27/10/2016

Oscillatore armonico.

$$H = \frac{p^2}{2m} + \frac{m\omega^2}{2}x^2 .$$

$$-\frac{\hbar^2}{2m}\psi'' + \frac{m\omega^2}{2}x^2\psi = E\psi$$

$$D\left[\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\right] = D\left[\frac{m\omega^2}{2}x^2\right]$$

$$\dashrightarrow D[x] = D\left(\sqrt{\frac{\hbar}{m\omega}}\right) \Rightarrow \sqrt{\frac{m\omega}{\hbar}}\,x = \text{adim}.$$

$$D[p] = D\left[\hbar\frac{d}{dx}\right] = \sqrt{m\omega\hbar} \Rightarrow \frac{p}{\sqrt{m\omega\hbar}} = \text{adim}.$$

Operatori di creazione/annichilaz.

$$\sqrt{\frac{m\omega}{2\hbar}}\,x + i\sqrt{\frac{1}{2m\omega\hbar}}\,p \equiv a ,$$

$$\sqrt{\frac{m\omega}{2\hbar}}\,x - i\sqrt{\frac{1}{2m\omega\hbar}}\,p \equiv a^\dagger ,$$

$$\Longrightarrow [a, a^\dagger] = 1 .$$

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger)$$

$$p = -i\sqrt{\frac{m\omega\hbar}{2}}(a - a^\dagger) ,$$

$$[x, p] = i\hbar , \quad \text{OK}.$$

$$H = \frac{1}{2m}(-)\frac{m\omega\hbar}{2}(a - a^\dagger)^2 + \frac{m\omega^2}{2}\frac{\hbar}{2m\omega}(a + a^\dagger)^2$$

$$= -\frac{\omega\hbar}{4}(a^2 + a^{\dagger 2} - aa^\dagger - a^\dagger a) + \frac{\omega\hbar}{4}(a^2 + a^{\dagger 2} + aa^\dagger + a^\dagger a)$$

$$= \frac{\omega\hbar}{2}(a^\dagger a + aa^\dagger)$$

$$= \omega\hbar(a^\dagger a + \frac{1}{2})$$

$$\langle H \rangle \geq \omega\hbar/2 .$$

E_n ?

$$H|\psi_n\rangle = E_n|\psi_n\rangle .$$

$$H(a|\psi_n\rangle) = aH|\psi_n\rangle - \omega\hbar\, a|\psi_n\rangle$$

$$\left([H, a] = \omega\hbar[a^\dagger a, a] = \omega\hbar[a^\dagger, a]a = -\omega\hbar a\right)$$

$$H(a|\psi_n\rangle) = (E_n - \omega\hbar)(a|\psi_n\rangle)$$

$\therefore$ $a|\psi_n\rangle$ è un autost. di H con $E = E_n - \omega\hbar$!

$\therefore$ $a^2|\psi_n\rangle$ è autost. H con $E_n - 2\omega\hbar$,
etc. ...

Ma visto che $E_n \geq 0$, $\forall n$,
ci deve essere uno stato $|\psi_0\rangle$ t.c.

$$a|\psi_0\rangle = 0 \Rightarrow \boxed{E_0 = \frac{\omega\hbar}{2}}$$

$$\Rightarrow \quad H|\psi_1\rangle = \frac{3\omega\hbar}{2}|\psi_1\rangle, \quad a|\psi_1\rangle \propto |\psi_0\rangle$$

$$H|\psi_2\rangle = \frac{5\omega\hbar}{2}|\psi_2\rangle \quad \text{etc.} ..$$

Meglio ancora:

$$a|0\rangle = 0, \quad \langle 0|0\rangle = 1.$$

$$H|0\rangle = \omega\hbar(a^\dagger a + 1/2)|0\rangle = \frac{\omega\hbar}{2}|0\rangle$$

$$H(a^\dagger|0\rangle) = \omega\hbar\left(a^\dagger a + \frac{1}{2}\right)a^\dagger|0\rangle = \frac{3}{2}\omega\hbar(a^\dagger|0\rangle).$$

$$(a^\dagger \underline{a a^\dagger}|0\rangle = a^\dagger[a, a^\dagger]|0\rangle = a^\dagger|0\rangle \quad \text{ok.}$$

etc.

$$H|n\rangle = E_n|n\rangle, \quad E_n = \omega\hbar(n + 1/2)$$

$$|n\rangle \equiv \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle, \quad n = 0, 1, 2, \ldots$$

$$\langle n|n\rangle = 1.$$

$$\langle n | n \rangle = \frac{1}{n!} \langle 0 | a^n a^{\dagger n} | 0 \rangle$$

Ora $\quad a a^{\dagger n} = [a, a^{\dagger n}] + a^{\dagger n} a$

$$= n\, a^{\dagger\, n-1} + a^{\dagger n} a$$

quindi:

$$\langle n | n \rangle = \frac{1}{(n-1)!} \langle 0 | a^{n-1} a^{\dagger\, n-1} | 0 \rangle = \cdots$$

$$= \langle 0 | a a^{\dagger} | 0 \rangle = 1$$

La funzione d'onda.

Lo stato fondamentale:

$$a|0\rangle = 0.$$

$$a = \sqrt{\frac{m\omega}{2\hbar}}\,x + \frac{i}{\sqrt{2m\omega\hbar}}\,p$$

$$a\,\psi_0(x) = 0,$$

$$\sqrt{\frac{\hbar}{2m\omega}}\frac{d}{dx}\psi_0(x) = -\sqrt{\frac{m\omega}{2\hbar}}\,x\,\psi_0(x)$$

$$\psi_0'/\psi_0 = -\frac{m\omega}{\hbar}x$$

$$\therefore\ \psi_0(x) = c\cdot e^{-\frac{m\omega}{2\hbar}x^2}$$

$$1 = \int dx\,|\psi_0(x)|^2 \to c^2\int dx\, e^{-\frac{m\omega}{\hbar}x^2} = c^2\cdot\frac{\sqrt{\hbar\pi}}{\sqrt{m\omega}} \quad \therefore\ c = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}.$$

$$|1\rangle = a^\dagger|0\rangle$$

$$\psi_1(x) = \left(\sqrt{\frac{m\omega}{2\hbar}}\,x - i\sqrt{\frac{1}{2m\omega\hbar}}\,p\right)\psi_0(x)$$

$$= 2\sqrt{\frac{m\omega}{2\hbar}}\,x\,\psi_0(x) = c_1\cdot x\,e^{-\frac{m\omega}{2\hbar}x^2}, \quad \text{etc.}$$

$$\psi_n(x) = c_n H_n(\alpha x)\,e^{-\frac{\alpha^2}{2}x^2}, \quad \left(\alpha \equiv \sqrt{\frac{m\omega}{\hbar}}\right)$$

Soluzioni dell'eq. di Schrödinger.

$$-\frac{\hbar^2}{2m}\psi'' + \frac{m\omega^2 x^2}{2}\psi = E\psi$$

$\frac{2}{\omega\hbar}\cdot$ $\quad \xi = \sqrt{\frac{m\omega}{\hbar}}\, x \equiv \alpha x$

$$-\psi'' + \xi^2\psi = \lambda\psi, \qquad \lambda = E/(\omega\hbar/2).$$

$\xi \to \pm\infty \qquad \psi'' \sim \xi^2\psi \quad \therefore \psi \sim e^{\pm\frac{\xi^2}{2}}$

Normalizzabilità $\Rightarrow \psi \sim e^{-\xi^2/2}$.

Pongo: $\psi = w(\xi)\, e^{-\xi^2/2}$, Def. $w(\xi)$ (No approssim.)

$$\Rightarrow \psi' = (w' - \xi w)\, e^{-\xi^2/2}$$

$$\psi'' = \left[(w'' - w - \xi w') - \xi(w' - \xi w)\right] e^{-\xi^2/2}$$

$$= (w'' - w - 2\xi w' + \xi^2 w)\, e^{-\xi^2/2}.$$

$$\Rightarrow \quad w'' - 2\xi w' + (\lambda - 1) w = 0.$$

Sol. con lo sviluppo in serie di potenze. (Frobenius) 1849–1917 Ferdinand Georg F.

$$w(\xi) = \sum_n a_n \xi^n$$

$$\Rightarrow w'' = \sum n(n-1) a_n \xi^{n-2} = \sum_n (n+2)(n+1) a_{n+2}\xi^n$$

$$-2\xi w' = -2\sum_n n a_n \xi^n.$$

$$(\lambda-1) w = (\lambda - 1)\sum_n a_n \xi^n$$

$$\Rightarrow (n+2)(n+1) a_{n+2} + \{\lambda - (2n+1)\} a_n = 0. \qquad (*)$$

$\Pi V(x) \Pi^{-1} = V(-x) = V(x)$

Parità: $\Rightarrow \psi_n =$ pari o dispari.

(i) Soluzioni pari. $\psi_n(-x) = \psi_n(x)$.

$a_0 \neq 0$. $a_0 \xrightarrow[(*)]{} a_2 \xrightarrow[(*)]{} a_4 \rightarrow \cdots$

La serie $<$ termina / non termina.

La serie non termina $\Rightarrow$

$n \gg 1$ $(*) \Rightarrow \cancel{n^2} a_{n+2} \approx 2n a_n$

$$a_{n+2} \simeq \frac{2}{n} a_n \sim \frac{2}{n}\frac{2}{n-2} a_{n-2} = \cdots \sim \frac{1}{m!} \quad (n \sim 2m).$$

$$w = \sum_{n \,(\text{pari})} a_n \xi^n \underset{\xi \to \infty}{\simeq} \sum_m \frac{1}{m!} \xi^{2m} \simeq e^{+\xi^2}$$

$$\therefore \psi = w(\xi) e^{-\xi^2/2} \underset{\xi \to \infty}{\sim} e^{+\xi^2/2} \quad !$$

Non normalizzabile!

(L'eq. ricorrenza ha generato la soluzione non normalizzabile.)

$\Rightarrow$ La serie deve terminare!

$$\lambda = 2n+1 \quad , \quad n \text{ pari}.$$

ii) Le soluzioni dispari.

Una considerazione analoga per

$$\psi = \sum_{n=1,3,5,\dots} a_n \xi^n \quad \text{deve terminare.}$$

$$(*) \Rightarrow \lambda = 2n+1 \quad , \quad n = 1, 3, 5, \dots$$

Conclusione:

$$E = \frac{\omega\hbar}{2}(2n+1) = \omega\hbar\left(n+\frac{1}{2}\right).$$

$$w'' - 2\xi w' + 2n w = 0.$$

$$H_n'' - 2\xi H_n' + 2n H_n = 0.$$

EQ di Hermite. Sol. polinomiali

= Polinomi di Hermite

Polinomi di Hermite : Funzione generatrice

$$S(\xi,s) = e^{-s^2+2\xi s} = e^{\xi^2-(\xi-s)^2}$$

$$\equiv \sum_{n=0}^{\infty} \frac{s^n}{n!} H_n(\xi) \quad \text{Def. } H_n(\xi). \qquad (*)$$

$H_n(\xi)$ = polinomio di grado n.

= polinomi con parità definita.

Dimostr. $H_n(\xi)$ soddisfa $(*)$.

$$\frac{\partial S}{\partial \xi}: \quad \sum_n \frac{s^n}{n!} H_n' = 2s\,S = 2s \sum \frac{s^n}{n!} H_n = \sum_n (2n) \frac{s^n}{n!} H_{n-1}$$

$$\therefore \quad H_n' = 2n\,H_{n-1} \qquad (\$)$$

$$\frac{\partial}{\partial s} S = \sum \frac{s^n}{n!} H_{n+1} = (2\xi - 2s)\,S = 2\xi \sum \frac{s^n}{n!} H_n - 2\sum n \frac{s^n}{n!} H_{n-1}$$

$$\therefore \quad H_{n+1} = 2\xi H_n - 2n\,H_{n-1} \qquad (*)$$

$$(\$)(*) \Rightarrow \quad H_{n+1} = 2\xi H_n - H_n' \Rightarrow H_{n+1}' = 2H_n + 2\xi H_n' - H_n''$$

$$2(n+1) H_n = 2H_n + 2\xi H_n' - H_n''$$

$$H_n'' - 2\xi H_n' + 2n\,H_n = 0, \qquad \text{ok.}$$

Explicit expressions. $e^{-s^2+2\xi s} =$

$$e^{\xi^2-(\xi-s)^2} = \sum_n \frac{s^n}{n!} H_n(\xi)$$

$$\Rightarrow H_n(\xi) = e^{\xi^2}\left(\frac{\partial^n}{\partial\xi^n} e^{-\xi^2}\right)(-)^n.$$

$H_0(\xi)=1$; $H_1(\xi) = 2\xi$

$H_2(\xi) = -2+4\xi^2$, etc

$H_n(\xi)=$ polin. di grado n, di parità definita.

$\therefore\ \psi_n(-x) = (-)^n \psi_n(x)$ ok.

La relaz. di ON dei polinomi di Hermite.

$$\int_{-\infty}^{\infty} d\xi\, e^{-\xi^2} H_n(\xi) H_m(\xi) = \delta_{nm} \sqrt{\pi}\, 2^n n!$$

$$\alpha \equiv \sqrt{\frac{m\omega}{\hbar}}$$

$$\Rightarrow \psi_n(x) = C_n H_n(\alpha x)\, e^{-\frac{\alpha^2 x^2}{2}} = C_n H_n\left(\sqrt{\frac{m\omega}{\hbar}}\,x\right) e^{-\frac{m\omega}{2\hbar}x^2}$$

con $C_n = \left(\frac{\alpha}{\pi^{1/2} 2^n n!}\right)^{1/4}$

$$\begin{cases} S(\xi,s) = e^{-s^2+2s\xi} = \sum_n \frac{s^n}{n!} H_n(\xi) \\ S(\xi,t) = e^{-t^2+2t\xi} = \sum_m \frac{t^m}{m!} H_m(\xi) \end{cases}$$

$$\int d\xi \, S(\xi,s) S(\xi,t) e^{-\xi^2}$$

$$\sum_{n,m} \frac{s^n t^m}{n! \, m!} \int d\xi \, e^{-\xi^2} H_n(\xi) H_m(\xi) \quad (*)$$

$$= \int d\xi \, e^{-\xi^2} e^{-s^2+2s\xi} e^{-t^2+2t\xi}$$

$$= \int d\xi \, e^{-(\xi-s-t)^2+(s+t)^2-s^2-t^2}$$

$$= \sqrt{\pi} \cdot e^{2st} = \sqrt{\pi} \sum_n \frac{(2st)^n}{n!} \quad (**)$$

$$(*) = (**) \Rightarrow$$

$$\int d\xi \, e^{-\xi^2} H_n(\xi) H_m(\xi)$$

$$= \delta_{n,m} \sqrt{\pi} \, 2^n n!$$

$$x = \sqrt{\frac{\hbar}{2m\omega}}\left(a + a^\dagger\right)$$

$$p = -i\sqrt{\frac{m\omega\hbar}{2}}\left(a - a^\dagger\right)$$

$\Rightarrow$ Elementi di matrice di x, p.

$$\langle n+1|x|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\langle n+1|a^\dagger|n\rangle$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\frac{1}{\sqrt{(n+1)!}}\frac{1}{\sqrt{n!}}\underbrace{\langle 0|a^{n+1}a^\dagger a^{\dagger n}|0\rangle}_{(n+1)!} = \sqrt{\frac{\hbar}{2m\omega}}\sqrt{n+1}$$

$$= \frac{1}{\alpha}\sqrt{\frac{n+1}{2}}$$

$$\langle n|x|n+1\rangle = \sqrt{}\quad\sqrt{n+1}\qquad\checkmark$$

$$\langle n+1|p|n\rangle = -\frac{i}{\sqrt{2m\omega\hbar}}\frac{1}{\sqrt{(n+1)!}}\frac{1}{\sqrt{n!}}\langle 0|a^{n+1}(a-a^\dagger)a^{\dagger n}|0\rangle$$

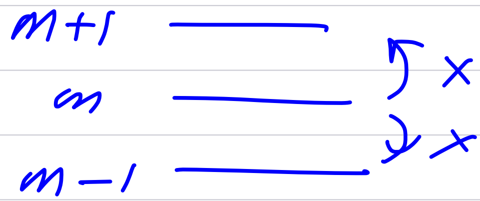
$$= \frac{i}{\sqrt{2m\omega\hbar}}\sqrt{n+1} = \frac{i}{\sqrt{m\omega\hbar}}\sqrt{\frac{n+1}{2}}$$

$$\langle n|p|n+1\rangle = -\frac{i}{\sqrt{2m\omega\hbar}}\sqrt{n+1}$$

$$(X)_{nm} = X_{nm} = \begin{pmatrix} 0 & * & & \\ * & 0 & * & \mathbb{O} \\ & * & 0 & \\ \mathbb{O} & & & \ddots \end{pmatrix}$$

Elementi di matrice di x^2, p^2.

$$x_{nm} \neq 0 \qquad n = m \pm 1$$



$$\Rightarrow \quad x^2_{nm} \quad ?$$

$$\langle n| x^2 |m\rangle = \sum_\ell \langle n|x|\ell\rangle\langle \ell|x|m\rangle$$

$$= \sum_\ell x_{n\ell}\cdot x_{\ell m} \, . \quad \text{(prodotto matriciale)}$$

$$\therefore \quad x^2_{nm} \neq 0 \quad \text{per } n = m \pm 2, \, m .$$

$$(x^2)_{n+2,n} = x_{n+2,n+1}\, x_{n+1,n} = \frac{1}{\alpha}\sqrt{\frac{n+2}{2}}\sqrt{\frac{n+1}{2}} = \frac{(n+2)(n+1)}{2\alpha}$$

$$(x^2)_{nn} = x_{n,n+1}\, x_{n+1,n} + x_{n,n-1}\, x_{n-1,n} = \frac{1}{\alpha}\left(\frac{n}{2} + \frac{n+1}{2}\right) = \frac{2n+1}{2\alpha}$$

$$(x^2)_{n-2,n} = (x^2)_{n,n-2} = \frac{n(n-1)}{2\alpha} .$$

$$(p^2)_{n+2,n} = p_{n+2,n+1}\, p_{n+1,n} = -(m\omega\hbar)\frac{(n+2)(n+1)}{2}$$

$$(p^2)_{nn} = p_{n,n+1}\, p_{n+1,n} + p_{n,n-1}\, p_{n-1,n} =$$

$$= m\omega\hbar\left(\frac{n+1}{2} + \frac{n}{2}\right) = m\omega\hbar\,\frac{2n+1}{2}$$

$$(p^2)_{n-2,n} = (p^2)_{n,n-2} = -m\omega\hbar\,\frac{n(n-1)}{2} ;$$

$$H_{nn} = \frac{1}{2m}(p^2)_{nn} + \frac{m\omega^2}{2}(x^2)_{nn}$$

$$= \frac{1}{2m} m\omega\hbar \, \frac{2n+1}{2} + \frac{m\omega^2}{2}\, \frac{\hbar}{m\omega}\, \frac{2n+1}{2}$$

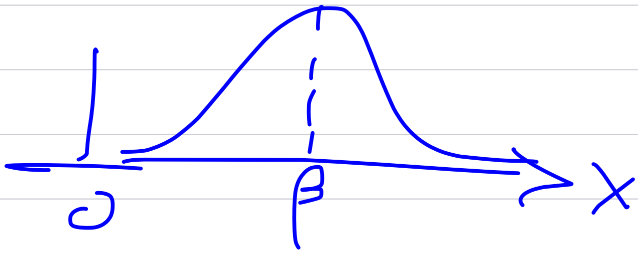
$$= \omega\hbar\left(n+\frac{1}{2}\right) \quad !$$

$$H_{n,m} = 0 \qquad n \neq m .$$

Evoluzione temporale.

$$\psi(x,0) \Rightarrow e^{-iHt/\hbar}\,\psi(x,0) = \psi(x,t)$$

$$\psi(x,0) = \psi_0(x) =$$



a $t=0$

$$P(x)\,dx = |\psi_0(x)|^2\,dx.$$

t ? Esiste t t.che la distrib. $|\psi(x,t)|^2 = |\psi(x,0)|^2$?

Ans. Sì. $t = \frac{2\pi n}{\omega}$ (Periodi classici)

$\because$ $\psi(x,0) = \sum_n c_n \psi_n(x)$, $\quad H\psi_n(x) = E_n \psi_n(x)$

$\Rightarrow$ $\quad E_n = \omega\hbar\left(n+\frac{1}{2}\right).$

$$\psi(x,t) = e^{-iHt/\hbar}\sum c_n \psi_n(x)$$

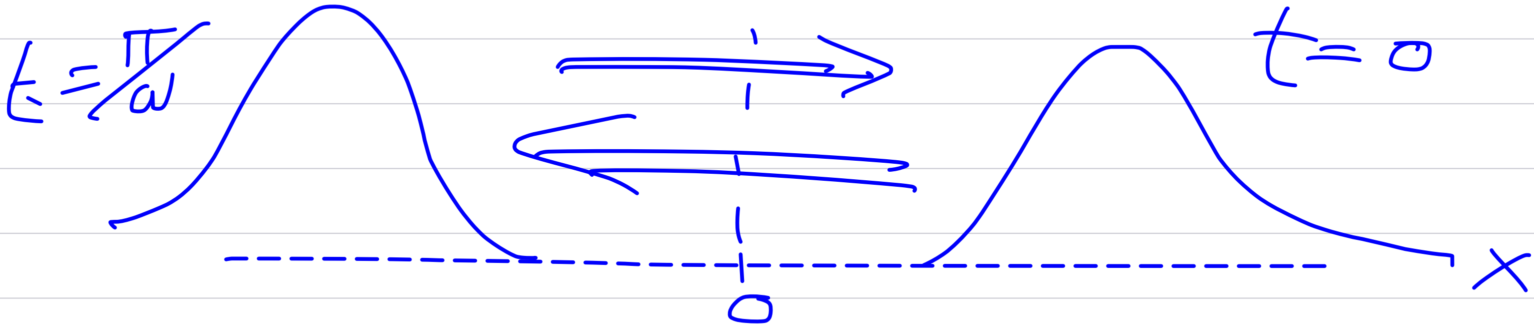
$$= \sum_n c_n e^{-iE_nt/\hbar}\psi_n(x) = e^{-\frac{i\omega t}{2}}\sum_n c_n e^{-i\omega n t}\psi_n(x)$$

$$\underset{t=2\pi n/\omega}{=} e^{-i\omega t/2}\sum_n c_n\psi_n(x) \propto \psi(x,0) \;!$$

Metà periodo ?

$t=\frac{\pi}{\omega}$ $\quad \Rightarrow \psi(x,t) = e^{-i\omega t/2}\sum_n c_n(-)^n\psi_n(x) = e^{-\frac{i\omega t}{2}}\sum_n c_n\psi_n(-x)$

$$\propto \psi(-x,0) \;!$$



$t = \pi/\omega$
$t = 0$
0
x

Stati coerenti.

$$a|\beta\rangle = \beta|\beta\rangle \qquad \beta\in\mathbb{C}$$

$$\langle\beta|\beta\rangle = 1.$$

$$|\beta\rangle = e^{-|\beta|^2/2}\, e^{\beta a^\dagger}\,|0\rangle$$

$$= \sum_n c_n\,|n\rangle$$

$$c_n = e^{-|\beta|^2/2}\,\frac{\beta^n}{\sqrt{n!}} \qquad \left(|n\rangle = \frac{a^{\dagger\,n}}{\sqrt{n!}}|0\rangle\right)$$

$$\Rightarrow \quad P_n = |c_n|^2 = e^{-|\beta|^2}\,\frac{|\beta|^{2n}}{n!}\ ; \quad \sum_n P_n = 1$$

(Distrib. di Poisson).

$$\psi_\beta(x) \Leftarrow a\psi_\beta = \beta\,\psi_\beta.$$

Evoluzione temporale di $|\beta\rangle$.