EQUAZIONE DI SCHRÖDINGER

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- $H\psi = E\psi \quad (H(q,p;\cancel{t}))$
- $i\hbar \frac{\partial}{\partial t}\psi = H\psi$

(i) Caso libero

$$H = \frac{p^2}{2m} = -\frac{\hbar^2}{2m}\nabla^2$$

$[p, H] = 0 \Rightarrow$ autostati comuni di $\{H, p\}$

$$\Rightarrow \hat{p}\psi = p\psi$$

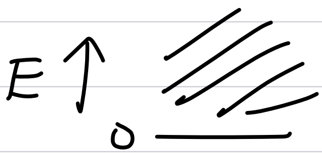
$$\psi = c \cdot e^{i p\cdot r/\hbar}$$

$$-\infty < p_i < \infty$$

$$H\cdot\left(e^{i p\cdot r/\hbar}\right) = E_p \left(e^{i p\cdot r/\hbar}\right)$$

con $E_p = p^2/2m$

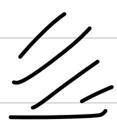
$$\boxed{E_p \geq 0}$$



1D

$$\psi_E = e^{ipx/\hbar}$$

$H\psi = E\psi$, $\quad E = p^2/2m$



Spettro: $E \geq 0$

Ogni livello è doppiamente degenere $p = \pm\sqrt{2mE}$

($\psi_{E=0}$ è singolo)

(ii) Particella in potenziale

$$H = \frac{\mathbf{p}^2}{2m} + V(\mathbf{r})$$

$$H\psi = E\psi$$

$$\left.\begin{aligned} -\frac{\hbar^2}{2m}\nabla^2\psi_n + V(\mathbf{r})\psi_n &= E_n\psi \\ \|\psi_n\| &= 1 \end{aligned}\right\}$$ Prob. agli autovalori

Soluzioni in casi semplici (dopo)

(iii) Sistemi "a due stati" (qubit)
polarizzazione del fotone

$|\psi\rangle \in |1\rangle, |2\rangle$

$|\psi\rangle = c_1|1\rangle + c_2|2\rangle$

$$H_0 = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} \Rightarrow \begin{aligned} H_0|1\rangle &= E_1|1\rangle \\ H_0|2\rangle &= E_2|2\rangle \end{aligned}$$

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad |2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$A = a\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad A^\dagger = A$$

Misura di $\underset{(t=0)}{A}$? $\Longrightarrow$ $\pm a$ (autovalori)

$A = a \rightarrow |a\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix};$

$ -a \rightarrow |-a\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Evoluzione temporale:

$$i\hbar \frac{\partial \psi}{\partial t} = H\psi$$

Un autostato di H, $\quad H\psi_n = E_n \psi_n$

$$i\hbar \frac{\partial \psi_n}{\partial t} = H\psi_n = E_n \psi_n \quad \therefore \quad \boxed{\psi_n(t) = e^{-i\frac{E_n t}{\hbar}} \psi_n(x)}$$

uno stato stazionario.

Uno stato generico $\quad H\phi_n = E_n \phi_n$

$$|\psi(0) = \sum_n c_n \phi_n$$

$$\Rightarrow \boxed{\psi(t) = \sum_n c_n \phi_n(t) = \sum_n c_n e^{-i\frac{E_n t}{\hbar}} \phi_n} \quad (*)$$

$\because$ $(*)$ soddisfa all'eq

$$i\hbar \frac{\partial \psi(t)}{\partial t} = \sum_n c_n E_n e^{-i\frac{E_n t}{\hbar}} \phi_n \quad (*)$$

$$H\psi(t) = \sum_n c_n H e^{-i\frac{E_n t}{\hbar}} \phi_n$$

$$= \sum_n c_n e^{-iE_n t/\hbar} H\phi_n$$

$$= \sum_n c_n e^{-iE_n t/\hbar} E_n \phi_n \quad (**)$$

$(*) = (**)$ $\therefore$ $i\hbar \frac{\partial \psi(t)}{\partial t} = H\psi(t)$

and $\psi(0) = \sum_n c_n \phi_n$ OK $\Big\}$ Soln

Esercizio Sist. a due stati

$$H = \begin{pmatrix} E_0 & 0 \\ 0 & -E_0 \end{pmatrix}$$

$$A = a\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$t=0$ La misura di A $\begin{cases} a : |a\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ -a : |-a\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{cases}$

Supp. che il risultato sia $A = a$

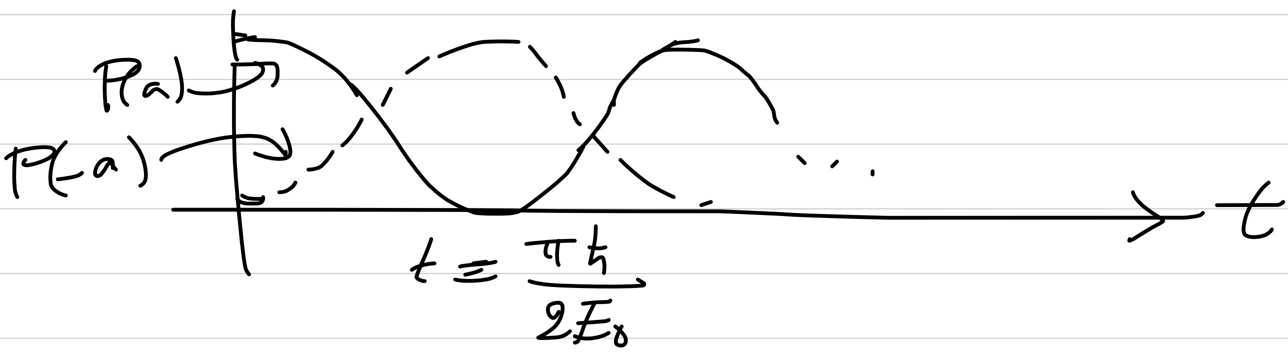
$$|\psi(0)\rangle = |a\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\left(|1\rangle + |2\rangle\right)$$

$$\Rightarrow \psi(t) = \frac{1}{\sqrt{2}}\left(e^{-iE_0t/\hbar}|1\rangle + e^{iE_0t/\hbar}|2\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(e^{-iE_0t/\hbar}\frac{|a\rangle + |-a\rangle}{\sqrt{2}} + e^{iE_0t/\hbar}\frac{|a\rangle - |-a\rangle}{\sqrt{2}}\right)$$

$$= \cos\frac{E_0t}{\hbar}|a\rangle + \sin\frac{E_0t}{\hbar}|-a\rangle$$

$$P_a(t) = \cos^2\frac{E_0t}{\hbar} \quad ; \quad P_{-a}(t) = \sin^2\frac{E_0t}{\hbar}$$



cfr Oscillazione di K ; Oscillazione di neutrini

Solutione formale dell'eq. di Schrödinger

$$i\hbar \frac{\partial}{\partial t}\psi(t) = H\psi(t) \quad \left(H = H(p, q; \cancel{t})\right)$$

$$\Rightarrow \quad |\psi(t)\rangle = e^{-iHt/\hbar}\,|\psi(0)\rangle$$

$$\boxed{e^{-iHt/\hbar}} = \sum_n \frac{1}{n!}\left(\frac{-it}{\hbar}\right)^n H^n$$

$H\psi_n = E_n\psi_n$

$$\therefore \quad e^{-iHt/\hbar}\,|\psi(0)\rangle = e^{-iHt/\hbar}\sum_n c_n|\psi_n\rangle$$

$$= \sum_n c_n e^{-\frac{iHt}{\hbar}}|\psi_n\rangle = \sum_n c_n e^{-\frac{iE_n t}{\hbar}}|\psi_n\rangle$$

$$= |\psi(t)\rangle \quad \text{ok.}$$

Proprietà generali dell'evoluzione temporale.

$$[\hat{F}, \hat{H}] = 0 \Rightarrow \text{Variabile } F \text{ è "conservata"}$$

(i) Se $\hat{F}\psi(0) = f\,\psi(0)$ (autostato di $\hat{F}$ con f)

$$\Rightarrow \hat{F}\,\psi(t) = \hat{F}\,e^{-iHt/\hbar}\,\psi(0)$$

$$= e^{-iHt/\hbar}\,\hat{F}\,\psi(0) = f\,\psi(t)$$

Un autostato di $\hat{F}$ ($[\hat{F}, \hat{H}] = 0$) lo resta, con lo stesso autovalore.

(ii) $\langle F \rangle = \langle \psi | \hat{F} | \psi \rangle \qquad \forall \psi$

$$\langle \dot{F} \rangle = \frac{\partial}{\partial t} \langle \psi | \hat{F} | \psi \rangle = 0$$

$$\because \; \text{G} = \frac{1}{i\hbar}\, i\hbar \frac{\partial}{\partial t} \langle \psi(t) | \hat{F} | \psi(t) \rangle$$

$$= \frac{1}{i\hbar} \left[\langle \psi(t) | \hat{F}\, i\hbar \frac{\partial}{\partial t} | \psi(t) \rangle + \left(i\hbar \frac{\partial}{\partial t} \langle \psi(t) | \right) \hat{F} | \psi(t) \rangle \right]$$

$$\left(\begin{aligned} & i\hbar \frac{\partial}{\partial t} \psi(t) = H\,\psi(t) \; ; \\ & -i\hbar \frac{\partial}{\partial t} \psi^*(t) = H^*\,\psi^*(t) \end{aligned} \right)$$

$$\rightarrow \frac{1}{i\hbar} \langle \psi(t) | [\hat{F}, H] | \psi(t) \rangle = 0 .$$

(iii) $|\psi\rangle = |\psi_n\rangle \Rightarrow$

$$\dot{f}_n = 0 \quad ; \quad f_n = \text{cost.}$$

(iv) Per un operatore $\hat{F} = \hat{F}(\hat{p}, \hat{q}; t)$ generico

$$i\hbar \frac{d}{dt}\langle \hat{F} \rangle = \langle \psi | \left(i\hbar \frac{\partial}{\partial t} \hat{F} \right) | \psi \rangle + \langle \psi | [\hat{F}, \hat{H}] | \psi \rangle$$

(v) $\hat{H} = \hat{H}(p, q; t)$

$$i\hbar \frac{\partial}{\partial t} \psi(q,t) = H(p,q,t)\, \psi(q,t)$$

$$\Rightarrow \psi(q,t) \neq e^{-i \frac{H(p,q,t)\,t}{\hbar}} \psi(q,0)$$

NO

Spettro continuo / funzione delta di Dirac.

* Spettro discreto

$$\hat{F}\psi_n = f_n \psi_n \qquad (n = 1, 2, 3 \dots) \quad \{f_n\}$$

$||\psi_n|| = 1$

Spettro continuo (e.g. $\hat{p}$ di una particella libera)

$$\hat{F}\psi_f = f\psi_f \qquad f_0 \le f < \infty$$

(E di una particella libera)

N.B. Un sistema può avere sia uno spettro continuo che discreto (e P ⇝ stati di diffusione, stati legati · H)

$$\hat{F}\psi_f(q) = f\psi_f(q)$$

$$\psi(q) = \int df\, c(f)\, \psi_f(q)$$

(Sviluppo in autostati di $\hat{F}$)

cfr $\sum_n c_n \psi_n(q)$

$$\Rightarrow \quad P(f)\,df = |c(f)|^2 df \qquad (P_n = |c_n|^2)$$

$$1 = \int df\, P(f) = \int df\, |c(f)|^2 \qquad \text{(Prob. totale)}$$

D'altra parte,

$$\int dq\, |\psi(q)|^2 = 1 \qquad \Longrightarrow$$

Relazione di ortogonalità (ON)

$$\hat{F}\psi_f = f\psi_f \qquad (1)$$

$$\hat{F}\psi_{f'} = f'\psi_{f'} \qquad (2)$$

$$\Rightarrow \qquad 0 = (f - f')\int dq\, \psi^*_{f'}(q)\psi_f(q)$$

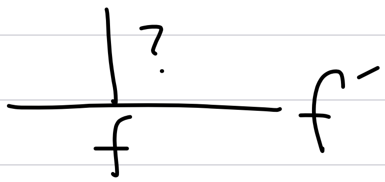
$$\therefore \quad \int dq\, \psi^*_{f'}\psi_f = 0 \qquad f' \neq f \qquad \checkmark$$

ma $\displaystyle\int dq\, \psi^*_f \psi_f = \int dq\, |\psi_f|^2 = 1 \;??$

$$1 = \int dq\, |\psi(q)|^2 = \int dq \int df\, C(f)\psi_f(q)\int df'\, C(f')^*\psi^*_{f'}(q)$$

$$= \int df\,df'\, C(f)\,C(f')^* \int dq\, \psi^*_{f'}(q)\psi_f(q)$$

$$= \underline{\int df\, |C(f)|^2} \int df' \underbrace{\int dq\, \psi^*_{f'}\psi_f}_{=0 \quad f' \neq f}$$



Se $\langle f'|f\rangle = 0 \quad f' \neq f$

$\langle f|f\rangle = 1$

$$1 = \int df\, |C(f)|^2 \cdot 0 = 0$$

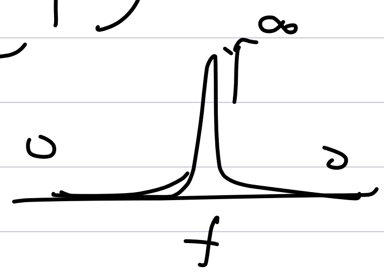
$$\Longleftrightarrow \qquad 1 = \int df\, |C(f)|^2 \quad \left(\text{Prob. totale} = 1\right)$$

Quello che ci vuole è :

$$\int dq\, \psi^*_{f'}\, \psi_f = 0 \qquad f' \neq f$$

$$\int dq\, \psi^*_{f'}\, \psi_f \Big|_{f'=f} = \infty \,, \quad \text{t.c.}$$

$$\int df' \Big(\underbrace{\int dq\, \psi^*_{f'}\, \psi_f}_{=} \Big) = 1$$



$$1 = \int dq\, |\psi(q)|^2$$

$$\Longleftrightarrow \int df\, |c(f)|^2 = 1 \qquad \text{OK!}$$

Funz. singolari con queste proprietà sono state introdotte da Dirac.

$$\delta(x) = 0 \qquad x \neq 0$$

$$\delta(0) = \infty$$

$$\int_a^b dx\, \delta(x)\, g(x) = g(0) \qquad a < x < b$$

$$= 0 \qquad 0 < a < b \,,$$

$$a < b < 0$$

Proprietà di $\delta(x)$ (Funzione generalizzata: Distribuzione)

- Ben definito come Kernel dell'integrazione
- Si può definire come limite di una funzione.

- $\delta(-x) = \delta(x)$
- $\int_a^b dx\, f(x)\,\delta(x-x_0) = f(x_0) \qquad a < x_0 < b$

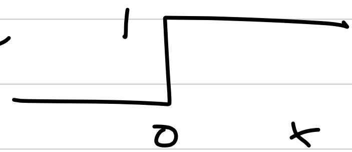
$$\int_0^a dx\, \delta(x) f(x) = \frac{1}{2} f(0)$$

$$f(g(x)) = \sum_{\substack{\text{radici}\\ g(x)=0}} \frac{1}{|g'(x_i)|}\,\delta(x-x_i)$$

$$\delta(a(x-x_0)) = \frac{1}{|a|}\,\delta(x-x_0)$$

$$\theta(x) = 1 \quad x \geq 0$$
$$= 0 \quad x < 0$$

funz. theta di Heavyside



$$\theta'(x) = \delta(x)$$

$$\int_a^b dx\, f(x)\,\theta'(x) = f(x)\,\theta(x)\Big|_a^b - \int_a^b dx\, f'\,\theta(x)$$

$$= f(b) - \int_0^b dx\, f' = f(0) \qquad \text{ok}$$

$(a < 0 < b)$

$$\delta(x) = \lim_{\epsilon \to 0} \frac{1}{\sqrt{\pi}\,\epsilon} e^{-x^2/\epsilon^2}$$

$$\delta(x) = \lim_{L \to \infty} \frac{\sin Lx}{\pi x}$$

$$\delta(x) = \lim_{L \to \infty} \frac{\sin^2 Lx}{\pi L x^2} \;;$$

$$\delta(x) = \frac{1}{\pi} \frac{\epsilon}{x^2+\epsilon^2}$$

$$\frac{1}{x - i\epsilon} = P\frac{1}{x} + \pi i\, \delta(x)$$

Funzione delta come limite di funzioni ma intesa come Kernel di un integrale

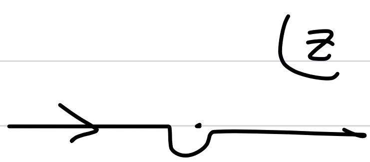
da prendere fuori l'integrazione!

$$\delta(x) \,\text{"}=\text{"}\, \text{"}\lim_{L \to \infty}\text{"} \frac{\sin Lx}{\pi x}$$

$$\lim_{L \to \infty} \int_a^b dx \frac{\sin Lx}{\pi x} f(x) = \lim \int_{aL}^{bL} \frac{\sin z}{\pi z} f\left(\frac{z}{L}\right)$$

$$\simeq \frac{f(0)}{\pi} \int_{-\infty}^{\infty} dz \frac{\sin z}{z}$$

$$\int_{-\infty}^{\infty} dz \frac{\sin z}{z} = \int_C dz \frac{\sin z}{z}$$



$$= \int_C dz \frac{e^{iz} - e^{-iz}}{2i\, z}$$

$$= \frac{1}{2i}\left[\int dz \frac{e^{iz}}{z} - \int dz \frac{e^{-iz}}{z} \right] = \frac{2\pi i}{2i} = \pi$$

$$\frac{1}{x-i\epsilon} = P\frac{1}{x} + \pi i\,\delta(x)$$

$$\frac{1}{x-i\epsilon} + \frac{1}{x+i\epsilon} = 2\cdot\frac{x}{x^2+\epsilon^2} \underset{\epsilon\to 0}{=} 2\cdot P\frac{1}{x}$$

$$\frac{1}{x-i\epsilon} - \frac{1}{x+i\epsilon} = 2i\,\frac{\epsilon}{x^2+\epsilon^2}$$

$$\lim_{\epsilon\to 0}\int dx\,\frac{\epsilon}{x^2+\epsilon^2}f(x) \simeq f(0)\int dx\,\frac{\epsilon}{x^2+\epsilon^2}$$

$$= f(0)\cdot\int_{-\infty}^{\infty} dx\,\frac{1}{x^2+1} = f(0)\,\tan^{-1}x\Big|_{-\infty}^{\infty}$$

$$= \pi f(0)$$

applicazione importante :

Trasformata di Fourier.

$$\boxed{\int_{-\infty}^{\infty} dx\, e^{i(k-k')x} = 2\pi\,\delta(k-k')}$$

$$1^\circ = \lim_{L\to\infty}\int_{-L}^{L} dx\, e^{i(k-k')x} = \frac{e^{i(k-k')L} - e^{-i(k-k')L}}{i(k-k')}$$

$$= 2\cdot\frac{\sin(k-k')L}{k-k'} \underset{L\to\infty}{=} 2\pi\,\delta(k-k') \quad ✓$$

Vice versa

$$\int dk\, e^{ikx}e^{-ikx'} = 2\pi\,\delta(x-x')$$

Trasformata inversa di Fourier

$$\psi(x) = \int_{-\infty}^{\infty} dk\, a(k)\, e^{ikx} \quad (1) \qquad \left(k = \frac{p}{\hbar}\right)$$

$$\Leftrightarrow \quad a(k) = \int \frac{dx}{2\pi}\, e^{-ikx}\, \psi(x) \quad (2)$$

Infatti: $\int \frac{dx}{2\pi} e^{-ikx} \cdot \int dk'\, a(k')\, e^{ik'x}$

$$= \int dk' \frac{a(k')}{2\pi} \underbrace{\int dx\, e^{i(k'-k)x}}_{= 2\pi\delta(k'-k)}$$

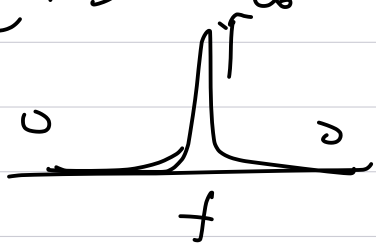
$$= \int dk'\, a(k') \cdot \delta(k'-k) = a(k) \quad \text{ok.}$$

Back to p. 10 :

$$\int dq\, \psi_{f'}^{*}\, \psi_f = 0 \qquad f' \neq f$$

$$\int dq\, \psi_{f'}^{*}\, \psi_f \Big|_{f'=f} = \infty, \quad \text{t.che}$$

$$\int df' \Big(\underbrace{\int dq\, \psi_{f'}^{*}\, \psi_f}_{\simeq} \Big) = 1 \qquad (\cdot\$\cdot)$$



(O.N.)

$$(\cdot\$\cdot) \Rightarrow \int dq\, \psi_{f'}^{*} \psi_f = \langle f' | f \rangle = \delta(f'-f) \;!$$

COMPLETEZZA.

$\boxed{\text{ON:}}$ $$\int dq\, \psi^*_f(q)\, \psi_{f'}(q) = \delta(f-f')$$
$$\int dq\, \psi^*_n\, \psi_m = \delta_{n,m}.$$

cfr. I vettori in $\mathbb{R}^3$:

$\hat{i} = (1,0,0)$ e $\hat{j} = (0,1,0)$ sono ON

ma non completi. Per esprimere un vettore qualsiasi come combinaz. lineare, ci vuole un terzo vettore (*)

$\mathbb{k} = (0,0,1)$. $\{\hat{i}, \hat{j}, \mathbb{k}\}$ insieme ON e completi

Viceversa, per $\boxed{\text{la completezza}}$ della base (*) non occorre che siano ON: $\tilde{\mathbb{k}} = \frac{1}{\sqrt{2}}(0,1,1)$ ✓

$\{\hat{i}, \hat{j}, \tilde{\mathbb{k}}\}$ sono completi (NON ON).

Analogamente, la condiz. che $\{\psi_f\}$ nel caso di spettro continuo, e $\{\psi_n\}$ nel caso discreto (in generale, $\{\psi_n, \psi_f\} \in \{f_n, f_0 \le f < \infty\}$) formino un insieme completo?

• Spettro discreto

$$\psi(q) = \sum_n c_n \psi_n(q) \quad , \quad c_n = \int dq' \, \psi_n^*(q') \, \psi(q')$$

$$|\psi\rangle = \sum_n c_n |n\rangle \qquad\qquad = \langle n | \psi \rangle$$

$$\psi(q) = \int dq' \underbrace{\left(\sum_n \psi_n(q) \psi_n^*(q') \right)}_{S} \psi(q')$$

$\circledast$ stato arbitrario

(S) indip. dallo stato; specifico x $\hat{F}$

$$\therefore \quad \sum_n \psi_n(q) \psi_n^*(q') = \delta(q - q')$$

$$|\psi\rangle = \sum_n |n\rangle \langle n | \psi \rangle$$

$$\therefore \quad \sum_n |n\rangle\langle n| = \mathbb{1}$$

→ rel. Completezza

3D $\delta(q-q') = \delta^3(\mathbf{r}-\mathbf{r}')$

Spettro continuo

$$\psi(q) = \int df \, c(f) \, \psi_f(q) =$$

$$= \int dq' \int df \, \psi_f(q) \left(\psi_f^*(q') \psi(q') \right)$$

$$\Rightarrow \int df \, \psi_f(q) \psi_f^*(q') = \delta(q - q')$$

N.B. In un sistema con sia spettro discreto che continuo

$$\sum_n \psi_n(q) \psi_n^*(q') + \int df \, \psi_f(q) \psi_f^*(q') = \delta(q - q')$$

$$\sum_n |n\rangle\langle n| + \int df \, |f\rangle\langle f| = \mathbb{1}$$

Unitarietà (Prob. tot. = 1) in MQ.

Spettro discreto

$$|\psi\rangle = \sum_n c_n |n\rangle \quad ; \quad \hat{F}|n\rangle = f_n |n\rangle$$

$$P_n = |c_n|^2$$

$$\sum_n P_n = \sum_n |c_n|^2 = \sum_n \langle\psi|n\rangle\langle n|\psi\rangle$$

$$= \langle\psi| \Big(\sum_n |n\rangle\langle n|\Big) |\psi\rangle \underset{\text{Compl.}}{=} \langle\psi|\psi\rangle = \underset{\text{OK.}}{1}$$

Operatore di proiezione. Un'ulteriore forma x P_n

$$\Pi_n \equiv |n\rangle\langle n| \quad ; \quad \Pi_n|\psi\rangle = c_n|n\rangle \ \big(\text{NO }\underline{\text{somma}}\big)$$

$$|\psi\rangle = \sum_n c_n |n\rangle$$

$$\langle\psi|\Pi_n|\psi\rangle = c_n\langle\psi|n\rangle = |c_n|^2 \qquad !$$

$$P_n = |c_n|^2 = |\langle n|\psi\rangle|^2 = \underline{\langle\psi|\Pi_n|\psi\rangle}$$

Es. autostati di impulso.

$$\psi_p(x) = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar} \quad ; \quad \hat{p}\,\psi_p = p\,\psi_p$$

ON: $\int dx\, \psi_{p'}^*(x)\,\psi_p(x) = \frac{1}{2\pi\hbar}\int dx\, e^{i(p-p')x/\hbar}$

$$= \frac{1}{2\pi\hbar}\, 2\pi\,\delta\big((p-p')/\hbar\big) = \delta(p-p')$$

Completezza.

$$\int dp\, \psi_p(x)\,\psi_p^*(x') = \delta(x-x')$$

autostati della posizione

$$\psi_{x_0}(x) = \delta(x-x_0)$$

ON $\int dx\, \psi_{x_0'}^*(x)\,\psi_{x_0}(x) = \int dx\, \delta(x-x_0')\,\delta(x-x_0)$

$$= \delta(x_0 - x_0') \quad \checkmark$$

Completezza $\int dx_0\, \psi_{x_0}(x)\,\psi_{x_0}(x') = \int dx_0\, \delta(x-x_0)\,\delta(x'-x_0)$

$$= \delta(x-x') \quad \checkmark$$

Operatore di impulso $\hat{p}$ come operatore di traslazione in x

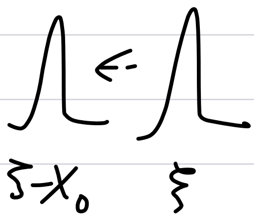
$$U(x_0) \equiv e^{i\hat{p}x_0/\hbar} \qquad \hat{p} = -i\hbar\frac{\partial}{\partial x}$$

$$\Rightarrow U(x_0)\psi(x) = e^{ipx_0/\hbar}\psi(x)$$

$$= \sum_{n=0}^{\infty}\frac{x_0^n}{n!}(\partial/\partial x)^n(x) = \underline{\psi(x+x_0)}$$

Sviluppo Taylor

$$\Rightarrow U H(p,x) U^{-1}$$



$$= H(UpU^{-1}, UxU^{-1})$$

$$= H(p, \quad x+x_0)$$

$$\because UxU^{-1} = e^{i\hat{p}x_0/\hbar}\, x\, e^{-i\hat{p}x_0/\hbar}$$

$$= \left[1 + \frac{i\hat{p}x_0}{\hbar} + \frac{1}{2!}\left(\frac{i\hat{p}x_0}{\hbar}\right)^2 + \dots\right] x$$

$$\left[1 - \frac{i\hat{p}x_0}{\hbar} + \frac{1}{2!}\left(\frac{i\hat{p}x_0}{\hbar}\right)^2 + \dots\right]$$

$$= x + \frac{ix_0}{\hbar}[\hat{p}, x] + \frac{1}{2!}\left(\frac{ix_0}{\hbar}\right)^2[\hat{p},[\hat{p},x]] + \dots$$

$$= x + x_0 \qquad \text{OK.}$$

Equazione di Schrödinger:
proprietà generali e applicaz. ni 1D.

Proprietà generali

$$H = -\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{r})$$

$$H\psi = E\psi.$$

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{r})\right)\psi = E\psi$$

Eq. differenziale di 2° grado $\Rightarrow$
condiz. al contorno.

Per stati legati

$$\int d^3r\, |\psi(\mathbf{r})|^2 < \infty \Rightarrow |\psi| \underset{r\to\infty}{<} r^{-3/2}$$

In generale,
CONTINUITÀ di $\{\psi, \nabla\psi\}$
attraverso discontinuità (finite) di V.

$\Leftarrow$ Densità e densità di corrente.

Densità (ρ) e densità corrente ($\vec{j}$)

$$\rho \equiv \psi^* \psi = |\psi|^2 .$$

$$\frac{d}{dt}\rho = \frac{1}{i\hbar}\left\{\psi^* i\hbar \psi - (-i\hbar \psi^*)\psi \right\}$$

$$= \frac{1}{i\hbar}\left[\psi^*\left(-\frac{\hbar^2}{2m}\nabla^2 + V(r)\right)\psi - \left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\psi^* \cdot \psi\right]$$

$$= \frac{i\hbar}{2m}\left\{\psi^* \nabla^2 \psi - (\nabla^2 \psi^*)\psi\right\}$$

$$= -\nabla \cdot \vec{j}$$

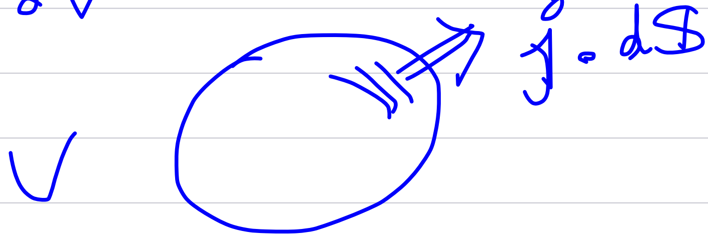
dove $$\vec{j} = \frac{i\hbar}{2m}\left((\nabla\psi^*)\psi - \psi^* \nabla\psi\right)$$

$$\Rightarrow \quad \frac{\partial}{\partial t}\rho + \nabla \cdot \vec{j} = 0 \qquad (*)$$

Eq. di continuità.

Integrando in un volume, si ha:

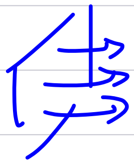
$$\frac{\partial}{\partial t}\int_V d^3r\, \rho = -\int_{\partial V} d\vec{S} \cdot \vec{j}$$



e.g. $\psi_p(x) = e^{ipx/\hbar}$

$$\Rightarrow \quad j = \frac{i\hbar}{2m}\left(\psi^{*\prime}\psi - \psi^{*}\psi'\right)$$

$$= \frac{p}{m} = v$$

$\Longrightarrow$  $=$ flusso

$$\psi = c\, e^{ipx/\hbar} \longrightarrow j = |c|^2 \cdot v$$

$$\psi_{\mathbf{p}}(\mathbf{r}) = e^{i\mathbf{p}\cdot\mathbf{r}/\hbar}$$

$$\vec{j} = \frac{\mathbf{p}}{m} = \vec{v}.$$

Continuità di ρ, $\vec{j}$ $\Rightarrow$

continuità di ψ, $\nabla\psi$.

Teorema di Ehrenfest

$\psi(r,t) \sim$ un pacchetto d'onda.

$$\Rightarrow \frac{d}{dt}\langle\psi|p|\psi\rangle = -\langle\psi|\nabla V|\psi\rangle \quad ①$$

$$\frac{d}{dt}\langle\psi|m r|\psi\rangle = \langle\psi|p|\psi\rangle . \quad ②$$

①: $\text{L.H.S} = \frac{1}{i\hbar}\, i\hbar \frac{d}{dt}\langle\psi|p|\psi\rangle$

$$= \frac{1}{i\hbar}\langle\psi|[p, H]|\psi\rangle \qquad \left(H = \frac{p^2}{2m} + V\right)$$

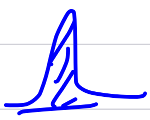
$$= -\langle\psi|\nabla V|\psi\rangle$$

$$② = \frac{m}{i\hbar}\langle\psi|[r, H]|\psi\rangle = \qquad \left([x, p^2] = [x, p_i p_i] = 2i\hbar p_x .\right)$$

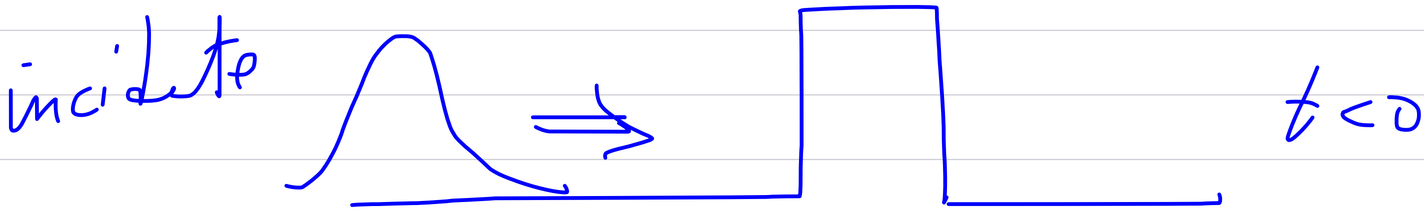
$$= \langle\psi|p|\psi\rangle .$$

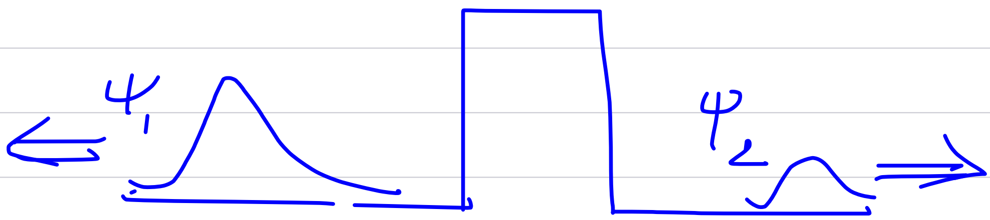
$$\begin{cases} [A, BC] = B[A, C] + [A, B]C . \\ [AB, C] = A[B, C] + [A, C]B . \end{cases}$$

<u>Eq. di Newton</u> ?

 $\sim$ una particella ?

Una particella (elettrone) non frazionalizza, mentre un pacchetto d'onda sì.





$|\psi|^2 \sim$ probabilità

Teo. Feynman - Hellman

$$H = \frac{p^2}{2m} + V(r; g)$$

$$\rightarrow \quad E_n = E_n(g) \qquad H\psi_n = E_n \psi_n$$

$$\frac{d}{dg} E_n(g) = \langle \psi_n | \frac{\partial V}{\partial g} | \psi_n \rangle .$$

Teo del Viriale.

$$\langle \psi_n | \frac{p^2}{2m} | \psi_n \rangle = \frac{1}{2} \langle \psi_n | r \cdot \nabla V | \psi_n \rangle .$$

$$\left([r \cdot p, H] \rightarrow \qquad \right) \qquad H\psi_n = E_n \psi_n$$