Teoria delle perturbazioni dipendenti dal tempo

- transizioni quantistiche

$$H(t) = H_0 + V(t)$$

$$H_0\, \psi_n^{(0)} = E_n^{(0)} \psi_n^{(0)}, \quad \{E_n, \psi_n^{(0)}\} \text{ note.}$$

$$\psi_n^0(t) = e^{-iE_n^{(0)}t/\hbar}\, \psi_n^{(0)}$$

$$i\hbar \frac{\partial \psi(t)}{\partial t} = H(t)\, \psi(t) \quad \not\Rightarrow \quad \underline{\psi(t) = e^{-iH(t)t/\hbar}}$$

$$\Rightarrow |\psi(t)\rangle = U(t)\, |\psi(0)\rangle \; ; \quad U(0) \equiv \mathbb{1}$$

con $\quad i\hbar \frac{\partial}{\partial t} U(t) = H(t)\, U(t).$

$$\dashrightarrow U(t) = \mathbb{1} - \frac{i}{\hbar}\int_0^t H(t')\,dt' + \left(-\frac{i}{\hbar}\right)^2 \int_0^t dt_1\, H(t_1) \int_0^{t_1} dt_2\, H(t_2)$$

$$+ \ldots$$

⇒ Proprietà generali di sistemi con $H(t)$

⇒ Casi perturbativi, $H(t) = H_0 + V(t)$

: sviluppo in potenze di V

$$|\Psi(t)\rangle = \sum_n a_n(t)\, e^{-iE_n^{(0)}t/\hbar}\, |\Psi_n^{(0)}\rangle \equiv |n\rangle$$

Eq. Schrödinger $\Rightarrow$ $$i\hbar \frac{\partial}{\partial t}|\Psi(t)\rangle = (H_0+V(t))\,|\Psi(t)\rangle$$

$$i\hbar \frac{\partial|\Psi(t)\rangle}{\partial t} = \sum_n \cancel{i\hbar}\, \dot a_n\, e^{-iE_n^{(0)}t}\, |n\rangle + \cancel{\sum_n a_n E_n^{(0)} e^{-iE_n^{(0)}t}|n\rangle}$$

$$(H_0+V(t))|\Psi(t)\rangle = \cancel{\sum_n a_n E_n^{(0)} e^{-iE_n^{(0)}t/\hbar}|n\rangle} + \sum_n a_n\, e^{-iE_n^{(0)}t/\hbar}\, V(t)|n\rangle$$

$\langle k|\, e^{iE_k^{(0)}t/\hbar}$:

(ancora esatto!)

$$i\hbar\, \dot a_k(t) = \sum_n a_n(t)\cdot e^{i\omega_{kn}t}\langle k|V(t)|n\rangle \;; \quad (*)$$

dove $\omega_{kn} \equiv (E_k^{(0)}-E_n^{(0)})/\hbar$.

$$a_k(0) = \delta_{k,i} \quad \left(\Leftrightarrow |\Psi(t)\rangle\big|_{t=0} = e^{-iE_i^{(0)}t/\hbar}\,|i\rangle\right)$$

$\Rightarrow$ Soluzione iterativa
(sviluppo in potenze di V)

Prima convertiamo l'eq. differ. (*) a

$$a_k(t) = \delta_{k,i} - \frac{i}{\hbar}\sum_n \int_0^t dt'\, V_{kn}(t')\, e^{i\omega_{kn}t'}\, a_n(t') \quad (**)$$

un'equazione integrale!

$\left(V_{kn}(t) \equiv \langle k|V(t)|n\rangle\right)$

(**) $\Longrightarrow$ (*) (verificare!)

(**) può essere risolta ordine × ordine in V.

Al primo ordine V^1:

$$\underbrace{\underbrace{a_k^{(1)}(t) = \delta_{k,i}}_{\text{Sol. } V^0} - \frac{i}{\hbar}\int^t dt' \, V_{ki}(t') e^{i\omega_{ki} t'}}_{\text{sol. fino al } V^1} \quad (S.)$$

La sol. al secondo ordine si ha, sostituendo (S.) al 2° membro di (**) !

$$a_k(t) = \delta_{k,i} - \frac{i}{\hbar}\sum_n \int_0^t dt' \, V_{kn}(t') e^{i\omega_{kn} t'} a_n(t')$$

$$a_k^{(2)} = \delta_{ki} - \frac{i}{\hbar}\int^t dt' \, V_{ki}(t') e^{i\omega_{ki} t'} + \left(-\frac{i}{\hbar}\right)^2 \sum_n \int^t dt' \, V_{kn}(t') e^{i\omega_{kn} t'} \cdot \int^{t'} dt'' \, V_{ni}(t'') e^{i\omega_{ni} t''},$$

etc. In generale, se

$$\text{"} a_k^{(L)} \text{"} = a_k^{(V^0)} + a_k^{(V^1)} + \cdots \, a_k^{(V^L)}$$

$$a_k^{(L+1)} = a_k^{(0)}(0) - \frac{i}{\hbar}\int^t dt' \, V_{kn}(t') e^{i\omega_{kn}(t')} a_n^{(L)}$$

$$a_k(t) = \lim_{L\to\infty} a_k^{(L)}(t) \quad (\text{esatto})$$

N.B. $a_k^{(L)} = \sum_{\ell=0}^{L} a_k^{(V^\ell)}$

cfr la notazione p. la teoria pert. indip. dal tempo.

Dunque: al primo ordine troviamo

$$\begin{cases} a^{(1)}_{k,i} = -\dfrac{i}{\hbar}\displaystyle\int^{t} dt'\, V_{k,i}(t')\, e^{i\omega_{ki} t'} & (k\neq i) \\[2ex] a^{(1)}_{i,i} = 1 - \dfrac{i}{\hbar}\displaystyle\int^{t} dt'\, V_{ii}(t') \end{cases}$$

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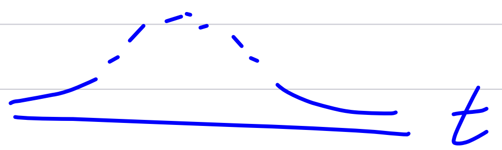
(1) Perturbazioni che durano per un intervallo finito; $V(t) \to 0 \quad t \to \pm\infty$.

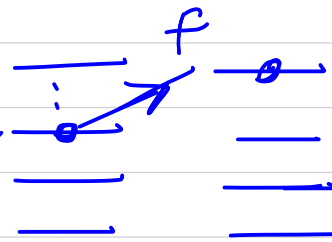


Integrando (a ∞) $(k = f)$

$t=\infty$

$$a^{(1)}_{fi}(\infty) = -\frac{i}{\hbar}\int^{\infty} dt'\, V_{fi}(t')\, e^{i\omega_{fi} t'}$$

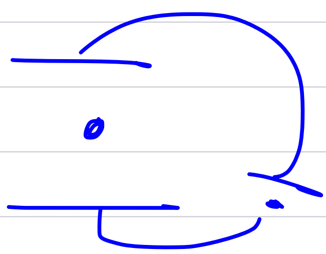
$$\therefore\ P_{fi} = P_{i\to f}(\infty) = \frac{1}{\hbar^2}\left|\int_{-\infty}^{\infty} dt'\, V_{fi}(t')\, e^{i\omega_{fi} t'}\right|^2$$



dove $V_{fi}(t) = \langle f|V(t)|i\rangle$

es. 1. Atomo di idrogeno posto in un condensatore che scarica.

$$\mathcal{E}(t) = \mathcal{E}_0\, e^{-t/t_0}\,\theta(t)$$



$$V(t) = -e\mathcal{E} z = -e\mathcal{E}_0 \cdot z \cdot e^{-t/t_0}\,\theta(t)$$

Supponiamo che $|\psi(0)\rangle = |1s\rangle$ (con $1s = 100$)

$P(1s \to 2p)$? $\quad \left(N.B. \quad P(1s \to 2s) = 0 \right)$

$$V_{fi} = e\mathcal{E}_0\, e^{-t/t_0}\,\theta(t)\,\langle 2p0|z|1s\rangle$$

$$= \quad '' \quad \langle 210|z|100\rangle$$

$$\langle 210|z|100\rangle = \int d^3r\, R_{21} Y^*_{1,0}\, r\cos\theta\, R_{10} Y_{00}$$

$$= \int dr\, r^2 \left(\frac{1}{2\sqrt{6}} r e^{-r/2}\right) \cdot r \cdot \left(2e^{-r}\right) \int d\Omega \sqrt{\frac{3}{4\pi}}\cos\theta \cdot \cos\theta \cdot \frac{1}{\sqrt{4\pi}} \cdot$$

$$= \frac{2\pi}{4\pi\sqrt{2}} \int_0^\infty dr\, r^4 e^{-3r/2} \int_{-1}^{1} d\cos\theta\, \cos^2\theta = \frac{1}{2}\cdot\left(\frac{2}{3}\right)^5 \Gamma(5)\cdot 2\cdot\frac{1}{3}\cdot$$

$$= \frac{1}{2\sqrt{2}}\left(\frac{2}{3}\right)^5 \cdot \frac{4!\cdot 2}{3}\, r_B = \frac{2^8}{3^5\sqrt{2}}\, r_B \equiv c_0\, r_B$$

$$c_0 \simeq 0.745$$

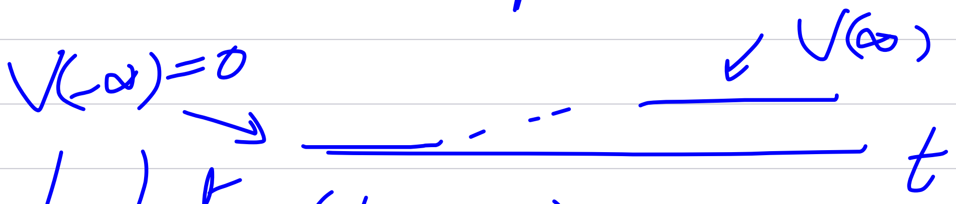
$$P_{1s\to 2p} = \frac{e^2\mathcal{E}_0^2}{\hbar^2} c_0^2 r_B^2 \left| \int dt\, e^{-t/t_0} \cdot e^{i\omega_{fi} t} \right|^2$$

$$\omega_{fi} = \frac{e^2}{2 r_B \hbar}\left(1 - \frac{1}{4}\right) = \frac{3e^2}{8 r_B \hbar}$$

$$\int dt \ldots = \int dt\, e^{-\frac{t}{t_0} + i\omega t} = \frac{1}{1/t_0 - i\omega} = \frac{t_0}{1 - i\omega t_0}$$

$$\left| \int dt \ldots \right|^2 = \frac{t_0^2}{1 + \omega_{fi}^2 t_0^2}$$

(2) Perturbazioni che tendono a un operatore costante.



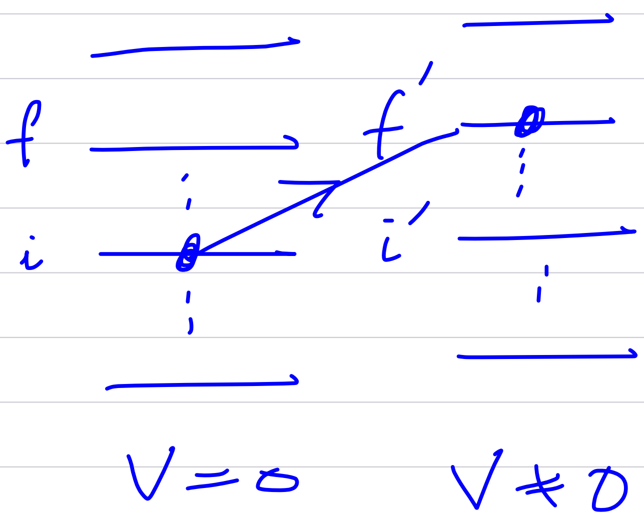
Il sistema nonperturbato ($t=-\infty$) e il sistema finale ($t=+\infty$) sono diversi.

$\exists$ Effetti di perturbazioni

Il problema è calcolare

$$P_{f'i} = |\langle f' | \psi(t)\rangle|^2$$

$$|\psi(0)\rangle = |i\rangle$$



Come si tiene conto del fatto che $|f'\rangle \neq |f\rangle$ (l'effetto delle perturb. statiche) ?

Risposta:

$$a^{(1)}_{ki} = -\frac{i}{\hbar}\int^t dt' \, V_{ki}(t') e^{i\omega_{ki} t} \qquad \left(\frac{1}{i\omega_{ki}}\partial_t e^{i\omega_{ki} t}\right)$$

integ. per parti:

$$= -\frac{V_{ki}(t)}{\omega_{ki}\hbar} e^{i\omega_{ki} t} + \frac{1}{\omega_{ki}\hbar}\underbrace{\int^t dt' \frac{\partial V_{ki}(t')}{\partial t'} e^{i\omega_{ki} t'}}_{\text{convergente!}}$$

$$a^{(1)}_{ii} = 1 - \frac{i}{\hbar} V_{ii}(\infty)\, t + \cdots$$

$$\Rightarrow \quad P_{f'i} = \frac{1}{(\omega_{fi}\hbar)^2}\left|\int_{-\infty}^{\infty} dt' \frac{\partial V_{fi}}{\partial t} e^{i\omega_{fi} t}\right|^2 \qquad (\bullet)$$

(i) Dimostrazione di (●).

$$|\psi^{(1)}\rangle = a^{(1)}_{ii}\, e^{-iE^{(0)}_i t/\hbar}|i\rangle + \sum_k a^{(1)}_{ki}\cdot e^{-iE^{(0)}_k t/\hbar}|k\rangle$$

$$= \left(1 - \frac{i}{\hbar}V_{ii}\cdot t + \dots\right) e^{-iE^{(0)}_i t/\hbar}|i\rangle$$

$$+ \sum_k{}' \left(-\frac{V_{ki}(t)}{\omega_{ki}\cdot\hbar} e^{i\omega_{ki} t}\right) e^{-iE^{(0)}_k t/\hbar}|k\rangle$$

$$+ \sum_k{}' \frac{1}{\omega_{ki}\hbar}\int^t dt' \frac{\partial V_{ki}(t')}{\partial t'} e^{i\omega_{ki} t'}|k\rangle$$

$$\simeq e^{-\frac{i}{\hbar}V_{ii} t}\, e^{-iE^{(0)}_i t/\hbar}|i\rangle$$

$$-\sum_k{}' \frac{V_{ki}}{E^{(0)}_k - E^{(0)}_i}|k\rangle\cdot e^{-iE^{0}_i t/\hbar} \qquad (!)$$

$$+\sum_k{}' \frac{1}{\omega_{ki}\hbar}\int^t dt' \frac{\partial V_{ki}(t')}{\partial t'} e^{i\omega_{ki} t'}|k\rangle$$

$$\simeq e^{-(E^{(0)}_i + E^{(1)}_i)t/\hbar}|i\rangle^{0+1} + \sum_k{}' (\bullet)|k\rangle$$

OK!

(ii) Questo sistema contiene sia l'effetto statico dovuto allo perturbazione $H_{t=\infty} = H_{t=-\infty} + V(\infty)$ che l'effetto dinamico di transizione quantistiche $(V = V(t))$.

(iii) $H = H(t) \Rightarrow$ L'energia non si conserva.
$\Rightarrow$ Transizione quantistiche.

(iv) Limite adiabatico. $\frac{\partial V_{ki}(t)}{\partial t}$ lento rispetto a $e^{i\omega_{ki} t}$, $P_{ki} \longrightarrow 0$

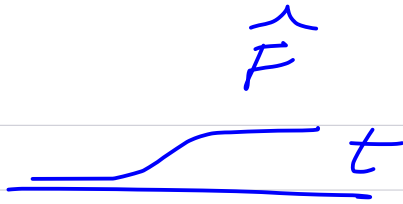
(v) Limite di variazione rapida $\boxed{0 < t < \epsilon}$

$$\underbrace{P_{fi} \simeq \frac{i}{\omega_{fi} \hbar} V_{fi}(\epsilon)}$$

È limite perturbativo di una formula più generale

$$P_{fi} \simeq |\langle f' | i \rangle|^2 \quad (\text{dopo})$$

Es. $V(t) = \int_{-\infty}^{t} dt' \, e^{-\kappa t'^2} \hat{F}$



$$V(\infty) = \sqrt{\frac{\pi}{\kappa}}\, \hat{F}$$

$$\frac{\partial V_{fi}}{\partial t} = e^{-\kappa t^2} \hat{F}_{fi}$$

$$\int dt \left(\partial_t V\right)_{fi} e^{i\omega_{fi} t} = \hat{F}_{fi} \int_{-\infty}^{\infty} dt \, e^{-\kappa t^2 + i\omega_{fi} t}$$

$$\Rightarrow \hat{F}_{fi} \int_{-\infty}^{\infty} dt \, e^{-\kappa\left(t - \frac{i\omega_{fi}}{2\kappa}\right)^2 - \frac{\omega_{fi}^2}{4\kappa}} = \sqrt{\frac{\pi}{\kappa}}\, \hat{F}_{fi} \, e^{-\frac{\omega_{fi}^2}{4\kappa}}$$

$$P_{fi} = \frac{1}{\omega_{fi}^2 \hbar^2} \frac{\pi}{\kappa} |\hat{F}_{fi}|^2 e^{-\frac{\omega_{fi}^2}{2\kappa}}$$

$$\xrightarrow[\kappa \to 0]{} 0 \quad \text{(Variazione adiabatica).}$$

$$\xrightarrow[\kappa \to \infty]{} \frac{1}{\omega_{fi}^2 \hbar^2} \frac{\pi}{\kappa} |\hat{F}_{fi}|^2 \quad \text{(Variazione rapida).}$$

Variazione istantanea: formula nonperturbativa
(Quantum quenching)

$$H_0 \longrightarrow H_0 + V$$
$$t<0 \qquad t=0_+$$
$$|\psi_i\rangle \longrightarrow |\psi_i\rangle$$

$$H_0 |\psi_i^{(0)}\rangle = E_i^{(0)} |\psi_i^{(0)}\rangle \qquad ~~(*)~~$$

$$(H_0+V)|\psi_n\rangle = E_n |\psi_n\rangle \qquad ~~(**)~~$$

La probabilità richiesta è

$$P_{i\to n}\big|_{t>0} = |\langle\psi_n|\psi_i^{(0)}\rangle|^2$$

$$\langle\psi_n|\cancel{(*)} - (\langle\psi_i|\cancel{E_p(**)}\rangle)^*$$

$$\begin{cases} \langle\psi_n|H_0|\psi_i^{(0)}\rangle = E_i^{(0)}\langle\psi_n|\psi_i^{(0)}\rangle \\ \langle\psi_n|H_0+V|\psi_i^{(0)}\rangle = E_n\langle\psi_n|\psi_i^{(0)}\rangle \end{cases}$$

$$\langle\psi_n|V|\psi_i^{(0)}\rangle = (E_n - E_i^{(0)})\langle\psi_n|\psi_i^{(0)}\rangle$$

$$\therefore \langle\psi_n|\psi_i^{(0)}\rangle = \frac{\langle\psi_n|V|\psi_i^{(0)}\rangle}{E_n - E_i^{(0)}}$$

$$\therefore P_{ni} = \left|\frac{\langle\psi_n|V|\psi_i^{(0)}\rangle}{E_n - E_i^{(0)}}\right|^2$$

$$\underset{V\ll H_0}{\Longrightarrow} \qquad P_{ni} \simeq \left|\frac{V_{ni}}{E_n^{(0)} - E_i^{(0)}}\right|^2 = \frac{1}{\omega_{ni}^2\hbar^2}|V_{ni}|^2$$

O.K.

Una digressione

06/03/17 da qui $\Rightarrow$

Teo. perturbazioni indip. dal tempo

$\Downarrow$

Teo perturbazioni dipendenti dal tempo $\Rightarrow$ Sistemi con $H = H(t)$

- Evoluzione temporale generale
- Variazione istantanea (Quantum quenching)
- Teo. adiabatico
- Fase di Berry
- Landau-Zener
- Teo. Oscill. ciclico
- Effetto Zenone quantistico

$\Downarrow$

perturbazioni periodiche

Formula d'oro di Fermi

Evoluzione temporale

$$i\hbar \frac{\partial}{\partial t}|\psi(t)\rangle = H(t)|\psi(t)\rangle$$

$$|\psi(t)\rangle = U(t,0)|\psi(0)\rangle$$

$$\left\{ \begin{array}{l} i\hbar \dfrac{\partial U(t,0)}{\partial t} = H(t)U(t,0) \\ U(0,0) = \mathbb{1} \end{array} \right\} \Longleftrightarrow$$

$$U(t,0) = \mathbb{1} - \frac{i}{\hbar}\int_0^t dt' H(t') U(t',0) \quad \text{(esatto)}$$

$\Rightarrow$ Soluzione

$$U(t) = \mathbb{1} - \frac{i}{\hbar}\int dt' H(t') + \left(-\frac{i}{\hbar}\right)^2 \int^t dt_1 H(t_1) \int_0^{t_1} dt_2 H(t_2)$$

$$+ \dots$$

$$= \mathbb{1} - \dots + \frac{1}{n!}\int_0^t dt_1 dt_2 \dots \int_0^t dt_n\, T\left(H(t_1)H(t_2)\cdots H(t_n)\right)$$

$$+ \dots$$

dove $T(H(t_1)\dots H(t_n)) \equiv$

$$H(t_1)\dots H(t_n) \quad \text{se } t_1 > t_2 > \dots t_n$$

$$H(t_{i_1})\dots H(t_{i_n}) \quad \text{se } t_{i_1} > t_{i_2} > \dots t_{i_n}.$$

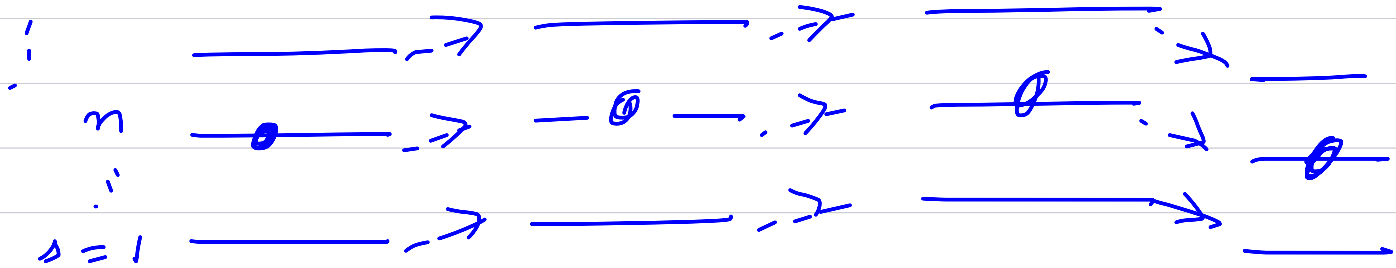
$$\equiv T\left\{e^{-\frac{i}{\hbar}\int_0^t dt\, H(t)}\right\} \quad \text{scritta simbolica}$$

Teorema adiabatico

autostati istantanei (adiabatici)

$$H(t)|\psi_s(t)\rangle = E_s(t)|\psi_s(t)\rangle \qquad (s=1,2,\dots)$$

N.B. $|\psi_s(t)\rangle$ non rappresenta l'evoluzione temporale in generale.



$E_n(t_1)$ $E_n(t_2)$ $E_n(t_3)$ $E_n(t_4)$

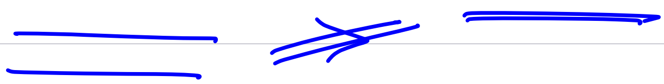
$$|\psi(0)\rangle = |\psi_n\rangle \,, \quad t=0$$

Teo adiabatico:

Nel limite di variazione lenta

$$|\psi(t)\rangle \simeq e^{i\Delta}|\psi_n(t)\rangle$$

Ipotesi: durante l'evoluzione temporale nessuna coppia di stati adiabatici diventano degeneri.



Dimostrazione:

$$|\Psi(t)\rangle = \sum_s c_s(t)\, e^{-\frac{i}{\hbar}\int_0^t dt' E_s(t')}\, |\psi_s(t)\rangle, \quad \text{con } c_s(0) = \delta_{s,n}$$

Def.
$$\delta_s(t) \equiv \frac{1}{\hbar}\int^t dt'\, E_s(t') \ : \ \text{fase dinamica}$$

Eq. Sch.: $i\hbar \frac{\partial}{\partial t}|\Psi(t)\rangle = H(t)\,|\Psi(t)\rangle$.

$$i\hbar\frac{\partial}{\partial t}|\Psi(t)\rangle = \sum_s \left(i\hbar \dot{c}_s |s\rangle + \cancel{E_s(t)\, c_s |s\rangle} + c_s\, i\hbar \frac{\partial}{\partial t}|s\rangle \right) e^{-\frac{i}{\hbar}\int}$$

$$= H(t)|\Psi(t)\rangle = \cancel{\sum_s c_s\, e^{-\frac{i}{\hbar}\int} E_s(t)\, |s\rangle}$$

$$\langle k(t)|\, e^{\frac{i}{\hbar}\int E_k} \Rightarrow$$

$$\dot{c}_k(t) = -\sum_s c_s(t)\, e^{\frac{i}{\hbar}\int dt' (E_k(t') - E_s(t'))} \langle k|\frac{\partial}{\partial t}|s\rangle$$

ancora esatto !!

$\langle k|\frac{\partial}{\partial t}|s\rangle$?

$$H(t)|s\rangle = E_s|s\rangle$$

$$\frac{\partial H(t)}{\partial t}|s\rangle + H(t)\frac{\partial}{\partial t}|s\rangle = \frac{\partial E_s}{\partial t}|s\rangle + E_s \frac{\partial}{\partial t}|s\rangle$$

$\langle k|$: $k \neq s$
$$\langle k|\frac{\partial H}{\partial t}|s\rangle + E_k \langle k|\frac{\partial}{\partial t}|s\rangle = E_s \langle k|\frac{\partial}{\partial t}|s\rangle$$

$$\therefore \ \langle k|\frac{\partial}{\partial t}|s\rangle = \frac{\langle k|\partial H/\partial t|s\rangle}{E_s - E_k}$$

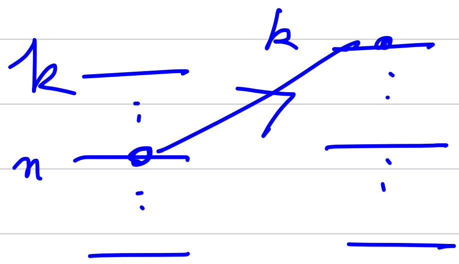
Per $k = s$, $\frac{\partial}{\partial t}\langle k|k\rangle = 0$. $\langle k|\frac{\partial}{\partial t}|k\rangle = -\frac{\partial}{\partial t}\langle k|\,k\rangle$

$$\therefore \ \langle k|\frac{\partial}{\partial t}|k\rangle = i\gamma_k \quad (\gamma_k \in \mathbb{R})$$

Ora, per $c_s(0) = \delta_{s,n}$, $|\psi(0)\rangle = |n\rangle$, $|\psi(t)\rangle \xrightarrow[t\to 0]{} |n\rangle$

$$\dot{c}_k = -\sum_{s\neq k} c_s(t)\cdot e^{\frac{i}{\hbar}\int dt' \left(E_k(t')-E_s(t')\right)} \frac{\langle k|\frac{\partial H}{\partial t}|s\rangle}{E_s - E_k} - i c_k \gamma_k$$

$$\underset{k\neq n}{\Longrightarrow} \quad \dot{c}_k \simeq -\underset{\simeq 1}{c_n} \frac{e^{\frac{i}{\hbar}\int^t (E_k - E_n)dt'}}{E_n - E_k} \langle k|\frac{\partial H}{\partial t}|n\rangle$$



$$\simeq -\frac{e^{\frac{i}{\hbar}(E_k - E_n)t}}{E_n - E_k}\langle k|\frac{\partial H}{\partial t}|n\rangle$$

$\Rightarrow$

$$\underset{=0}{c_k(t) - c_k(0)} \simeq +\frac{\hbar}{i}\frac{e^{\frac{i}{\hbar}(E_k-E_n)t} - 1}{(E_n - E_k)^2}\langle k|\frac{\partial H}{\partial t}|n\rangle$$

$\therefore$

$c_k(t) \simeq 0$ $(k\neq n)$ se $\boxed{\left|\frac{\hbar\langle k|\frac{\partial H}{\partial t}|n\rangle}{(E_n - E_k)^2}\right| \ll 1}$

N.B. Ipotesi $E_k \neq E_n$ $(k\neq n)$
(Non degeneracy).

$k=n \Rightarrow$ Fase di Berry.

Particella confinata in $[0, L]$

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin k_n x$$

$$k_n = \frac{\pi n}{L},$$

$$E_n = \frac{k_n^2 \hbar^2}{2m} = \frac{\pi^2 \hbar^2}{2mL^2} n^2$$

$$\Delta E_n \propto \frac{\hbar^2}{m^2 L(t)^2}$$

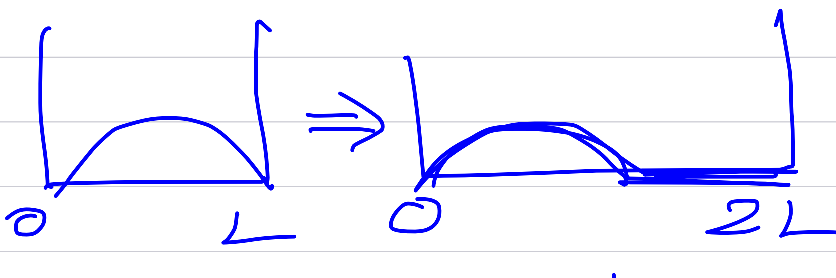
$v = \dot{L}$ fisso. $L(t) = L_0 + vt$

$v \ll 1$: variazione adiabatica

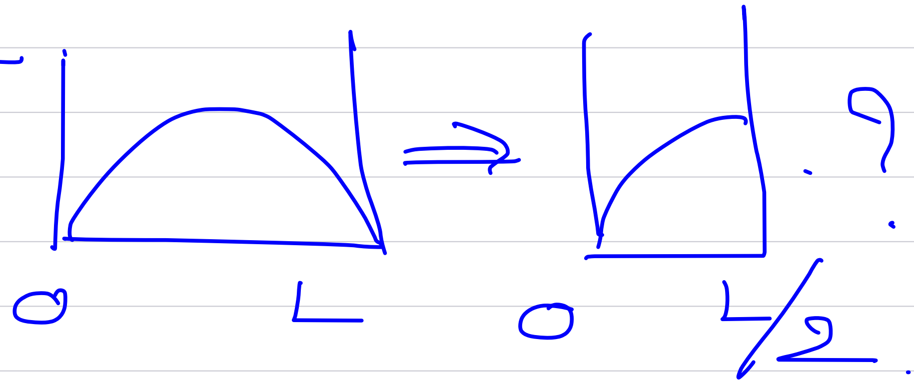
• ma a $t \to \infty$?

Variazione rapida.

Espansione



Contrazione



Landau-Zener transition

Landau, Zener, Stuckelberg, Majorana (1932)

Sistemi a due stati con

$$H = \begin{pmatrix} E_1(t) & F \\ F^* & E_2(t) \end{pmatrix}, \quad \text{con } E_1 = E_0 - \frac{\nu}{2} t, \quad E_2 = E_0 + \frac{\nu}{2} t, \quad (\nu > 0)$$

$\therefore \; E_2(t) - E_1(t) = \nu t.$

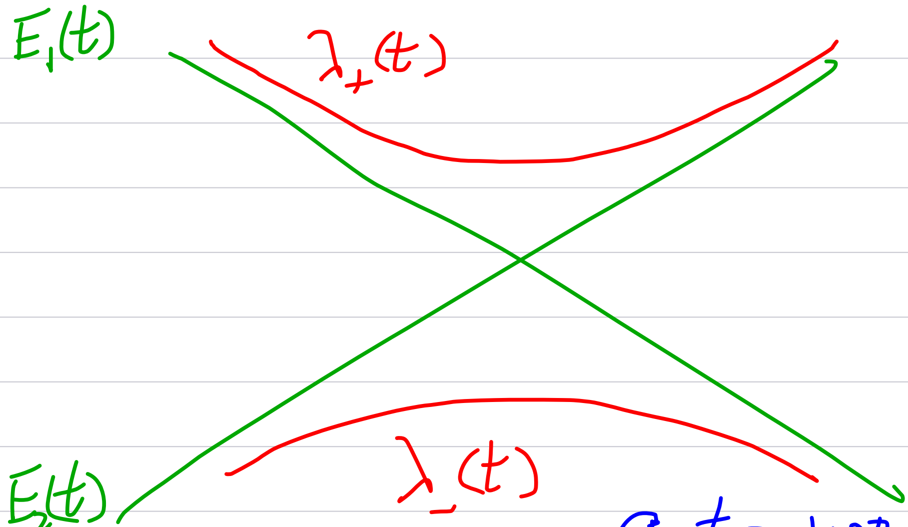
$\Rightarrow$ autovalori istantanei

$$\det \begin{vmatrix} E_1 - \lambda & F \\ F^* & E_2 - \lambda \end{vmatrix} = 0 \Rightarrow$$

$$\lambda^2 - (E_1 + E_2)\lambda + E_1 E_2 - |F|^2 = 0.$$

$$\lambda_\pm = \frac{1}{2}\left\{ E_1 + E_2 \pm \sqrt{(E_1 - E_2)^2 + 4|F|^2} \right\} \equiv \lambda_\pm(t).$$

autovalori adiabatici



a $t \to -\infty$

$\lambda_-(t) \longrightarrow E_2$ (st. fond)

$\lambda_+(t) \longrightarrow E_1$ (st. ecc.)

a $t \to +\infty$

$\lambda_-(t) \to E_1$ (st. fond.)

$\lambda_+(t) \to E_2$ (st. ecc.)

N.B. Nè $E_i(t)$ nè $\lambda_\pm(t)$ rappresentano di per se lo stato fisico e evoluzione.

Supp. che $|\psi(-\infty)\rangle = |\lambda_-\rangle$

Q: $P\Big|_{|\psi(\infty)\rangle = |\lambda_+\rangle}$?

Soluzione: pongo

Wittig (2005)

$$|\psi(t)\rangle = \begin{pmatrix} A(t)\, e^{-\frac{i}{\hbar}\int^t dt' E_1(t')} \\ B(t)\, e^{-\frac{i}{\hbar}\int^t dt' E_2(t')} \end{pmatrix}$$

Con $\begin{pmatrix} A(-\infty) = 0 \\ B(-\infty) = 1 \end{pmatrix} \Leftrightarrow |\psi(-\infty)\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

$$P_{ecc.}(\infty) = |B(\infty)|^2$$

Risultato (esatto): $P_{ecc.}(\infty) = e^{-2\pi|F|^2/v\hbar}$ $\xrightarrow{v\to 0}$ 0 (lim. adiabatico); $\xrightarrow{v\to\infty}$ 1 (variaz. rapida).

$\because\ i\hbar \frac{\partial}{\partial t}|\psi(t)\rangle = H(t)\,|\psi(t)\rangle$

$$\begin{pmatrix} i\hbar \dot{A}\, e^{-i\int E_1} + \underline{A \cdot E_1 e^{-i\int E_1}} \\ i\hbar \dot{B}\, e^{-i\int E_2} + \underline{B\, E_2 e^{-i\int E_2}} \end{pmatrix} =$$

$$\begin{pmatrix} E_1(t) & F \\ F^* & E_2(t) \end{pmatrix} \begin{pmatrix} A e^{-i\int E_1} \\ B e^{-i\int E_2} \end{pmatrix} = \begin{pmatrix} \underline{A E_1 e^{-i\int E_1}} + B F e^{-i\int E_2} \\ A F^* e^{-i\int E_1} + \underline{B E_2 e^{-i\int E_2}} \end{pmatrix}$$

$$i\hbar \dot A\, e^{-i\int E_1} = F B\, e^{-i\int E_2}$$
$$i\hbar \dot B\, e^{-i\int E_2} = F^* A\, e^{-i\int E_1} \quad \Longrightarrow$$

$$\begin{cases} i\hbar \dot A = F\cdot B\, e^{-i\int \omega(t')} \\ i\hbar \dot B = F^* A\, e^{i\int \omega(t')} \end{cases} \qquad \left(\omega(t) \equiv (E_2(t)-E_1(t))/\hbar = \frac{\nu t}{\hbar} \right)$$

$$\Rightarrow \begin{cases} i\hbar \ddot A = F \dot B e^{-i\int\omega} - i\omega F B e^{-i\int\omega} = \frac{1}{i\hbar}|F|^2 A + \omega\hbar \dot A \\ i\hbar \ddot B = F^* \dot A e^{i\int\omega} + i\omega F^* A e^{i\int\omega} = \frac{1}{i\hbar}|F|^2 B - \omega\hbar \dot B \end{cases}$$

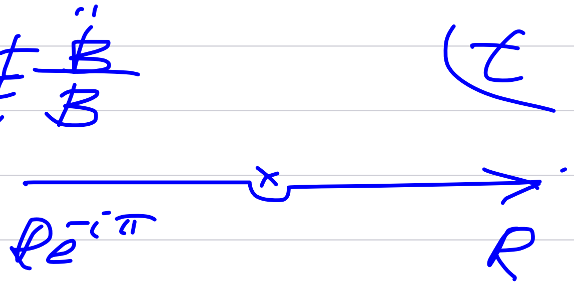
$$\Rightarrow \begin{cases} \ddot A + \frac{|F|^2}{\hbar^2} A + i\omega \dot A = 0 ; \\ \ddot B + \frac{|F|^2}{\hbar^2} B - i\omega \dot B = 0 \end{cases}$$

$B(-\infty)=1$, $A(-\infty)=0$; $\omega(t)=\nu t/\hbar$

$$\frac{\ddot B}{B} + i\omega \frac{\dot B}{B} + \frac{|F|^2}{\hbar^2} = 0 \qquad (*)$$

$$\int \frac{dt}{t}(*) \Rightarrow 2^\circ = \frac{i\nu}{\hbar}\int_0^\infty dt\, \frac{\dot B}{B} = \frac{i\nu}{\hbar}\lg B(\infty)$$

$$\therefore \frac{i\nu}{\hbar}\lg B(\infty) = \frac{|F|^2}{\hbar^2}\int_C \frac{dt}{t} + \int_C \frac{dt}{t}\frac{\ddot B}{B}$$



$$1^\circ = \frac{|F|^2}{\hbar^2}\lg\frac{R}{Re^{-i\pi}} = i\pi\frac{|F|^2}{\hbar^2}$$

$$2^\circ = 2\pi i\, \mathrm{Res}\Big|_{t=0} = 2\pi i \frac{\ddot B}{B}\Big|_{t=0} = -2\pi i \frac{|F|^2}{\hbar^2}$$

$$\therefore \frac{i\nu}{\hbar} \log B(\infty) = -\pi i \frac{|F|^2}{\hbar^2}$$

$$\therefore B(\infty) = e^{-\pi|F|^2/\nu\hbar} \quad \text{(esatto!)}$$

$$\begin{cases} \to 0 & \nu \to 0 \ \text{(adiab.)} \\ \to 1 & \nu \to \infty. \ \text{(variaz. rapida)} \end{cases}$$

Ci sono moltissime applicazioni.

Fisica nucleari e atomiche,

Sistemi laser-atomi, atomi in reticolo ottico;

Oscillazione di neutrini; ...

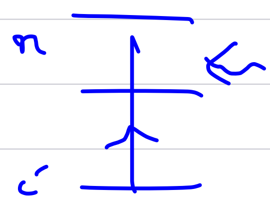
Perturbazioni periodiche

$$V(t) = \theta(t)\left\{ \hat{F} e^{-i\omega t} + \hat{F}^{\dagger} e^{i\omega t} \right\}$$

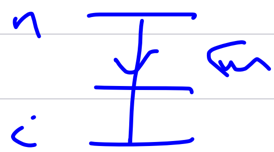
$$a_{ni}^{(1)} = -\frac{i}{\hbar}\int_0^t dt'\, V_{ni}(t')\, e^{i\omega_{ni} t'}$$

$$= -\frac{1}{\hbar}\left\{ F_{ni}\, \frac{e^{i(\omega_{ni}-\omega)t} - 1}{\omega_{ni}-\omega} + \hat{F}^{\dagger}_{ni}\, \frac{e^{i(\omega_{ni}+\omega)t} - 1}{\omega_{ni}+\omega} \right\}$$

1° termine dominante se $\omega_{ni} - \omega \simeq 0$

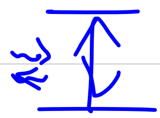


2° " " se $\omega_{ni} + \omega \simeq 0$

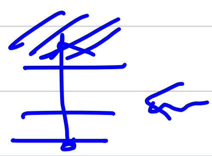


Studieremo: 3 tipi di transizioni

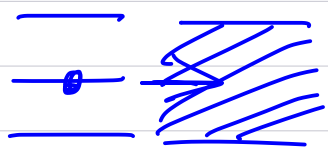
(1) transizioni tra due livelli discreti;



(2) Discreto → continuo (ionizzazione)



(3) Decadimento



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Preparazione.

$$\frac{e^{i(\omega_{ni}-\omega)t}-1}{\omega_{ni}-\omega} = e^{\frac{i(\omega_{ni}-\omega)t}{2}} \cdot 2i\,\frac{\sin\frac{(\omega_{ni}-\omega)t}{2}}{\omega_{ni}-\omega}$$

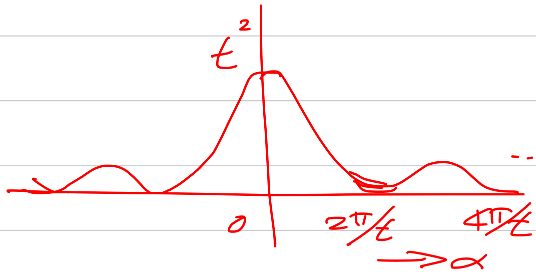
$$a^{(1)}_{ni} = -\frac{i}{\hbar}\left[e^{\frac{i(\omega_{ni}-\omega)t}{2}} F_{ni}\frac{\sin\frac{(\omega_{ni}-\omega)t}{2}}{(\omega_{ni}-\omega)/2} + F^{t}_{ni}\cdot\begin{pmatrix}\omega_{ni}-\omega\\ \to \omega_{ni}+\omega\end{pmatrix}\right]$$

Def: $D(t,\alpha) = \left(\frac{\sin\alpha t/2}{\alpha/2}\right)^2$, $\alpha = \omega_{ni}-\omega \; \{\omega_{ni}+\omega\}$

$$P_n(t) = |a^{(1)}_n|^2 = \frac{1}{\hbar^2}\Big[|F_{ni}|^2 D(t,\omega_{ni}-\omega) + |F_{in}|^2 D(t,\omega_{ni}+\omega)$$
$$+ 2\,\mathrm{Re}\,F_{ni}F_{in}\,e^{-i\omega t}\sqrt{D(t,\omega_{ni}-\omega)D(t,\omega_{ni}+\omega)}\Big]$$

Proprietà di D.

$$D(t,\alpha) = \left(\frac{\sin\alpha t/2}{\alpha/2}\right)^2$$
$$= t^2 \quad (\alpha = 0)$$
$$= 0 \quad (\alpha = 2\pi n/t)$$



$$D(t,\alpha) \xrightarrow[t\to\infty]{} 2\pi t\,\delta(\alpha)$$

(Rem. $\lim_{L\to\infty}\frac{\sin^2 xL}{\pi L x^2} = \delta(x)$)

(i) ω non risonante $(\neq \pm\omega_{ni})$

$t \gg 1/\omega \Rightarrow \overline{\sin^2 x} = 1/2$, $\overline{\cos x} = \overline{\sin x} = 0$

$D \simeq \frac{2}{\alpha^2}$,

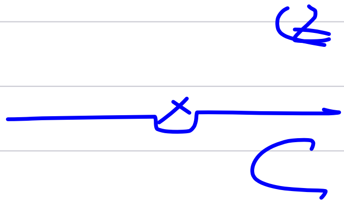
$\to P_{ni} \simeq \frac{4|F_{ni}|^2}{\hbar^2\omega^2} \quad (\omega \gg |\omega_{ni}|)$

$$\lim_{L\to\infty}\int_a^b \frac{\sin^2 Lx}{\pi L x^2} f(x)\,dx \qquad \begin{pmatrix} a<0 \\ b>0 \end{pmatrix}$$

$$= \lim \frac{1}{\pi}\int_{aL}^{bL} dz\, \frac{\sin^2 z}{z^2} f\left(\frac{z}{L}\right)$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} dz\, \frac{\sin^2 z}{z^2}\left\{ f(0) + O\left(\frac{1}{L}\right)\right\}$$

$$= \frac{f(0)}{\pi}\underbrace{\int_{-\infty}^{\infty} dz\, \frac{\sin^2 z}{z^2}}$$



$$= -\frac{1}{4}\int_C \frac{\left(e^{iz}-e^{-iz}\right)^2}{z^2} = -\frac{1}{4}\left\{\int_C \frac{e^{2iz}}{z^2} - \int_C \frac{2\,dz}{z^2} + \int_C \frac{e^{-2iz}}{z^2}\right\}$$

$$= -\frac{1}{4}\cdot(2\pi i)\cdot 2i + 0 + 0 .$$

$$= \pi .$$

② Frequenza risonante. $(\omega \simeq \pm\omega_{ni})$

$\boxed{\omega = \omega_{ni}} \Rightarrow P_{ni} \simeq \frac{|F_{ni}|^2}{\hbar^2} t^2 \to \infty$

chiaramente la formula pert. è valida solo

per $\frac{|F_{ni}|^2}{\hbar^2} t^2 \ll 1$. $t \ll \left|\frac{\hbar}{F_{ni}}\right|$

$\Rightarrow$ Oscillazione tra due livelli.

$\Rightarrow$ Approssimazione alternativa.

- Trascurare tutti gli altri livelli, ma altrimenti trattare il problema esattamente, senza l'app. perturbat.

$$\Rightarrow H = \begin{pmatrix} E_0/2 & F e^{-i\omega t} \\ F^* e^{i\omega t} & -E_0/2 \end{pmatrix} \qquad \begin{aligned} E_f &\equiv E_0/2 \\ E_i &\equiv -E_0/2 \end{aligned}$$

Pongo $E_0 \equiv \omega_0 \hbar$; $F \equiv \omega_1 \hbar/2$

$$\Rightarrow H = \frac{\hbar}{2}\begin{pmatrix} \omega_0 & \omega_1 e^{-i\omega t} \\ \omega_1 e^{i\omega t} & -\omega_0 \end{pmatrix} \Rightarrow i\hbar \frac{\partial}{\partial t}|\psi(t)\rangle = H(t)|\psi(t)\rangle$$

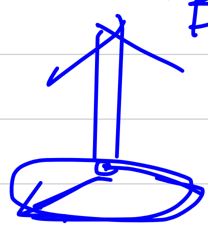
$$|\psi(0)\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

N.B. È equivalente al sistema di spin 1/2

con $\mu = -g S$, posto in campo magnetico

$\mathbf{B} = B_0 \hat{z} + \mathbf{B}_\perp$, $\quad H = -\mu \cdot \mathbf{B}$,

con $\mathbf{B}_\perp = (B_1 \cos\omega t, B_1 \sin\omega t, 0)$



$$H = -\frac{g\mu}{2} \sigma \cdot \mathbf{B}$$

$$= -\frac{g\mu}{2} \left\{ B_0 \sigma_z + B_1 \cos\omega t \, \sigma_1 + B_1 \sin\omega t \, \sigma_2 \right\}$$

$$= -\frac{g\mu}{2} \begin{pmatrix} B_0 & e^{-i\omega t} B_1 \\ e^{i\omega t} B_1 & -B_0 \end{pmatrix}$$

Ponendo

$$\hbar\omega_0 = g\mu B_0 \, ;$$

$$\hbar\omega_1 = g\mu B_1$$

$$H = \frac{\hbar}{2} \begin{pmatrix} \omega_0 & \omega_1 e^{-i\omega t} \\ \omega_1 e^{i\omega t} & -\omega_0 \end{pmatrix} \qquad !$$

$\Longleftrightarrow$ NMR

$$|B_1| \ll |B_0|$$

$$|\psi(t)\rangle = \begin{pmatrix} |\psi_1(t)\rangle \\ |\psi_2(t)\rangle \end{pmatrix} \underset{t=0}{=} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$i\frac{\partial}{\partial t}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \omega_0 & \omega_1 e^{-i\omega t} \\ \omega_1 e^{i\omega t} & -\omega_0 \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

Pongo: $\begin{cases} \psi_1(t) = e^{-i\omega t/2}\, u_1(t) \\ \psi_2(t) = e^{i\omega t/2}\, u_2(t) \end{cases}$

$$\Rightarrow \quad i\frac{\partial u_1}{\partial t} + \frac{\omega}{2}u_1 = \frac{\omega_0}{2}u_1 + \frac{\omega_1}{2}u_2$$

$$i\frac{\partial u_2}{\partial t} - \frac{\omega}{2}u_2 = \frac{\omega_1}{2}u_1 - \frac{\omega_0}{2}u_2$$

or

$$\begin{cases} i\frac{\partial u_1}{\partial t} = \frac{\omega_0-\omega}{2}u_1 + \frac{\omega_1}{2}u_2 \\ i\frac{\partial u_2}{\partial t} = \frac{\omega_1}{2}u_1 - \frac{\omega_0-\omega}{2}u_2 \end{cases}$$

or

$$i\frac{\partial}{\partial t}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \tilde{H}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$ con

$$\tilde{H} = \frac{1}{2}\begin{pmatrix} \omega_0-\omega & \omega_1 \\ \omega_1 & -(\omega_0-\omega) \end{pmatrix} = \frac{1}{2}\left\{(\omega_0-\omega)\sigma_3 + \omega_1\sigma_1\right\}$$

$$= \mathbf{b}\cdot\boldsymbol{\sigma}, \quad \mathbf{b} = \left(\frac{\omega_1}{2}, 0, \frac{\omega_0-\omega}{2}\right)$$

Visto che "l'Hamiltoniana" $\tilde{H}$, è indip. da t,

la sol. è:

$$\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = e^{-i\tilde{H}t} \begin{pmatrix} u_1(0) \\ u_2(0) \end{pmatrix} \to \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$\left(e^{i\mathbf{a}\cdot\boldsymbol{\sigma}} = \cos|\mathbf{a}| + i\frac{\mathbf{a}\cdot\boldsymbol{\sigma}}{|\mathbf{a}|}\sin|\mathbf{a}| \right)$

dove

$$e^{-i\tilde{H}t} = e^{-it(\mathbf{b}\cdot\boldsymbol{\sigma})} = \cos bt - i\frac{\mathbf{b}\cdot\boldsymbol{\sigma}}{b}\sin bt$$

$$= \begin{pmatrix} \cos bt - i\frac{\omega-\omega_0}{2\Omega}\sin bt & -i\frac{\omega_1}{2\Omega}\sin bt \\ -i\frac{\omega_1}{2\Omega}\sin bt & \cos bt + i\frac{\omega_0-\omega}{2\Omega}\sin bt \end{pmatrix}$$

$$\mathbf{b} = \left(\frac{\omega_1}{2}, 0, \frac{\omega_0-\omega}{2}\right)$$

$$b = \sqrt{\mathbf{b}^2} = \frac{1}{2}\sqrt{\omega_1^2 + (\omega_0-\omega)^2} \equiv \Omega$$

$\therefore$

$$\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = \begin{pmatrix} -i\frac{\omega_1}{2\Omega}\sin bt \\ \cos bt + i\frac{\omega_0-\omega}{2\Omega}\sin bt \end{pmatrix}$$

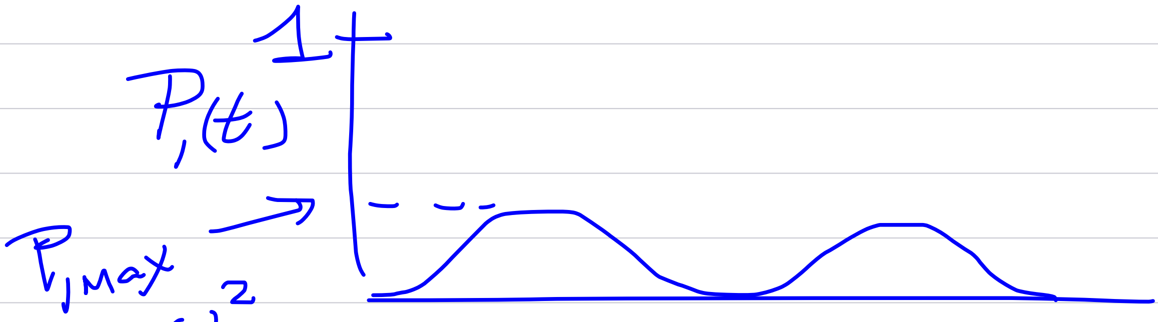
$\therefore$

$$\begin{cases} P_1(t) = \frac{\omega_1^2}{4\Omega^2}\sin^2 bt \\ P_2(t) = \cos^2 bt + \frac{(\omega_0-\omega)^2}{4\Omega^2}\sin^2 bt \end{cases}$$

$P_1(t) + P_2(t) = 1$, ok.

$$P_1 = P_{1\,Max} = \frac{\omega_1^2}{4\Omega^2} \quad \text{a} \quad t = \frac{\pi n}{b}$$

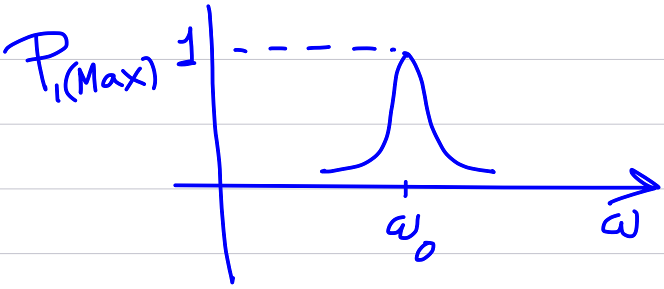
$$= \frac{\omega_1^2}{\omega_1^2 + (\omega_0-\omega)^2}$$



$$= \frac{\omega_1^2}{4\Omega^2} = \omega_1^2 / \omega_1^2 + (\omega_0 - \omega)^2$$

$P_{1Max} \ll 1$ se $\omega_1 \ll \omega_0$ $\left(\begin{matrix} F \ll E_0 ; \\ B_\perp \ll B_0 \end{matrix} \right)$

ma per frequenze risonanti



$E_0 \equiv \omega_0 \hbar$; $F \equiv \omega_1 \hbar / 2$

$$P_{fi} = P_1 = \frac{\omega_1^2}{4\Omega^2} \sin^2 \Omega t = \frac{4|F|^2 \sin^2 \Omega t}{\hbar^2 \left\{ (\omega_0 - \omega)^2 + \frac{4|F|^2}{\hbar^2} \right\}}$$

- $\Omega \equiv$ Frequenza di Rabi
- NMR,
- Determinazione precisa di μ, Q, ...
  (metodo di risonanza)

Transizione discreto $\to$ continuo : ionizzazione.

$$P_{fi} = \frac{|F_{fi}|^2}{\hbar^2} \frac{\sin^2 \frac{\omega_{fi}-\omega}{2} t}{\left(\frac{\omega_{fi}-\omega}{2}\right)^2}$$

@ $t \to \infty$,

$$\frac{\sin^2 \frac{\omega_{fi}-\omega}{2} t}{\left(\frac{\omega_{fi}-\omega}{2}\right)^2} \longrightarrow 2\pi t\, \delta(\omega_{fi}-\omega)$$

$$= 2\pi\hbar\, t\, \delta(E_f - E_i - \omega\hbar)$$

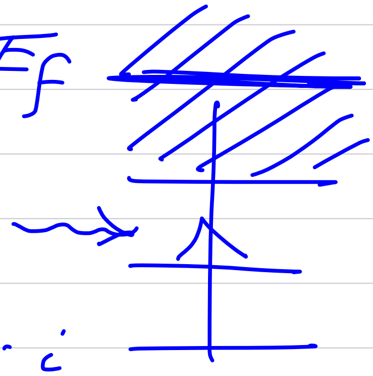
$$\therefore P_{fi}(t) = \frac{|F_{fi}|^2}{\hbar^2} 2\pi\hbar\, \delta(E_f - E_i - \omega\hbar) \cdot t \rightarrow \;?$$

$\Downarrow$ conservazione dell'energia.

$$\Rightarrow w_{fi} = \frac{2\pi |F_{fi}|^2}{\hbar} \delta(E_f - E_i - \omega\hbar) \equiv \text{prob. transiz. per unità di } t \text{ (rate)}.$$

Ma non basta. $\Rightarrow$

$$dw_{fi} = \frac{2\pi}{\hbar} |F_{fi}|^2 \delta(E_f - E_i - \omega\hbar)\, d\Phi$$



dove

$$d\Phi = [\#\text{ di stati in } (E_f, E_f + dE_f)]$$

$\equiv$ densità di stati

$\equiv \rho(E)\, dE\, d\rho \rightarrow$ altri # quantici

Particella in 3D

$$\psi_p(r) = e^{i p \cdot r/\hbar}\,.$$

$$\langle p' | p \rangle = (2\pi\hbar)^3 \delta^3(p'-p)\,,$$

$$\sum_n |n\rangle\langle n| = \mathbb{1}$$

$$\Rightarrow \quad d\Phi_p = \frac{d^3 p}{(2\pi\hbar)^3} \Longleftarrow \quad \sum_p |p\rangle\langle p| = \mathbb{1}$$

Infatti:

$$\langle p_2 | p_1 \rangle = \int \frac{d^3 p}{(2\pi\hbar)^3} \langle p_2 | p \rangle \langle p | p_1 \rangle \qquad \int d\Phi_p |p\rangle\langle p| = \mathbb{1}$$

$$= \int \frac{d^3 p}{(2\pi\hbar)^3} \left((2\pi\hbar)^3\right)^2 \delta(p_2 - p)\, \delta(p - p_1)$$

$$= (2\pi\hbar)^3 \delta(p_2 - p_1) \quad \text{ok}.$$

$$E = \frac{p^2}{2m}$$

$$dE = \frac{p\, dp}{m}$$

$$d\Phi_p = \frac{d^3 p}{(2\pi\hbar)^3} = \frac{dp\, p^2 d\Omega}{(2\pi\hbar)^3} = \frac{m p\, dE\, d\Omega}{(2\pi\hbar)^3}\,,$$

$$= \rho(E)\, dE\, d\Omega$$

$$\rho(E) = \frac{m p}{(2\pi\hbar)^3}$$

N.B. $\psi_p(r) = C\, e^{i p \cdot r/\hbar}\,, \quad |p\rangle \to C |p\rangle$

$$\Rightarrow \quad \langle p' | p \rangle = C^2 (2\pi\hbar)^3 \delta^3(p' - p)$$

$$d\Phi_p \to d\Phi_p / C^2\,,$$

ma $|F_{fi}|^2 = |\langle p | \hat{F} | i \rangle|^2 \to C^2 |F_{fi}|^2$

$\therefore \; dW_{fi}$ indip. da C !

$$dW_{fi} = \frac{2\pi}{\hbar} |F_{fi}|^2 \delta(E_f - E_i - \omega\hbar)\, d\Phi$$

$$= \frac{2\pi}{\hbar} |F_{fi}|^2 \delta(E_f - E_i - \omega\hbar)\, \rho(E_f)\, dE_f\, d\Omega$$

$$= \underbrace{\frac{2\pi}{\hbar} |F_{fi}|^2 \rho(E_f)\, d\Omega}\Big|_{E_f = E_i + \omega\hbar}$$

Formula d'oro di

FERMI

fin qui 07/03/17

Densità di stati in 1D.

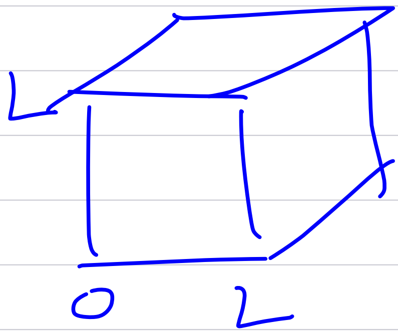
$E = \frac{p^2}{2m}$. $\psi_p(x) = e^{ipx/\hbar}$; $\langle p'|p\rangle = 2\pi\hbar\,\delta(p'-p)$ $\quad dE = \frac{p\,dp}{m}$

$$d\Phi = \frac{dp}{2\pi\hbar} = \frac{m\,dE}{2\pi\hbar\,p} \underset{p=\pm|p|}{\Longrightarrow} \frac{m}{\pi\hbar p}\,dE = \int \rho(E)\,dE$$

$$\rho(E) = \frac{m}{\pi\hbar p} = \frac{m}{\pi k \hbar^2}$$

Densità di stati dalla quantizzazione in una scatola periodica

3D: $\psi_{\{n\}} = L^{-3/2} e^{i k_n \cdot r}$ $\left(k_x = \frac{2\pi n_x}{L}\ \text{etc.}\right)$



$$E_{(n_x \cdots n_z)} = \frac{\pi^2}{2mL^2}(n_x^2+n_y^2+n_z^2)$$

$\#$ stati $(n_x, n_x+dn_x)(n_y, n_y+dn_y)(n_z, n_z+dn_z)$

$$= dn_x\,dn_y\,dn_z = \left(\frac{L}{2\pi}\right)^3 d^3k = L^3 \frac{d^3p}{(2\pi\hbar)^3} = d\Phi$$
$$= \rho\,dE\,d\Omega$$

$$dw = \frac{2\pi}{\hbar}|F_{fi}|^2 \rho\,d\Omega$$

$\rho = L^3 \rho_{cont.}$, ma $|F_{fi}|^2 = |\langle p_n|F|i\rangle|^2$

$$= L^{-3}\left|\int dr\, e^{-ip\cdot r} F \psi_i\right|^2$$

$$= |F_{fi}|^2_{cont}.$$

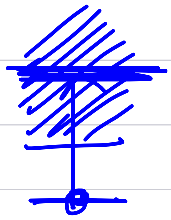
analogamente per 1D

Esercizio $H = H_0 + V$

$$H_0 = \frac{p^2}{2m} - g\delta(x) \; ; \; V = \hat{F}\cdot e^{-i\omega t} + h.c.$$

$$\hat{F} = f\cdot x$$

$$\psi_0 = \sqrt{\kappa}\, e^{-\kappa|x|}$$



$$\kappa = mg/\hbar^2 : \quad E_0 = -\frac{mg^2}{2\hbar^2}$$

• Prendo $\psi_p(x) = e^{+ipx/\hbar} = e^{+ikx}$ $(k \geq 0)$.

$$F_{fi} = \int dx\, e^{-ikx} f\cdot x \sqrt{\kappa}\, e^{-\kappa|x|}$$

$$= f\sqrt{\kappa}\cdot\left(\int_0^\infty dx\, x e^{-\kappa x - ikx} + \int_{-\infty}^0 dx\, x e^{\kappa x - ikx}\right)$$

$$= f\sqrt{\kappa}\cdot\left(-\frac{\partial}{\partial\kappa}\int_0^\infty dx\, e^{-(\kappa+ik)x} + \frac{\partial}{\partial\kappa}\int dx\, e^{(\kappa-ik)x}\right)$$

$$= f\sqrt{\kappa}\cdot\left(-\frac{\partial}{\partial\kappa}\frac{1}{\kappa+ik} + \frac{\partial}{\partial\kappa}\frac{1}{\kappa-ik}\right) =$$

$$= f\sqrt{\kappa}\left(\frac{1}{(\kappa+ik)^2} - \frac{1}{(\kappa-ik)^2}\right)$$

$$= f\sqrt{\kappa}\,\frac{-4ik\kappa}{(\kappa^2+k^2)^2}$$

$$\therefore\; w = \frac{2\pi}{\hbar}|F_{fi}|^2 \rho(E_f)\Big|_{E_f = E_i + \omega\hbar}$$

$$= \frac{2\pi}{\hbar}\cdot f^2\frac{16k^2\kappa^3}{(\kappa^2+k^2)^4}\cdot\frac{m}{\pi k\hbar^2} = \frac{32mf^2}{\hbar^3}\,\frac{k\kappa^3}{(\kappa^2+k^2)^4}$$

dove $k = \frac{p}{\hbar} = \frac{\sqrt{2m}}{\hbar}\sqrt{\omega\hbar - \frac{mg^2}{2\hbar^2}}$

Osservazioni

(i) $\exists$ soglia per l'ionizzazione, $\omega\hbar > -E_0 = \frac{m\rho^2}{2\hbar^2}$

(al di sotto, non succede nulla!).

(ii) Verifica dimensione. $V = f \cdot x = [E]$

$$D[w] = D\left[\frac{m f^2}{\hbar^3} \cdot k^{-4}\right] = m \cdot \frac{(f k^{-1})^2 k^{-2}}{\hbar^3}$$

$$\Rightarrow M \left(\frac{ML^2}{T^2}\right)^2 \cdot L^2 \frac{1}{\left(\frac{ML^2}{T}\right)^3} = \frac{1}{T}, \text{ OK}$$

(iii). Paradosso?

$w \neq 0$ anche per $\hat{F} = f$ (costante)?

$$V = \hat{F} \cdot e^{-i\omega t} + h.c.$$

$$V_{fi} = f \langle \psi^{(k)} | \psi^{(i)} \rangle$$

$$= f\sqrt{K}\left(\int_0^{\infty} dx\, e^{-Kx - ikx} + \int_{-\infty}^{0} dx\, e^{Kx - ikx}\right)$$

$$= f\sqrt{K} \cdot \left(\frac{1}{K+ik} + \frac{1}{K-ik}\right) = f\sqrt{K}\, \frac{2K}{K^2+k^2} \neq 0$$

$\Longrightarrow$??

--> Richiede un trattamento corretto di ψ_{fin}.

(iv) Stati del continuo (corretto). $V = -g\,\delta(x)$

$$-\frac{\hbar^2}{2m}\psi'' - g\,\delta(x)\,\psi = E\psi$$

$$-\frac{\hbar^2}{2m}(\psi'_+ - \psi'_-) - g\,\psi(0) = 0$$

$$\psi'_+ - \psi'_- = -\frac{2mg}{\hbar^2}\psi(0)$$

Right mover

$$\psi^{(R)}(x) = e^{ikx} + B e^{-ikx} \qquad x<0$$

$$= C\, e^{ikx} \qquad x>0$$

ψ: $\quad 1 + B = C$. $\quad (*)$

ψ': $\quad ikC - ik(1-B) = -\frac{2mg}{\hbar^2}\,C$

$$\hookrightarrow \quad 1 - B = C + \frac{2mg}{ik\hbar^2}\,C \qquad (**)$$

$(*)+(**)$

$$C = \frac{1}{1 - \frac{mg}{k\hbar^2}\,i} \equiv \frac{1}{1 - i\kappa/k}$$

$$= \frac{ik/\kappa}{1 + ik/\kappa} = \frac{ik}{\kappa + ik}$$

$$B = C - 1 = -\frac{1}{1 + ik/\kappa} \equiv B(k)$$

$$= -\frac{\kappa}{\kappa + ik}$$

NB B/C = imag. puro.

$$|B|^2 + |C|^2 = 1, \quad \text{OK!}$$

Right-mover:

$$\psi^{(R)} = e^{ikx} + B(k)e^{-ikx} \qquad (x<0)$$

$$= (1+B)\, e^{ikx} \qquad (x>0)$$

left mover

$$\psi^{(L)} = (1+B)\, e^{-ikx} \qquad (x<0)$$

$$= e^{-ikx} + B\, e^{ikx} \qquad (x>0)$$

Verifica:

$$\langle \psi^{(R)} | \psi^{(L)} \rangle =$$

$$= \int_0^{\infty} dx\, e^{-Kx} (1+B^*)\, e^{-ikx}$$

$$+ \int_{-\infty}^{0} dx\, e^{Kx} \left(e^{-ikx} + B^* e^{ikx} \right)$$

$$= \left(\frac{1}{K+ik} + \underbrace{\frac{1}{K-ik}} \right) + B^* \left(\frac{1}{K+ik} + \frac{1}{K+ik} \right)$$

$$= \frac{2K}{K^2+k^2} - \underbrace{\frac{K}{K-ik}} \cdot \frac{2}{K+ik}$$

$$= 0 \qquad \text{OK !!} \implies \text{No ionizza. per } \underline{V = f e^{-i\omega t}}$$

$$\left(\langle \psi^{cont} | \psi^{st.\,legato} \rangle = 0 \right)$$

Ma allora il calcolo di w_{fi} era errato ?

per $V = f \cdot x\, e^{-i\omega t} + h.c.$

In verità :

$$\psi^{(R)} = \begin{cases} e^{ikx} + B e^{-ikx} & (x<0) \\ e^{ikx} + B e^{ikx} & (x>0) \end{cases}$$

quindi $\psi^{(R)} = e^{ikx} + (\text{fn. pari in } x)$

$$\int dx\, \psi^{(R)*} \cdot x\, e^{-K|x|}$$

$$= \int dx\, e^{-ikx} \cdot x\, e^{-K|x|} \quad \text{OK.}$$

(Il termine $\propto B^*$ non contribuisce !)

$$\langle \psi^{(R)} | f \cdot x | \psi^{(0)} \rangle = \langle \psi^{(\text{onda piana})} | f \cdot x | \psi^{(0)} \rangle !$$

$\Rightarrow$ In generale, per $V = f \cdot x^{2m+1}$,

$m \in \mathbb{Z}$, $\psi^{(f)} \sim e^{ikx}$ OK,

(V) $V = f \cdot x^2$?

$$F_{fi} = \langle \psi^{(k)} | f \cdot x^2 | \psi^{(0)} \rangle \quad + (L)$$

$$= \int dx \left(e^{-ikx} + B e^{ik|x|} \right)^* \cdot x^2 \sqrt{\kappa}\, e^{-\kappa |x|}$$

$$= f\sqrt{\kappa} \cdot \left\{ 2 \cdot \left(\frac{1}{(\kappa+ik)^3} + \frac{1}{(\kappa-ik)^3} \right) + 2 B^* \frac{1}{(\kappa-ik)^3} \right\}$$

$$\therefore \; V_{fi} = V_{fi}\Big|_{\text{onda piana}} \left(1 + O(B)\right)$$

ma $B = -\frac{1}{1+ik/\kappa} \ll 1$, se $\underbrace{\frac{k}{\kappa} \gg 1}$

(vi) $\psi^{(R)}(x)$, $\psi^{(L)}(x)$ non intuitivi come stati finali.

$$\psi^{(R)}, \psi^{(L)} \Longleftrightarrow \psi^{(pari)}, \psi^{(disp)}$$

$$\psi^{(Pari)} = \psi^{(R)} + \psi^{(L)}$$

$$\begin{cases} = e^{ikx} + (2B+1)e^{-ikx} & x<0 \\ = e^{-ikx} + (2B+1)e^{ikx} & x>0 \end{cases}$$

$$\psi^{(Disp)} = \psi^{(R)} - \psi^{(L)}$$

$$= \frac{1}{\sqrt{2}}\left(e^{ikx} - e^{-ikx}\right) \qquad \forall x \ !$$

$$\langle \psi_k^{(Disp)} | V | \psi^{(0)} \rangle$$

$$= \frac{f\sqrt{K}}{\sqrt{2}} \int_{-\infty}^{\infty} dx \left(e^{-ikx} - e^{ikx}\right) \times e^{-K|x|}$$

$$= \frac{f\sqrt{K}}{\sqrt{2}} \cdot 2 \int_0^{\infty} dx \ \times \left(e^{-ikx} - e^{ikx}\right) e^{-Kx}$$

$$= \sqrt{2} f \sqrt{K} \cdot \left\{ -\frac{\partial}{\partial K} \int_0^{\infty} dx \, e^{-(K+ik)x} - \frac{\partial}{\partial K}\int dx \, e^{-(K-ik)x} \right\}$$

$$= \sqrt{2} f \sqrt{K} \left(-\frac{\partial}{\partial K} \frac{1}{K+ik} + \frac{\partial}{\partial K} \frac{1}{K-ik} \right)$$

$$= \quad '' \quad \left(\frac{1}{(K+ik)^2} - \frac{1}{(K-ik)^2} \right)$$

$$= \sqrt{2} f \sqrt{K} \cdot \frac{-4ikK}{(K^2+k^2)^2}$$

$\hookrightarrow$!

$$w = \frac{2\pi}{\hbar} |F_{fi}|^2 \frac{m}{2\pi \hbar p}$$

No factor 2!

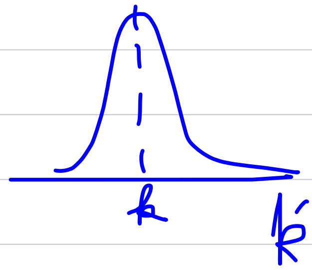
$$= \frac{32 m |f|^2 k K^3}{\hbar^3 (K^2 + k^2)^4}$$

OK.

Evoluzione temporale con il pacchetto d'onda

(Problema della diffusione con l'eq. Schrödinger dipendente da t)

$$\psi = e^{ikx} \Rightarrow \int dk'\, N\, e^{-\frac{(k'-k)^2}{\Delta}}\, e^{ik'x}$$



La funz d'onda dip. da t

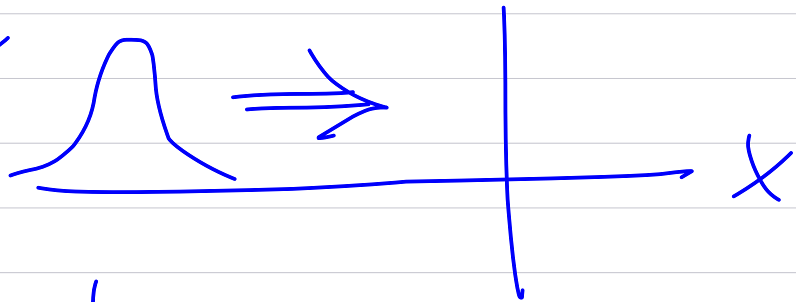
$$E = \frac{k^2\hbar^2}{2m}, \qquad e^{-iEt/\hbar} = e^{-i\frac{k^2\hbar t}{2m}}$$

$$\psi_k(x,t) = \int dk'\, N\, e^{-(k'-k)^2/\Delta}\, e^{ik'x - i\frac{k'^2\hbar t}{2m}}.$$

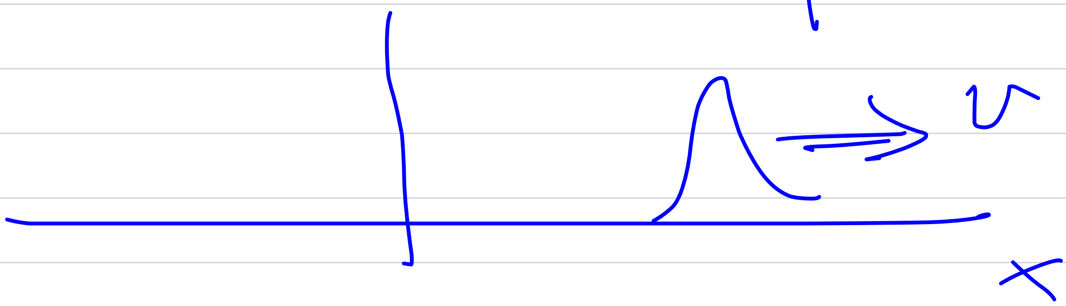
$t \to -\infty$ Fase stazionaria:

$$\frac{\partial}{\partial k}\left(kx - \frac{k^2\hbar t}{2m}\right) \simeq 0 \quad \therefore \quad x - \frac{k\hbar}{m}t \simeq 0.$$

$$x \simeq \frac{k\hbar}{m}t = vt$$



$t \to +\infty$



OK.

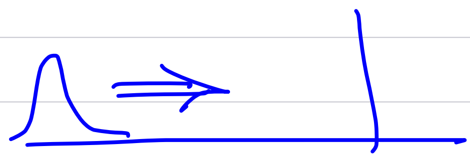
Sol. "right-mover"

$$\psi_k(x) = e^{ikx} + B e^{-ikx}, \qquad x<0$$
$$= C e^{ikx}, \qquad x>0$$

$$\psi(x,t) = \int dk' \, e^{-(k'-k)^2/\Delta} \, \underbrace{\psi_{k'}(x) \, e^{-i\frac{k'^2\hbar t}{2m}}}$$

$\boxed{t\to -\infty:}$ $x<0$

1° termine $x - \frac{k\hbar}{m} t \simeq 0$



(part. incidente).

2° $-x - \frac{k\hbar}{m}t = 0$ NO

$x>0$ $x - \frac{k\hbar}{m}t = 0$ NO

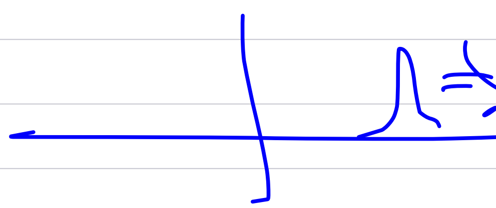
$\boxed{t\to\infty}$ $x<0$

1° termine $x - \frac{k\hbar}{m}t = 0$ NO

2° termine $x = -\frac{k\hbar}{m}t$

$x>0$

$x = \frac{k\hbar}{m}t$ OK.



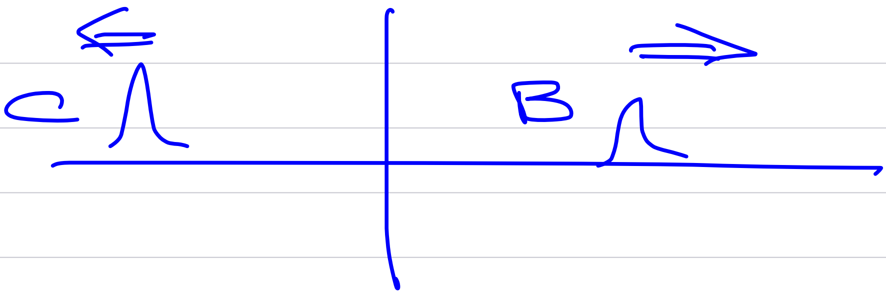
Analogamente, per il left-mover,

$$\psi^{(2)} \begin{cases} = C\, e^{-ikx} & x<0 \\ = e^{-ikx} + B\, e^{ikx} & x>0 \end{cases}$$

la sovrappos. $\Rightarrow$

$t \to -\infty$

$t \to +\infty$



$\psi^{(R)}$ e $\psi^{(L)}$ sono adatte $\times$ il problema della diffusione.

Nel problema di ionizzazione, possiamo prendere

$$\psi^{(Pari)} = \psi^{(R)} + \psi^{(L)}$$

$$\begin{cases} = e^{ikx} + (2B+1) e^{-ikx} & x < 0 \\ = e^{-ikx} + (2B+1) e^{ikx} & x > 0 \end{cases}$$

$$\psi^{(Disp)} = \psi^{(R)} - \psi^{(L)}$$

$$= \frac{1}{\sqrt{2}} \left(e^{ikx} - e^{-ikx} \right) \qquad \forall x \; !$$

a $t \to \infty$

$$e^{ikx} : \partial_k \left(-\frac{k^2 \hbar t}{2m} + kx \right) = 0 \Rightarrow x = \frac{k\hbar}{m} t$$

$$e^{-ikx} : \partial_k \left(-\frac{k^2 \hbar t}{2m} - kx \right) = 0 \Rightarrow x = -\frac{k\hbar}{m} t$$

$\therefore$ Sia $\psi^{(pari)}$ che $\psi^{(dispari)}$,

a $t \to \infty$, descrivono ~~solo~~ i pacchetti d'onda che si allontanano

OK.

Decadimenti e Formula di Breit-Wigner

$$H = H_0 + V$$

H_0 : Spettro discreto e continuo.

V : accoppia i due.

liv. discreto

No fotoni

$|i\rangle = |n, 0\rangle$

$|f\rangle = |m, p\rangle$ → 1 fotone con p

$E_n > E_m$

$$dW = \frac{2\pi}{\hbar}|V_{fi}|^2 \delta(E_f - E_i)\, d\Phi .$$

$$H = \frac{1}{2m}\left(p - \frac{e}{c}A\right)^2 - \frac{e^2}{r} \equiv H_0 + V ;$$

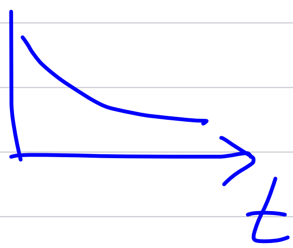
$$V = -\frac{e}{2mc}(p\cdot A + A\cdot p) \to -\frac{e}{mc}A\cdot p$$

$$W \equiv \sum_f W_{i\to f} \equiv \frac{\Gamma}{\hbar} = \frac{1}{\tau}$$ (larghezza del decadimento, $\tau \equiv$ la vita media)

$|\psi(0)\rangle = |i\rangle$

$$\frac{dP}{dt} = -\frac{\Gamma}{\hbar}P(t) \Rightarrow P(t) = e^{-\Gamma t/\hbar}P(0) = e^{-t/\tau}P(0)$$

$|\psi_i\rangle$ NON è un autostato di energia.



Interpretazione $E_i \to E_i - i\Gamma/2$ di modo che

$$|\psi_i(t)\rangle \Rightarrow e^{-i\frac{E_i t}{\hbar} - \frac{\Gamma t}{2}}|\psi_i(0)\rangle$$

$$\therefore\ P_i(t) = \||\psi_i(t)\rangle\|^2 = P_i(0)\cdot e^{-\Gamma t/\hbar} \quad (*)$$

Un paradosso ?

cfr. $a_n^{(1)} = -\frac{i}{\hbar}\int_0^t dt' \, V_{ni}(t') e^{i\omega_{ni} t} \underset{t \to 0}{\simeq} -\frac{i}{\hbar} V_{ni}(0) \cdot t$

$|a_n^{(1)}|^2 \propto t^2$.

$\therefore \; P_i(t) = 1 - c \cdot t^2 + \cdots$ (**)

(cfr) . (*) $\Rightarrow P_i(t) \sim 1 - C_0 t + \cdots$???

- In realtà
  (*) è una formula valida a grande t.
- (**) è valida generalmente, a $t \simeq 0$

$\boxed{\Longrightarrow \text{Effetto/paradosso Zenone quantistico.}}$

Parentesi: Effetto Zenone quantistico.

Sia $|\psi_0\rangle, |\psi_n\rangle, n \geq 1$ un insieme completo ON.

Sia H tale che $|\psi_0\rangle$ non sia un autostato

Sia $|\psi(t)\rangle\Big|_{t=0} = |\psi_0\rangle$.

$$|\psi(t)\rangle = e^{-iHt/\hbar}|\psi_0\rangle$$

$$P_i = |\langle\psi_i|\psi(t)\rangle|^2 = |\langle\psi_i|e^{-iHt/\hbar}|\psi_i\rangle|^2$$

$$= |\langle\psi_i|\left(1 - i\frac{Ht}{\hbar} + \frac{1}{2!}\left(-\frac{it}{\hbar}\right)^2 H\cdot H + \dots\right)|\psi_i\rangle|^2$$

$$= \left|1 - \frac{it}{\hbar}H_{ii} - \frac{t^2}{2\hbar^2}\langle i|H\cdot \underset{\substack{\uparrow \\ \sum_n |n\rangle\langle n| = \mathbb{1}}}{} H|i\rangle + \dots\right|^2$$

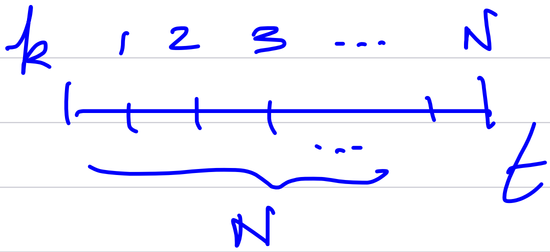
$$= \left|1 - \frac{it}{\hbar}H_{ii} - \frac{t^2}{2\hbar^2}\left\{H_{ii}^2 + \sum_n{}' |H_{ni}|^2\right\} + \dots\right|^2$$

$$= 1 + \frac{t^2}{\hbar^2}H_{ii}^2 - \frac{t^2}{\hbar^2}\left\{H_{ii}^2 + \sum_n{}' |H_{ni}|^2\right\} + \dots$$

$$= 1 - \frac{t^2}{\hbar^2}\left(\sum_n{}' |H_{ni}|^2\right) + \dots$$

$$= 1 - c\,t^2, \qquad c > 0.$$

t piccolo.



$\Rightarrow$ Misura ripetuta N volte di ψ_i.

$P_i^{(N)}(t) =$ prob. di trovare il sistema nello stato $|i\rangle$

$$= \prod P_i\left(\frac{t}{N}\right)^N = \left(1 - \frac{c\,t^2}{N^2}\right)^N$$

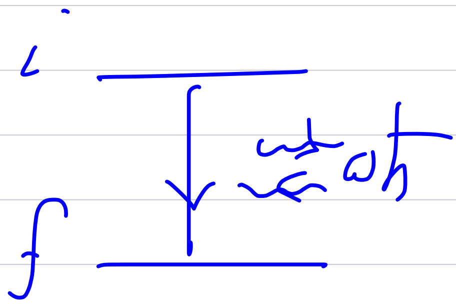
$$\underset{N \gg 1}{\simeq} e^{-\frac{c\,t^2}{N}} \xrightarrow[N \to \infty]{} 1 \quad !$$

(Effetto Zenone quantistico !)

Breit-Wigner.

$$i\hbar \dot{a}_n = \sum_k V_{nk}(t)\, e^{\frac{i}{\hbar}(E_n - E_k)t}\, a_k$$

set $a_n(0) = \delta_{n,i}$



$$E = E_f + \omega\hbar$$

$$\begin{cases} i\hbar \dot{a}_f = V_{fi}\, e^{-\frac{i(E_i - E)t}{\hbar}}\, a_i, & (*) \\ i\hbar \dot{a}_i = \sum_f V_{if}\, e^{\frac{i(E_i - E)t}{\hbar}}\, a_f & (**) \end{cases}$$

Risolvere $(*)$ e $(**)$ con Ansatz

$$a_i(t) = e^{-\frac{it}{\hbar}\left(\delta E - \frac{i\Gamma}{2}\right)} \qquad \text{(Wigner-Weisskopf)}$$

$$(*) \Rightarrow a_f(t) = \frac{V_{fi}}{E_i - E + \delta E - \frac{i\Gamma}{2}} \left\{ e^{-\frac{it}{\hbar}\left(E_i - E + \delta E - \frac{i\Gamma}{2}\right)} - 1 \right\}$$

$$\Rightarrow$$

$$(**) \quad \delta E - i\frac{\Gamma}{2} = \sum \frac{|V_{fi}|^2}{E_i - E + \delta E - i\frac{\Gamma}{2}} \left\{ 1 - e^{\frac{it}{\hbar}\left(E_i - E + \delta E - \frac{i\Gamma}{2}\right)} \right\}$$

$$\Rightarrow$$

$$O(V^2) \quad \delta E - i\frac{\Gamma}{2} = \sum_f \frac{|V_{fi}|^2}{E_i - E}\left(1 - e^{\frac{it}{\hbar}(E_i - E)}\right)$$

Ora

$$\frac{1 - e^{\frac{it}{\hbar}(E_i - E)}}{E_i - E} \xrightarrow[t \to \infty]{} P\frac{1}{E_i - E} - i\pi\,\delta(E_i - E)$$

N.B. $\frac{1}{x+i\epsilon} = \mathcal{P}\frac{1}{x} - \pi i\delta(x)$.

$$\frac{1-e^{\frac{it}{\hbar}(E_i-E)}}{E_i-E} = -\frac{i}{\hbar}\int_0^t e^{\frac{it'}{\hbar}(E_i-E)} \Rightarrow -\frac{i}{\hbar}\int_0^t dt'\, e^{\frac{it'}{\hbar}(E_i-E+i\epsilon)}$$

$$= -\frac{i}{\hbar}\left(-\frac{\hbar}{i}\right)\cdot\frac{1}{E_i-E+i\epsilon} = \mathcal{P}\frac{1}{E_i-E} - \pi i\delta(E_i-E).$$

$$\therefore\ \delta E - \frac{i\Gamma}{2} = \sum_f{}' \frac{|V_{fi}|^2}{E_i-E} - i\pi\sum_f |V_{fi}|^2\delta(E_i-E)$$

Cioè:

$$\delta E = \mathcal{P}\sum_f \frac{|V_{fi}|^2}{E_i-E} = \text{correzione all'energia al } 2^\circ \text{ ordine in } V.$$

$$\Gamma = 2\pi\sum_f |\langle f|V|i\rangle|^2\delta(E_i-E)$$

$$= 2\pi\int d\beta\, \rho(E,\beta)\, |\langle\beta,E|V|i\rangle|^2\Big|_{E_f+\hbar\omega-E_i=0}.$$

Formula di Fermi!

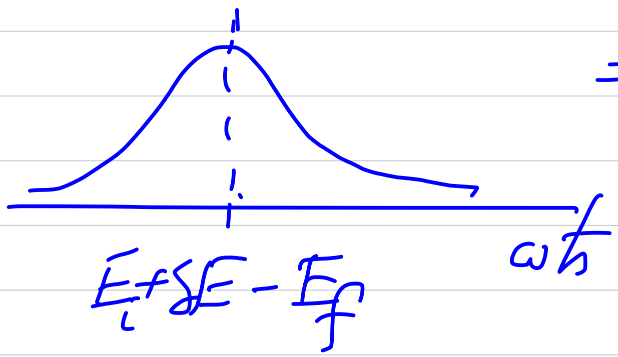
$$\lim_{t\to\infty} |a_f(t)|^2 = \frac{|\langle f|V|i\rangle|^2}{(E_i - E_f + \delta E)^2 + \Gamma^2/4}$$

$\Rightarrow$

$$P_E\, dE = dE \int d\beta \, \frac{\rho(E,\beta)\,|\langle f|V|i\rangle|^2}{(E_i - E_f + \delta E)^2 + \Gamma^2/4}$$

$$= \frac{1}{\pi} \frac{\Gamma/2}{(E_i - E + \delta E)^2 + \Gamma^2/4}$$

(Breit-Wigner)



$\Rightarrow \Delta E \sim \Gamma/2$

$\tau \sim \hbar/\Gamma$.

$\Delta E \cdot \tau \sim \hbar$.