

A THEOREM ON CYCLIC HARMONIC OSCILLATORS

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Abstract:

It is proven that the energy of a quantum mechanical harmonic oscillator with a generically time-dependent but cyclic frequency, $\omega(t_0) = \omega(0)$, cannot decrease on an average if the system is originally in a stationary state, after the system goes through a full cycle. The energy exchange always takes place in the direction from the macroscopic system (environment) to the quantum microscopic system.

1. The Theorem

Many physical systems reduce effectively to that of a harmonic oscillator with a time-dependent frequency (we set $m = 1$),

$$H(t) = \frac{p^2}{2} + \frac{1}{2} \omega(t)^2 q^2, \quad (1.1)$$

which is thus of certain interest [1]. In particular, the case of a periodic (cyclic) variation of ω such that

$$\omega(t_0) = \omega(0) \quad (1.2)$$

is of great interest. We shall not specify the time variation of $\omega(t)$ otherwise. Suppose furthermore that the system is initially in one of the stationary states, $|\psi\rangle$, with energy E_{in} . We prove below, independently of how the system goes through the cycle, that

$$E_{fin} = \bar{E}(t_0) = \langle \psi(t_0) | H | \psi(t_0) \rangle \geq E_{in}, \quad (1.3)$$

i.e. on an average the microscopic system can never give excess energy to the external environment. Such a result might appear surprising at first sight, since during a generic time variation of external parameters, the system can either give the energy away to or absorb it from the environment. Also, such a theorem certainly does not hold in general in a system with a finite number of independent states, such as a spin $\frac{1}{2}$ system in a varying magnetic field.

The proof is easiest in the Heisenberg picture. Heisenberg equations of motion tell us that $q(t)$, $p(t)$ are linear combinations of q , p ; they furthermore preserve the commutation relation (we set $\hbar = 1$)

$$[q(t), p(t)] = i. \quad (1.4)$$

In other words the time evolution is described by an $Sp(2)$ transformation

$$y(t)_\alpha = S_{\alpha\beta}(t) y(0)_\beta, \quad S_{\alpha\beta} S_{\gamma\delta} \varepsilon_{\beta\delta} = \varepsilon_{\alpha\gamma} \quad S \varepsilon S^T = \varepsilon, \quad (1.5)$$

where we wrote $y_\alpha = (q, p)$. Or writing

$$S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a, b, c, d \text{ real}$$

the condition (1.5) reduces to

$$ad - bc = \det(S) = 1. \quad (1.6)$$

The energy expectation value at time t is given by

$$\bar{E}(t) = \langle \psi | H(t) | \psi \rangle = \frac{1}{2} \langle \psi | p(t)^2 + \omega(t)^2 q(t)^2 | \psi \rangle. \quad (1.7)$$

At the end of a cycle (we set $\omega(0) = 1$),

$$E_{fin} = \frac{1}{2} \langle \psi | p_H(t_0)^2 + q_H(t_0)^2 | \psi \rangle. \quad (1.8)$$

By using Eq. (1.5) and by defining

$$D = \frac{1}{2}(qp + pq)$$

the quadratic form becomes

$$\begin{aligned} (S_{11}q + S_{12}p)^2 + (S_{21}q + S_{22}p)^2 &= \\ &= (S_{11}^2 + S_{21}^2)q^2 + (S_{12}^2 + S_{22}^2)p^2 + 2(S_{11}S_{12} + S_{21}S_{22})D. \end{aligned}$$

By using the Virial theorem

$$\frac{1}{2} \langle \psi | p^2 | \psi \rangle = \frac{1}{2} \langle \psi | q^2 | \psi \rangle = \frac{1}{2} E_{in} \quad (1.9)$$

and the fact (valid for real wave functions) that

$$\langle \psi | D | \psi \rangle = 0 \quad (1.10)$$

the final energy is given by

$$E_{fin} = \frac{1}{2} E_{in} [S_{11}^2 + S_{21}^2 + S_{12}^2 + S_{22}^2]. \quad (1.11)$$

The problem is then to find the minimum of a quadratic form $Q = a^2 + b^2 + c^2 + d^2$ under a constraint $ad - bc = 1$. Upon introduction of a Lagrange multiplier, the extremum of $a^2 + b^2 + c^2 + d^2 + 2\lambda(ad - bc - 1)$ is given by ¹

$$\lambda = -1, \quad a = d, \quad b = -c, \quad S = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}, \quad a^2 + b^2 = 1,$$

that is, when S is orthogonal. In that case the quadratic form takes the value 2 and therefore in general

$$E_{fin} = \frac{1}{2} E_{in} Q \geq E_{in}. \quad (1.12)$$

¹As the set of the evolution matrices is unbounded, this extremum can only be a minimum. In an alternative proof given in (C.2) this fact is evident.

Remarks

Let us make some remarks and discuss to which extent our theorem can be generalized.

In the adiabatic limit, the system “follows” the variation of the spectrum while staying in the “initial” eigenstate, and comes back to the original state, so we expect $E_f \rightarrow E_{in}$. In the sudden limit, the wave function does not make it to change as the external parameter goes through a (too) quick cyclic variation, so that $E_f \rightarrow E_{in}$ again. These two extreme cases are obviously included as special cases and do satisfy our theorem, but the point is that the latter holds for any periodic time variation of the external parameters (mass, frequencies, etc).

Our result is valid also in the case of a forced oscillator. Consider

$$H = \frac{1}{2}(p^2 + \omega(t)^2 q^2) - \kappa(t) q, \quad (1.13)$$

where $\kappa(t)$ is an arbitrary function with $\kappa(0) = \kappa(T) = 0$. The Heisenberg equations are

$$\dot{p} = i[H, p] = -\omega^2 q + \kappa, \quad \dot{q} = i[H, q] = p \quad (1.14)$$

so

$$\ddot{q} = -\omega^2 q + \dot{\kappa}. \quad (1.15)$$

Let us now consider a solution $Q_c(t)$ of (1.15) with the boundary condition

$$Q_c(0) = \dot{Q}_c(0) = 0;$$

it follows immediately that q is a sum of the homogeneous solution plus the particular one Q_c

$$q_H(t) = Q_c(t) + q_H^{(\kappa=0)}(t), \quad p_H(t) = \dot{Q}_c(t) + p_H^{(\kappa=0)}(t). \quad (1.16)$$

One has then

$$E_f = E_f^{(\kappa=0)} + \frac{1}{2} \left(Q^2 + \dot{Q}^2 \right) + (Q_c \langle q_H^{(\kappa=0)}(t) \rangle + \dot{Q}_c \langle p_H^{(\kappa=0)}(t) \rangle).$$

As the expectation values of q_H and p_H vanish in the initial stationary state it follows that

$$E_f = E_f^{(\kappa=0)} + \frac{1}{2} \left(Q^2 + \dot{Q}^2 \right) \geq E_i, \quad (1.17)$$

where the result of the preceding paragraphs has been used.

In perturbation theory the theorem can be easily seen to be valid. By writing $\omega(t)^2 x^2 = \omega_0^2 x^2 + \delta\omega(t) x^2$, $\delta\omega(t_0) = \delta\omega(0) = 0$, the first-order transition probability

$$P_{fi} = \frac{1}{\hbar^2} \left| \int_0^{t_0} dt \frac{\delta\omega(t)}{2} e^{i\omega_{fi}t} (x^2)_{fi} \right|^2 \quad (1.18)$$

is larger for the process $n \rightarrow n + 2$ than for $n \rightarrow n - 2$, hence $\langle H \rangle \geq E_{in}$.

In perturbation theory, one can actually verify the theorem for a more general class of perturbing potentials. Indeed, it can be easily seen that for any perturbation of the form,

$$\Delta V(t, x) = \delta\omega(t) x^N, \quad \delta\omega(t_0) = \delta\omega(0) = 0, \quad (1.19)$$

the theorem is valid, as $|(x^N)_{n+m,n}| \geq |(x^N)_{n-m,n}|$ ($m > 0$). This would suggest that actually the theorem holds for a wider class of periodic potentials, $V(t_0, x) = V(0, x)$ (*i.e.*, not only for harmonic oscillators).

The theorem does not hold for a generic initial pure state. Since in some cases the final energy expectation value (*i.e.*, in the pure state $U(t_0) |\psi(0)\rangle$) is strictly higher than the initial value (Sec. 2.), it suffices to consider the time-reversed process of such an evolution (by time-reversing the external variations also), to find a counter example to the theorem ². On the other hand, for an initial state which is not an eigenstate of energy, the operational meaning of the theorem itself would become somewhat unclear.

This fact does not however mean that the theorem is inapplicable for *any* pure initial state which is not an energy eigenstate: our assumption (an initial stationary state) is sufficient but not necessary. In fact, we have found another sufficient condition: for initial states of the form $\psi = \sum_n a_n \psi_n$, $n = n_0 + 4m$, $m \in \mathbb{Z}$, with a fixed n_0 , the theorem can be shown to hold by a slight generalization of the proof given here.

A somewhat related issue concerns a system which goes through many cycles. Our theorem holds in this case as a multiple cyclic variation *is* a cyclic one. On the other hand, if $\bar{E}_n = \langle H \rangle_n$ denotes the energy expectation value computed after the system has done n cycles, no claim is being made on the relative magnitudes among \bar{E}_n 's, *e.g.*, $\bar{E}_1 \leq \bar{E}_2 \leq \dots$, which certainly is false in general.

Nevertheless, a cumulative effect does take place: the maximum of the average energy gain factor, $R \equiv \langle \psi | H | \psi \rangle / E_{in}$, which occurs at appropriately tuned variation

²We thank Tomas Tyc for a useful communication on this point.

parameters, becomes exponentially large when the system undergoes many cycles, as can be verified by studying explicit examples (see the point (iii) in Section 2). This might suggest a possibility for some physical application of our theorem.

The energy gain factor R is universal, in the sense that it does not depend on the particular initial stationary state chosen (it depends of course on the parameters of external variations). This may have to do with the peculiarity of harmonic oscillators.

Note that the (classical) canonical equations of motion and Heisenberg equations have the same form, and the evolution matrix S are the same in both cases. As a result, an analogous theorem holds in classical mechanics, if one takes an average over random initial values (p, q) over a given (fixed-energy) trajectory.

Finally, the theorem (obviously) applies to a system of N independent oscillators

$$H = \sum_{i=1}^N \left[\frac{p_i^2}{2} + \frac{1}{2} \omega_i(t)^2 q_i^2 \right], \quad (1.20)$$

with arbitrary, periodic variations of ω , $\omega_i(t_0) = \omega_i(0)$, if the initial state is in a stationary state $|n_1, n_2, \dots, n_N\rangle$.

There is in fact an important case in which the theorem applies for a initial mixed state. Consider the N oscillator system of (1.20) and suppose that the system is originally in the thermal equilibrium with a heat bath at temperature T . The theorem then clearly applies in a statistical and quantum average sense. (Section 3.)

For an interacting N oscillator system, instead, the generalization is not obvious. We content ourselves with a slightly weaker theorem here: the number of the phonons in the system can be shown not to decrease on an average (Appendix C).

2. Example: Inverse Linear Variation of the Frequency

Consider the oscillator Eq.(1.1) with frequency varying as

$$\omega(t) = \frac{\omega_0}{\lambda(t)}, \quad \lambda(t) = 1 + vt, \quad (2.1)$$

where ω_0 and v are constants. From Heisenberg equations one gets

$$\frac{d^2 q}{d\lambda^2} + \Omega^2 \frac{q}{\lambda^2} = 0, \quad \Omega = \frac{\omega_0}{v}.$$

By setting $q(t) = \lambda(t)^\beta$, one gets a characteristic equation

$$\beta(\beta - 1) + \Omega^2 = 0,$$

with solutions

$$\beta_{1,2} = \frac{1}{2} \pm \delta, \quad \delta \equiv \sqrt{\frac{1}{4} - \Omega^2}, \quad (2.2)$$

so that the general solution of the Heisenberg equations reads

$$q_H = c_1 \lambda^{\beta_1} + c_2 \lambda^{\beta_2}, \quad p_H = v (c_1 \beta_1 \lambda^{\beta_1} + c_2 \beta_2 \lambda^{\beta_2}). \quad (2.3)$$

By imposing the initial condition one finds

$$\begin{aligned} q_H(t) &= \frac{1}{2\delta} \left\{ (\beta_2 \lambda^{\beta_1} - \beta_1 \lambda^{\beta_2}) q + \frac{1}{v} (\lambda^{\beta_2} - \lambda^{\beta_1}) p \right\}, \\ p_H(t) &= \frac{v}{2\delta\lambda} \left\{ \beta_1 \beta_2 (\lambda^{\beta_2} - \lambda^{\beta_1}) q + \frac{1}{v} (\beta_2 \lambda^{\beta_2} - \beta_1 \lambda^{\beta_1}) p \right\}. \end{aligned} \quad (2.4)$$

As a check, consider the adiabatic limit, $v \rightarrow 0$, $\Omega \rightarrow \infty$. One has $\beta_{1,2} \simeq \pm i\omega_0/v$, so that

$$\lambda^{\beta_{1,2}} \rightarrow (1 + vt)^{\mp i\omega_0/v} \rightarrow e^{\mp i\omega_0 t},$$

and

$$q_H(t) \rightarrow \frac{q}{2} (e^{-i\omega_0 t} + e^{i\omega_0 t}) + \frac{p}{2i\omega_0} (-e^{-i\omega_0 t} + e^{i\omega_0 t}) = q \cos \omega_0 t + \frac{p}{\omega_0} \sin \omega_0 t$$

which is the correct result.

Writing Eqs.(2.4) in the form of Eq.(1.5) with $S = S(\omega_0, v, \lambda)$ one gets, by inserting this in Eq.(1.7) and by using the Virial theorem,

$$\begin{aligned} \langle \psi | H | \psi \rangle &= \frac{1}{2} E_0 \left[\frac{1}{\lambda^2} (S_{11}^2 + S_{12}^2) + S_{21}^2 + S_{22}^2 \right] = \\ &= \frac{E_0}{2} \frac{\lambda^{2\delta} + \lambda^{-2\delta} + 2(4\delta^2 - 1)}{4\lambda\delta^2}. \end{aligned} \quad (2.5)$$

At time t , $\lambda = 1 + vt$, the energy mean value (2.5) can be larger or smaller than the original energy depending on the sign of the velocity v (hence whether the scale factor λ is smaller or larger than unity).

However, as we are most interested in cyclic variations of ω , let us consider the evolution from the original frequency ω_0 to a final frequency $\frac{\omega_0}{\lambda}$, and then back

to ω_0 . The second half of the evolution is simply described by the transformation $S = S(\frac{\omega_0}{\lambda}, -v, \frac{1}{\lambda})$ so that the total evolution is described by the Heisenberg evolution

$$\begin{pmatrix} q_H(t_0) \\ p_H(t_0) \end{pmatrix} = S^{cyc} \begin{pmatrix} q_H(0) \\ p_H(0) \end{pmatrix}, \quad S^{cyc} \equiv S(\frac{\omega_0}{\lambda}, -v, \frac{1}{\lambda}) \cdot S(\omega_0, v, \lambda) \quad (2.6)$$

where S is defined by Eq.(2.4).

The results for more general frequency variations are given in Appendix A and in Appendix B.

In the case of the linear variation (2.5), (2.6), we have analysed numerically the energy gain factor $R(\omega_0, v, \lambda) = \langle \psi | H | \psi \rangle / E_0 = \frac{1}{2} \text{Tr} [S^{cyc} (S^{cyc})^T]$ for various values of ω_0 , v and λ , λ being the scale factor ($\omega = \omega_0/\lambda$) at the moment of the maximal contraction ($\lambda < 1$) or expansion ($\lambda > 1$). We find $R(\omega_0, v, \lambda) \geq 1$ always as expected, but find also that:

- (i) At fixed λ , $R(\omega_0, v, \lambda) \rightarrow 1$ both in the adiabatic ($v \rightarrow 0$) and impulse approximation ($v \rightarrow \infty$) limits, as expected;
- (ii) There are various resonance effects at small v (see Fig. 1); in particular, as a function of the velocity v at a fixed λ , R reaches a maximum of order of $O(1/\lambda)$ (for $\lambda < 1$) or $O(\lambda)$ (for $\lambda > 1$), and then rather smoothly approaches the impulse-approximation value 1 asymptotically;
- (iii) When the system goes through N cycles, the maximal energy gain factor (which occurs at certain critical velocity) behaves as $R \propto (1/\lambda)^N$ or $R \propto \lambda^N$, a huge factor if λ is large;
- (iv) There are values of (v, λ) at which R attains values either exactly equal to or very close to unity (Fig. 1).
- (v) As a function of λ , the maximum of R grows indefinitely as $\lambda \rightarrow 0$ or as $\lambda \rightarrow \infty$.

3. Planck Distribution inside an Oscillating Cavity

As a possible physical application of our considerations, let us consider the black body spectrum confined in a perfectly reflecting three-dimensional box of linear dimension L . The electromagnetic field energy is described by the Hamiltonian per

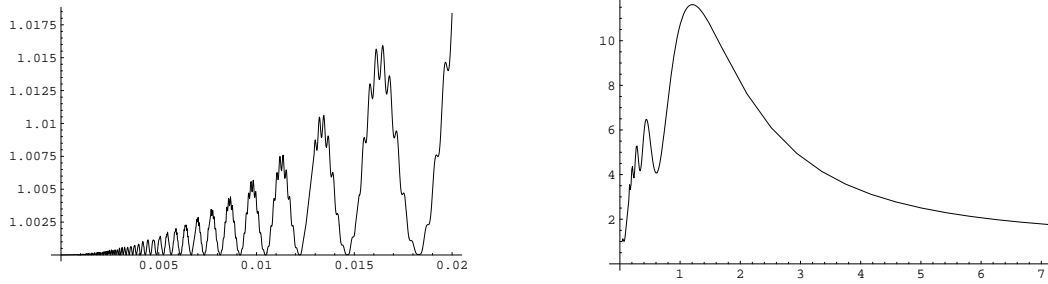


Figure 1: Energy gain factor as a function of v for fixed $\lambda(= 10)$, at small and large v .

unit volume

$$H = \sum_{\mathbf{k}} \left(\frac{c^2}{4} \mathbf{p}_{(1)}^2 + \mathbf{k}^2 \mathbf{q}_{(1)}^2 \right) + \sum_{\mathbf{k}} \left(\frac{c^2}{4} \mathbf{p}_{(2)}^2 + \mathbf{k}^2 \mathbf{q}_{(2)}^2 \right),$$

where $\mathbf{k}_{\mathbf{n}} \equiv \pi \mathbf{n} / L$. At temperature T the energy distribution is

$$u(\nu) d\nu = \frac{8\pi}{c^3} \frac{h\nu}{e^{h\nu/kT} - 1} \nu^2 d\nu. \quad (3.1)$$

Suppose that at certain moment the box starts to contract or expand with a constant linear velocity. How does the energy distribution change with the linear size of the box?

At time $t = 0$ various modes are distributed according to the Planck distribution, (3.1). Each mode simply transforms as in Eq.(2.4): ($|\delta| \gg 1$, $\delta = i|\delta|$),

$$E_n \rightarrow \langle H \rangle = \frac{E_n}{2} \frac{\lambda^{2\delta} + \lambda^{-2\delta} + 2(4\delta^2 - 1)}{4\lambda\delta^2} \simeq \frac{E_n}{\lambda} \quad (3.2)$$

Since

$$\omega = \frac{\pi |\mathbf{n}| c}{L_0}, \quad L(t) = L_0(1 + vt), \quad |V| = L_0 |v| \ll c, \\ \Omega = \frac{\omega}{v} = \frac{\pi |\mathbf{n}| c}{|V|} \gg 1, \quad \delta = \sqrt{\frac{1}{4} - \Omega^2} \simeq i \frac{\pi |\mathbf{n}| c}{|V|} \quad (3.3)$$

the process is adiabatic for all modes. The distribution remains Planckian; each mode is either red-shifted (in an expanding box) or shifted towards ultraviolet (a contracting box), simply by the scale factor λ .

In fact, the system is not so interesting as an application of the general theorem mentioned in Section 1, as it is entirely adiabatic; the only reason we discuss it here is its possible relevance to the so-called sonoluminescence phenomenon.

In the single-bubble sonoluminescence [2], a gas bubble trapped in a liquid is made oscillating radially by acoustic waves, and at the moments its radius attains the minimum the bubble emits pulse of light in the visible to ultraviolet wavelength range. In a typical experiment, the linear dimension of the bubble contracts up to a factor of $\lambda = 10^{-4}$: this is compatible with the observed increase of an effective temperature of a factor 10^4 or more. The energy density increases according to the Stefan's law,

$$U = \sigma T^4; \quad \sigma = 7.64 \cdot 10^{-15} \text{ erg cm}^{-3} \text{ K}^{-4} \quad (3.4)$$

but since the volume itself decreases by a factor λ^3 the total energy of blackbody radiation increases only by a factor $\frac{1}{\lambda}$. It is possible that this excess energy is released in the form of visible light pulse at each cycle ³.

4. Conclusion

A harmonic oscillator with a time-dependent frequency (or an ensemble of such oscillators) experiences a full cycle, with the system originally in a stationary state. After the cycle, the energy expectation value is predicted never to decrease, independently of the way the parameters vary with time during the cycle. The theorem deals with an apparently simple system familiar to everybody; nevertheless it involves various subtle issues as discussed here, and to the best of our knowledge, has not been discussed before, at least in this form.

Our result is somewhat reminiscent of the law of entropy increase, but concerns the evolution of quantum mechanical pure states, and no information loss is implied. A generic non-adiabatic “disturbance” by the external force always does work on the microscopic system on an average, increasing its energy. The energy flow is always in the direction from the macroscopic system to the microscopic one. In other words, the quantum mechanical harmonic oscillator cannot act as a perpetual machine, nor produce a net energy gain. A spaceship cannot continue her journey forever, getting its energy supply from the inexhaustible zero-point energy of the vacuum [3].

³In an experiment at Lawrence Livermore National Laboratory [2] $10^5 - 10^6$ photons are observed within a pulse, with the energy of order of 10^{-7} erg which seems to be compatible with such a rough estimate. Note also that the visible lights correspond to the effective temperature of $T = 10^4 - 10^5$ or to the energy for a single photon of order of $h\nu \sim kT \sim 10^{-12}$ erg.

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The authors are indebted to David M. Brink for many enlightening discussions (in particular on the applicability of the theorem in perturbation theory and in classical mechanics), and for sharing their enthusiasm. Thanks are also due to Francesco Maccarrone for discussions on the sonoluminescence. Numerical analyses have been done by use of Mathematica (Wolfram Research) and MatLab (MathWorks).

Appendix A General Power Dependent Frequencies

Consider a case of a generic power-behaved frequency,

$$H(t) = \frac{1}{2m}p^2 + \frac{1}{2}m\omega_0^2 z^{k-2}x^2, \quad z = 1 + vt. \quad (\text{A.1})$$

For $k = 0$ we recover the case discussed in the text. In the following, we shall set $m = 1, \omega_0 = 1$. Note that v has the dimension of a frequency, so that to recover the dependence on ω_0 it suffices to replace it by v/ω_0 . The Heisenberg equation of motion gives

$$\ddot{q}_H + z^{k-2}q_H = 0.$$

Multiplying this by $1/v^2$ one gets

$$\frac{d^2}{dz^2}q_H + \frac{1}{v^2}z^{k-2}q_H = 0. \quad (\text{A.2})$$

Its general solution is of the form,

$$q_H(t) = A\sqrt{z}J_{1/k}\left(\frac{2}{kv}z^{\frac{k}{2}}\right) + B\sqrt{z}Y_{1/k}\left(\frac{2}{kv}z^{\frac{k}{2}}\right) \quad (\text{A.3})$$

where J, Y are the Bessel functions of the first and second kind, respectively. Differentiation with respect to time yields

$$p_H(t) = v\frac{d}{dz}q_H(t).$$

The coefficients A and B are determined by imposing the initial conditions

$$q_H|_{z=1} = q, \quad p_H|_{z=1} = p :$$

$$\begin{aligned} A &= -\frac{\pi}{kv} \left[q \cdot \left(Y_{1+\frac{1}{k}}\left(\frac{2}{kv}\right) + (p - vq) \cdot Y_{\frac{1}{k}}\left(\frac{2}{kv}\right) \right); \\ B &= \frac{\pi}{kv} \left[q \cdot \left(J_{1+\frac{1}{k}}\left(\frac{2}{kv}\right) + (p - vq) \cdot J_{\frac{1}{k}}\left(\frac{2}{kv}\right) \right) \right]. \end{aligned} \quad (\text{A.4})$$

The expression for

$$E(t) = \langle 0 | \frac{1}{2} p_H(t)^2 + \frac{1}{2} z^{k-2} q_H(t)^2 | 0 \rangle. \quad (\text{A.5})$$

is quite complicated but can be analyzed numerically. We note that:

- 1) The case $k = -2$ reduces to the known result;
- 2) The behavior is qualitatively similar for all k , with a phase of monotonous increase following the initial, oscillating phase, as a function of v (see Fig. 2);
- 3) At fixed final “physical scale”, $z_{fin} = \lambda^{-\frac{2}{k-2}}$, $z_{fin}^{k-2} = \lambda^{-2}$, the asymptotic behavior in v turns out to be always the same as in the case $k = -2$, *i.e.*,

$$E_f \simeq \frac{\omega}{4} \left(1 + \frac{1}{\lambda^2}\right). \quad (\text{A.6})$$

This assertion is based on a graphical evidence (see Fig. 2), for the moment ⁴.

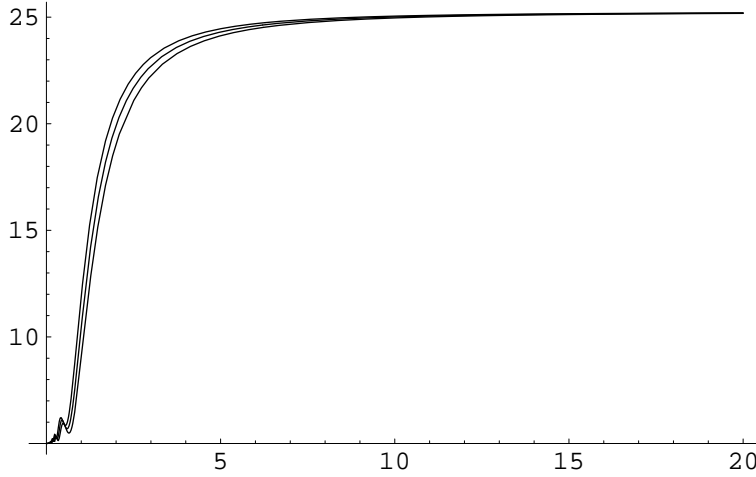


Figure 2: Energy as a function of v for a fixed $\lambda = 1/10$, with $k = -2, -3, -4$. Expected asymptotic value is $1/4(1 + 100) = 25.25$.

Appendix B Exponential Dependence

As a second example, let us consider the Hamiltonian,

$$H = \frac{1}{2} p^2 + \frac{1}{2} e^{2vt} q^2. \quad (\text{B.1})$$

⁴For generic order, the functions J, Y have essential singularities at $v = \infty$, which prevent us from analyzing either adiabatic or large v limit analytically.

The Heisenberg equation is

$$\ddot{q}_H + e^{2vt} q_H = 0.$$

Changing the variable to $\tau = vt$ one has

$$\frac{d}{d\tau^2} q_H + \frac{1}{v^2} e^{2\tau} q_H = 0$$

which has

$$\begin{aligned} q_H(t) &= A J_0\left(\frac{1}{v}e^{vt}\right) + B Y_0\left(\frac{1}{v}e^{vt}\right) \\ p_H(t) &= \frac{dq_H}{dt} = -A e^{vt} J_1\left(\frac{1}{v}e^{vt}\right) - B e^{vt} Y_1\left(\frac{1}{v}e^{vt}\right) \end{aligned} \quad (\text{B.2})$$

as general solutions. As before the coefficients are determined by the initial condition:

$$A = -\frac{\pi}{2v} \left[p Y_0\left(\frac{1}{v}\right) + q Y_1\left(\frac{1}{v}\right) \right], \quad B = \frac{\pi}{2v} \left[p J_0\left(\frac{1}{v}\right) + q J_1\left(\frac{1}{v}\right) \right].$$

The mean energy can be computed as in the preceding cases.

$$\begin{aligned} E(t) &= \frac{\pi^2 z^2}{16\pi^2} \left\{ -2J_0\left(\frac{z}{v}\right)Y_0\left(\frac{z}{v}\right) \left[J_0\left(\frac{1}{v}\right)Y_0\left(\frac{1}{v}\right) + J_1\left(\frac{1}{v}\right)Y_1\left(\frac{1}{v}\right) \right] \right. \\ &\quad + J_0^2\left(\frac{z}{v}\right) \left[Y_0^2\left(\frac{1}{v}\right) + Y_1^2\left(\frac{1}{v}\right) \right] + J_1^2\left(\frac{z}{v}\right) \left[Y_0^2\left(\frac{1}{v}\right) + Y_1^2\left(\frac{1}{v}\right) \right] \\ &\quad + 2J_1\left(\frac{z}{v}\right)Y_1\left(\frac{z}{v}\right) \left[J_0\left(\frac{1}{v}\right)Y_0\left(\frac{1}{v}\right) + J_1\left(\frac{1}{v}\right)Y_1\left(\frac{1}{v}\right) \right] + \\ &\quad \left. + \left[J_0^2\left(\frac{1}{v}\right) + J_1^2\left(\frac{1}{v}\right) \right] \left[Y_0^2\left(\frac{z}{v}\right) + Y_1^2\left(\frac{z}{v}\right) \right] \right\} \end{aligned} \quad (\text{B.3})$$

where $z = e^{vt}$. It is also possible to get the asymptotic behavior in v at fixed z :

$$E_{as} = \frac{1}{4}(1 + z^2) - \frac{1}{16v^2}(z^2 - 1)^2, \quad (\text{B.4})$$

which is compatible with (A.6) as $z = 1/\lambda$.

Appendix C Creation and Annihilation Operators

The whole problem can be analyzed by use of creation and annihilation operators. We introduce at each instance the variables $q_i(t)$, $p_i(t)$ in which the frequency is diagonal, $\omega_{ij}(t) = \delta_{ij}\omega_i(t)$; then define $a_i(t)$, $a_i^\dagger(t)$ in the standard manner in terms of

$q_i(t)$, $p_i(t)$. The time evolution introduces a linear transformation among $a_i(t)$, $a_i^\dagger(t)$, which has a general form,

$$a_i^\dagger \rightarrow A_{ik} a_k^\dagger + B_{ik} a_k, \quad a_i \rightarrow A_{ik}^* a_k + B_{ik}^* a_k^\dagger.$$

The coefficients must be such that the canonical commutation relations are preserved (in a matrix form):

$$AA^\dagger - BB^\dagger = 1, AB^T - BA^T = 0. \quad (\text{C.1})$$

For a single oscillator, the theorem of Section 1. can be immediately proven:

$$\begin{aligned} E_f &= \frac{\omega}{2} + \omega \langle n | (a')^\dagger a' | n \rangle = \frac{\omega}{2} + \omega \langle n | (|A|^2 + |B|^2) a^\dagger a | n \rangle = \\ &= \frac{\omega}{2} + n(1 + 2|B|^2)\omega \geq \omega \left(n + \frac{1}{2}\right) = E_{in}. \end{aligned} \quad (\text{C.2})$$

For N oscillators which are originally in a stationary state (ignoring the zero point energy),

$$\begin{aligned} E_i &= \langle \Psi | \sum_i \omega_i a_i^\dagger a_i | \Psi \rangle = \sum_i \omega_i n_i; \\ E_f &= \langle \Psi | \sum_i \omega_i a_i'^\dagger a_i' | \Psi \rangle = \langle \Psi | \sum_i \omega_i (A_{ik} a_k^\dagger + B_{ik} a_k) (A_{i\ell}^* a_\ell + B_{i\ell}^* a_\ell^\dagger) | \Psi \rangle = \\ &= \sum_{i,k} \omega_i (A_{ik} A_{ik}^* n_k + B_{ik} B_{ik}^* (n_k + 1)). \end{aligned}$$

For the diagonal matrices A e B (uncoupled oscillators), (C.2) trivially generalizes. For coupled oscillators, the generalization is not obvious.

We content ourselves here with a somewhat weaker result: it states that the occupation number (number of phonons) does not diminish under cyclic variations of frequency. Indeed,

$$\begin{aligned} N_i &= \langle \Psi | \sum_i a_i^\dagger a_i | \Psi \rangle = \sum_i n_i; \\ N_f &= \langle \Psi | \sum_i a_i'^\dagger a_i' | \Psi \rangle = \langle \Psi | \sum_i (A_{ik} a_k^\dagger + B_{ik} a_k) (A_{i\ell}^* a_\ell + B_{i\ell}^* a_\ell^\dagger) | \Psi \rangle = \\ &= \sum_{i,k} (A_{ik} A_{ik}^* n_k + B_{ik} B_{ik}^* (n_k + 1)). \end{aligned}$$

Eq.(C.1) implies

$$\sum_i A_{ik} A_{ik}^* = \sum_i B_{ik} B_{ik}^* + 1,$$

therefore

$$N_f = \sum_k n_k + \sum_{i,k} B_{ik} B_{ik}^* (2n_k + 1) \geq N_i. \quad (\text{C.3})$$

This (more general) result perhaps explains the physical meaning of our result on the energy expectation value.

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