$

$$\mathcal{L} = \frac{1}{4\pi} \operatorname{Im}\left[\left(\frac{4\pi i}{g^2} + \frac{\theta}{2\pi}\right)\int d^6 z \, \frac{1}{2}(W^a)^2\right] + \int d^8 z \, \Phi_i^\dagger e^{V} \Phi_i + \int d^6 z \, P(\Phi) + h.c.$$

$d^6 z \equiv d^4 x \, d\theta^2$

$d^8 z \equiv d^4 x \, d\theta^2 d\bar{\theta}^2$

is inv. under the super gauge transformation $(*)$, $(**)$.

Go to the Wess-Zumino gauge, defined by

$$V^3 = 0$$

and rescaling $V \to 2gV$ at the end,

$\Rightarrow$

$$\mathcal{L} = -\frac{1}{4}\left(F_{\mu\nu}^{(a)}\right)^2 - i\,\bar{\lambda}^{(a)}\bar{\sigma}^\mu \partial_\mu \lambda^{(a)} + \frac{1}{2}\left(D^{(a)}\right)^2 + g D^{(a)} \sum_i A_i^\dagger t^a A_i$$
$$+ \sum_i D^\mu A_i^\dagger D_\mu A_i - i \sum_i \bar{\psi}_i \bar{\sigma}^\mu D_\mu \psi_i$$
$$+ \sqrt{2}\, i g \left\{ A_i^\dagger (\lambda^a t^a) \psi_i - \bar{\psi} (\bar{\lambda}^a t^a) A_i \right\} \qquad (*)$$
$$+ \theta \frac{g^2}{32\pi^2} F_{\mu\nu}^a \tilde{F}^{a\,\mu\nu} + \sum_i F_i^\dagger F_i + P_i' F_i$$
$$- \frac{1}{2} P_{ij}'' \psi_i \psi_j + h.c.$$

Observations

(i) standard kinetic terms.

(ii) the "Yukawa" interaction term $(*)$

(Susy partner of the minimal coupling)

(iii) The scalar potential is given by

$$V = \underbrace{\frac{g^2}{2} \sum_a \left| \sum_i A_i^\dagger t^a A_i \right|^2}_{\text{"D"-term}} + \underbrace{\sum_i \left| P_i' \right|^2}_{\text{"F"-term}}.$$

(iii) bis Fayet-Illiopoulos term.

$$\delta\mathcal{L} = \gamma V\Big|_{\theta\theta\bar{\theta}\bar{\theta}} = \frac{1}{2}\gamma D \Rightarrow \underbrace{\frac{g_0^2}{2}\left(A_i^\dagger A_i + \gamma\right)^2}$$

(iv) θ - term.

(v) mass + Yukawa + scalar pot. $\in \overset{(F)}{} P(\Phi)$

(vi) $V_{scalar} \geq 0 \leftarrow$ Supersymmetry

If $V_{scalar} > 0$ Susy spontaneously broken at the tree level.

If $V_{scalar} = 0$, Susy OK classically, but may or may not be broken quantum-mechanically. (necessarily a non-perturbative effect: cannot be broken perturbatively.)

(vii) $m_\psi = m_A$ etc. If SUSY unbroken.

(viii) $U_R(1)$ symmetry

$\psi_i \to e^{i\alpha}\psi_i$; $\lambda \to e^{i\alpha}\lambda$
$A_i \to e^{2i\alpha}A_i$, $F_{\mu\nu} \to F_{\mu\nu}$ $\Bigg\} \Rightarrow$ In general broken by anomaly

or $\Phi \to e^{2i\alpha}\Phi(e^{-i\alpha}\theta)$;

$W \to e^{i\alpha}W(e^{-i\alpha}\theta)$

(ix) Susy standard model.

$$\lambda_u (\bar{\Psi}_L \phi) u_R + \lambda_d (\bar{\Psi}_L \phi^*) d_R \rightarrow \begin{array}{c}\text{Non}\\ \text{SUSY.}\end{array}$$

$$\Rightarrow \lambda_u (\bar{\Psi}_L \phi_1) u_R + \lambda_d (\bar{\Psi}_L \phi_2) d_R + \cdots$$

$\Rightarrow$ 2 Higgs doublets needed.

(x) SQCD

$$\mathcal{L} = \frac{1}{4\pi} \mathrm{Im}\left[\left(\frac{4\pi i}{g^2} + \frac{\theta}{2\pi}\right) \int \frac{1}{2} W^{a2} \, d^6 z \right]$$

$$+ \int d^6 z \left(Q_i^\dagger e^V Q_i + \tilde{Q}_i e^{-V} \tilde{Q}_i^\dagger \right) +$$

$$+ \int d^6 z \; m Q \tilde{Q} + h.c.$$

where $Q_i = (q_i, \psi_{Li}) \rightarrow q_L$ quarks

$\tilde{Q}_i = (\tilde{q}_i, \psi^c_{Ri})$ $\quad q^c_R$ quarks.

$\underline{m=0}$ $\quad G_F = SU_L(N_f) \times SU_R(N_f) \times U_V(1) \times U_R(1)$

$(\times U_A(1)) \rightarrow$ anomalous.

$\downarrow$

$U_X(1)$