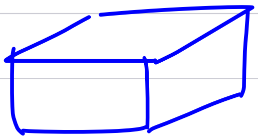
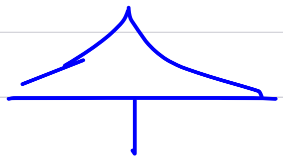
Exercitazioni 31/10/2016. Niente Ponte!

- HO in 2D / 3D e degenerazioni
- 
-  $t=0$ si toglie il pot. delta. Evoluz temporale?
- $\psi(x) = \sqrt{\kappa}\, e^{-\kappa|x|}$

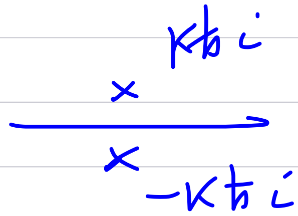
$$= \int dp\, a(p)\, e^{ipx/\hbar}$$

$$a(p) = \int \frac{dx}{2\pi\hbar} e^{-ipx/\hbar} \psi(x)$$

$$= \sqrt{\kappa}\left[\int_0^\infty \frac{dx}{2\pi\hbar} e^{-(\kappa+\frac{ip}{\hbar})x} + \int_{-\infty}^0 \frac{dx}{2\pi\hbar} e^{(\kappa-\frac{ip}{\hbar})x}\right]$$

$$= \frac{\sqrt{\kappa}}{2\pi\hbar}\left(\frac{1}{\kappa+ip/\hbar} + \frac{1}{\kappa-ip/\hbar}\right)$$

$$= \frac{\sqrt{\kappa}}{2\pi\hbar} \frac{2\kappa}{\kappa^2 + p^2/\hbar^2}$$



$$\psi(x,t) = \int dp\, a(p)\, e^{\frac{ipx}{\hbar} - i\frac{p^2 t}{2m\hbar}}$$

$$\underset{x>0}{\Longrightarrow} \frac{2\kappa\sqrt{\kappa}\,\hbar^2}{2\pi\hbar} \frac{1}{2\kappa\hbar i} 2\pi i\, e^{-\kappa x + i\frac{\kappa^2\hbar t}{2m}}$$

$$= \sqrt{\kappa}\, e^{-\kappa x + i\kappa^2\hbar t/2m}$$

Oscillatori accoppiati

$$H = \frac{P_1^2}{2m} + \frac{P_2^2}{2m} + \frac{m\omega^2}{2}(x^2+y^2) + g\,xy$$

$$\frac{x+y}{2} = X \qquad x-y = z$$

$$x = X + \frac{z}{2} \qquad p_1 = -i\hbar\frac{\partial}{\partial x} = -i\hbar\left(\frac{1}{2}\frac{\partial}{\partial X} + \frac{\partial}{\partial z}\right)$$

$$y = X - z/2 \qquad p_2 = -i\hbar\frac{\partial}{\partial y} = -i\hbar\left(\frac{1}{2}\frac{\partial}{\partial X} - \frac{\partial}{\partial z}\right)$$

term. cin =

$$-\frac{\hbar^2}{4m}\frac{\partial^2}{\partial X^2} - \frac{\hbar^2}{m}\frac{\partial^2}{\partial z^2} = \frac{P_X^2}{2M} + \frac{p^2}{2\mu}$$

$$M = m_1 + m_2 = 2m \;;\quad \mu = \frac{m_1 m_2}{m_1+m_2} = \frac{m}{2}\;;$$

mentre $\frac{m\omega^2}{2}(x^2+y^2) = \frac{m\omega^2}{2}\left(2X^2 + \frac{z^2}{2}\right) = \underbrace{\frac{M\omega^2}{2}X^2 + \frac{\mu\omega^2}{2}z^2}$

$$g\,xy = g\left(X^2 - \frac{z^2}{4}\right)$$

$$H = \frac{P^2}{2M} + \left(\frac{M\omega^2}{2} + g\right)X^2 + \frac{p^2}{2\mu} + \left(\frac{\mu\omega^2}{2} - \frac{g}{4}\right)z^2$$

$$\frac{\mu\omega^2}{2} \to \frac{m\omega^2}{4}$$

$\Rightarrow$ 2 H.O. indip. se

$$-\frac{M\omega^2}{2} < g < 2\mu\omega^2 \;; \qquad \left(\frac{\mu}{2} + \frac{\mu}{2}\right)\omega^2 \quad \mu\omega^2$$

$$-m\omega^2 < g < m\omega^2$$

$g = -\frac{M\omega^2}{2}$ 1 osc. 1 part libera $\;\to \tilde{\omega} = \sqrt{2}\,\omega$

$g = m\omega^2$, 1 osc., 1 libera

$|g| > m\omega^2 \Rightarrow$ instabile !

Cristallo 1D . 

$$L = \sum_n \left(\frac{m}{2}\dot{x}_n^2 - \frac{\kappa}{2}(x_{n+1}-x_n)^2 \right)$$

$x_N \equiv x_0$; $\quad x_{i+N} = x_i$

$\# = N$

Tras. Fourier

x

$$x_n = \frac{1}{\sqrt{N}}\sum_k A_k e^{ikna} , \quad A_k = A^*_{-k}$$

$$k = \frac{2\pi}{Na} l , \begin{cases} l = 0, \pm 1, \dots \pm \frac{N-1}{2} & (N \text{ dispari}) \\ l = 0, \pm 1, \dots \pm\left(\frac{N}{2}-1\right), \frac{N}{2} & (N \text{ pari}) \end{cases}$$

$A_0, A_{N/2} = A_{-N/2}$ real.

A_0 : traslazione totale.

gr. di libertà (N dispari) $1 + 2 \cdot \frac{N-1}{2} = N$ ok.

$$\frac{1}{N}\sum_{n=1}^{N} e^{ina(k-k')} = \frac{1}{N}\sum_{n=1}^{N} e^{2\pi i \frac{(l-l')}{N} n} = \delta_{ll'}$$ complet.

$$\frac{1}{N}\sum_k e^{ik(n-n')a} = \delta_{n,n'}$$ O.N

$$\sum_n (x_{n+1}-x_n)^2 = \sum_n \frac{1}{N}\sum_{kk'} A_k A_{k'}\, e^{ikna}\left(e^{ika}-1\right) e^{ik'na}\left(e^{ik'a}-1\right)$$

$$= \sum_k A_k A_{-k}\left(e^{ika}-1\right)\left(e^{-ika}-1\right) = 4\sum_k A_k A_{-k} \sin^2\frac{ka}{2}$$

$$\sum \frac{m}{2}\dot{x}_n^2 = \frac{m}{2}\sum_n \frac{1}{N}\sum_{kk'} \dot{A}_k \dot{A}_{k'}\, e^{ikna} e^{ik'na} = \frac{m}{2}\sum_k \dot{A}_k \dot{A}_{-k}$$

ma $A_k A_{-k} = |A_k|^2 = |a_k + i b_k|^2 = a_k^2 + b_k^2$

$\dot{A}_k \dot{A}_{-k} = \dot{a}_k^2 + \dot{b}_k^2$, $k = 1, 2 \cdots \frac{N-1}{2}$,

$$\therefore L = \frac{m}{2}\dot{A}_0^2 + \sum_{l=1}^{(N-1)/2}\left(\frac{m}{2}\dot{a}_k^2 - \frac{m\omega_k^2}{2}a_k^2\right) + a_k \leftrightarrow b_k .$$

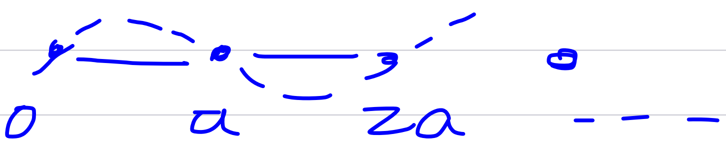
$$\omega_k^2 = \frac{4K}{m^2}\sin^2\frac{ka}{2}$$

→ N oscillatori indipendenti.
$(N-1)$

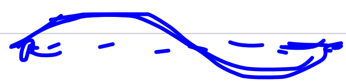
NB. $k_{max} = \frac{2\pi}{a}\frac{(N-1)/2}{N} \sim \frac{\pi}{a}$

$ikx \sim 2\pi i x/\lambda$ $\boxed{\lambda_{min} = \frac{2\pi}{k_{max}} \sim 2a}$

$k_{min} \sim \frac{2\pi}{aN} \to$



$\lambda \sim Na$



Livelli di Landau

Un elettrone (carica $-e$) si muove nel piano ($x-y$)

Campo mag $\mathbb{B} = (0, 0, B)$

Spettro:

$$H = \frac{1}{2m}\left(\vec{p} + \frac{e}{c}\mathbb{A}\right)^2$$

$$\mathbb{A} = (0, xB, 0) \qquad \mathbb{B} = \nabla \times \mathbb{A} \text{ ok.}$$

$$H = \frac{1}{2m}\left[p_x^2 + \left(p_y + \frac{e}{c}B\cdot x\right)^2\right]$$

$$[p_y, H] = 0 \Rightarrow$$

$$\Psi(x,y) = \psi_p(x)\, e^{ipy/\hbar} \leftarrow \text{autostato di } \hat{p}_y$$

$$H\Psi = E\Psi \Rightarrow$$

$$\left[\frac{1}{2m}p_x^2 + \frac{1}{2m}\left(p + \frac{eB}{c}x\right)^2\right]\psi_p(x) = E\psi_p(x)$$

$$\left[\,'' + \frac{1}{2m}\left(\frac{eB}{c}\right)^2\left(x + \frac{cp}{eB}\right)^2\right]\psi_p = E\psi_p$$

Ma questa è un osc. armonico con

$$\omega = \frac{eB}{mc} \quad !$$

(frequenza di ciclotrone Larmor)

$$\psi_p(x) = \psi^{(n)}(\tilde{x}) \qquad \tilde{x} = x + \frac{pc}{eB}$$

con $E = E_n = \omega\hbar\left(n + \frac{1}{2}\right)$ indip de p

$$\Psi_n = \psi^{(n)}(\tilde{x})\, e^{ipy/\hbar}$$

$3\omega\hbar/2$

$\omega\hbar/2$

Livelli di Landau

Oscillatore con delta

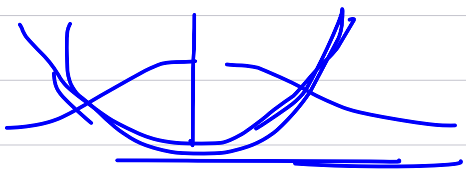
$$V(x) = \frac{1}{2} m\omega^2 x^2 + g\,\delta(x)$$

Non so calcolare tutto lo spettro
ma alcuni stati sì

$\psi_n(x)$ n dispari $\Rightarrow$

$$H\psi_n = E_n \psi_n \quad \text{ok}$$

Lo stato fondamentale ?

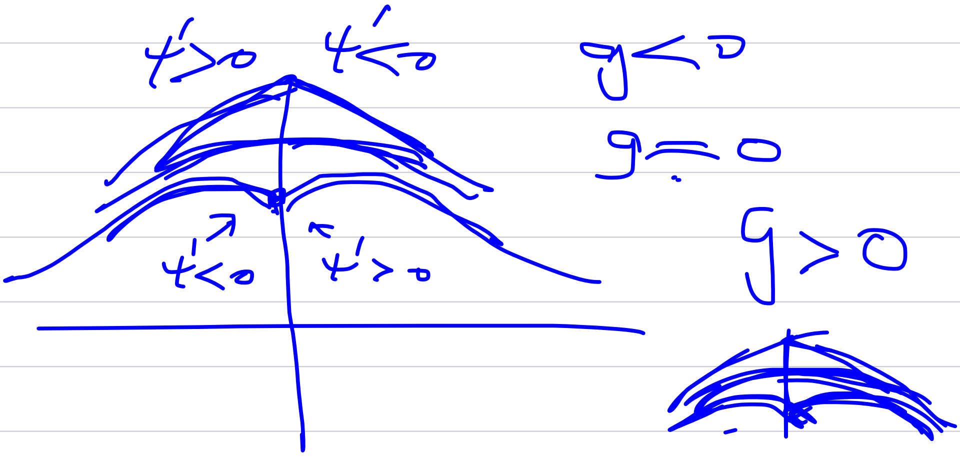


$$-\frac{\hbar^2}{2m}\psi'' + \left(g\,\delta(x) + \frac{m\omega^2}{2}x^2\right)\psi = E\psi$$

$g \geq 0$ $\qquad \boxed{E_0 \gtreqless \omega\hbar/2 \quad ?}$

$$-\frac{\hbar^2}{2m}(\psi'_+ - \psi'_-) + g\,\psi_0 = 0$$

$$\psi'_+ - \psi'_- = \frac{2mg}{\hbar^2}\psi_0 > 0$$



$$-\frac{\hbar^2}{2m}\psi'' + \left(g\delta + \frac{1}{2}m\omega^2 x^2\right)\psi = E\psi$$

$$\psi_0(x) = e^{-\frac{m\omega}{2\hbar}x^2}$$

$$\psi' = -\frac{m\omega}{\hbar}x\, e^{-\frac{m\omega}{2\hbar}x^2}$$

$$\psi''\big|_0 = -\frac{m\omega}{\hbar}$$

$$-\frac{\hbar^2}{2m}\psi''_0 = \frac{\omega\hbar}{2}$$ ok

$$(\psi'_+ - \psi'_-) = \frac{2mg}{\hbar^2} \gtrless 0$$

$$\psi'_+ = -\psi'_-$$

$$\psi_+ = \psi(\epsilon) \qquad \psi_- = \psi(-\epsilon)$$

$$= \psi(\epsilon)$$