Planck 1900

Luce come particelle : "Quanto di energia"

Problema del corpo nero e formula di Planck

Corpo nero :

Cavità completamente chiusa ; con le parete tenuta a temperatura T

che spettro di luce ? (quale colore e quale intensità)

Calcolo classico : catastrofe ultravioletta

$$C = \frac{\partial U}{\partial T} = \infty$$

$$U \propto (\infty \cdot) kT$$ → costante di Boltzmann

↳ # di modi

Prima consideriamo un analogo problema del calore specifico della materia .

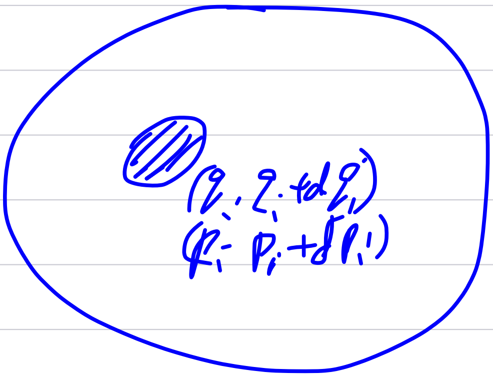
Teoria classica del calore specifico

Legge di equipartizione

Una sostanza di Materia tenuta a temperatura T

$\sim$ Insieme canonico:

$$P(\{q\},\{p\},T)\prod_i dq_i dp_i \qquad \{q_i p_i\} \quad i=1\ldots s$$



$$= \frac{1}{N} e^{-H(\{q_i\},\{p_i\})/kT} \prod_i dq_i dp_i$$

$k = 1.38 \cdot 10^{-23}\ J/K$
$= 1.38 \cdot 10^{-16}\ erg/K$

Supp. che H abbia la forma

$$H = \sum_i (a_i p_i^2 + b_i q_i^2) \qquad (\text{gas, solidi}, \ldots)$$

con a_i, b_i costanti $\Longrightarrow$

$$\langle q_i^2 \rangle = \langle p_i^2 \rangle = \frac{1}{2}kT$$

(Legge di equipartizione).

Dimostrazione:

$$\langle a_1 p_1^2 \rangle = \frac{\int dp_1\, a_1 p_1^2 e^{-a_1 p_1^2/kT}}{\int dp_1\, e^{-a p_1^2/kT}} = -\frac{\partial}{\partial(1/kT)} \int e^{-ap^2/kT} \Big/ \int e^{-ap^2/kT} = -\frac{\partial}{\partial \beta} \sqrt{\frac{\pi}{a\beta}}.$$

$$= \frac{1}{2\beta} = \frac{1}{2}kT \quad \text{ok}.$$

Esempi.

(1) Gas monoatomici (He, Ne, Ar, ... : gas rari)

$$H = \sum_{i=1}^{N} \frac{1}{2m}\left(p_{ix}^2 + p_{iy}^2 + p_{iz}^2\right)$$

$$1 \text{ mol} \Rightarrow N = N_{avg} \simeq 6 \cdot 10^{23}$$

$$\Rightarrow U \equiv \langle E \rangle = \frac{3}{2} N k T$$

$Nk \equiv R =$ costante di gas.

$$\sim \underbrace{6}\cdot 10^{23} \cdot \underbrace{1.38}\cdot 10^{-23} J/K \simeq \overbrace{8.31 \; J/mol/K}.$$

$$\simeq \underbrace{1.987} \; Cal/K$$

$$\therefore C = \frac{\partial U}{\partial T} = \frac{3}{2} Nk = \frac{3}{2} R \simeq 2.98 \; Cal/mol/K$$

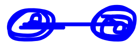
1 mol = 1 grammo molecola = q'tà di sostanza che contiene tante entità elementali (atomi, molecole, ...) quante sono gli atomi di C in 12 g di carbonio

H_2 : 1 mol = 2 grammi

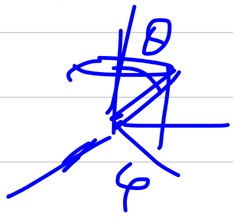
He : 1 mol = 4 grammi

$H \rightarrow$ 1 mol = 1 grammo $\rightarrow$ contiene N_A protoni

$$\therefore N_A = \frac{1\,gr}{m_p} = \frac{1\,gr.}{\underbrace{1.6 \cdot 10^{-24}}\,\underbrace{gr}} \sim \underbrace{6 \cdot 10^{23}}$$

(2) Gas bi-atomici  $(H_2, O_2, N_2 \ldots)$

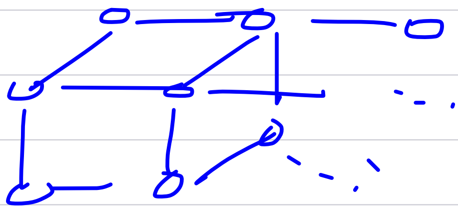
$$H = \sum_i \frac{1}{2m}\left(p_{ix}^2 + p_{iy}^2 + p_{iz}^2\right) + \frac{1}{2I}\left(p_{\theta}^2 + \frac{p_{i\varphi}^2}{\sin^2\theta_i}\right)$$



$$\Rightarrow U = \frac{5}{2}NkT = \frac{5}{2}RT$$

$$C = \frac{5}{2}R \simeq \frac{5}{2}\cdot(1.98) \simeq 4.96 \text{ Cal/mol/K}$$

(3) Solidi



$$H = \sum_{oscil.}\left(\frac{p_i^2}{2m} + \frac{m\omega_i^2 q_i^2}{2}\right)$$

$$\Longrightarrow U = \sum \langle E\rangle = \frac{k}{2}T(6N-6)$$

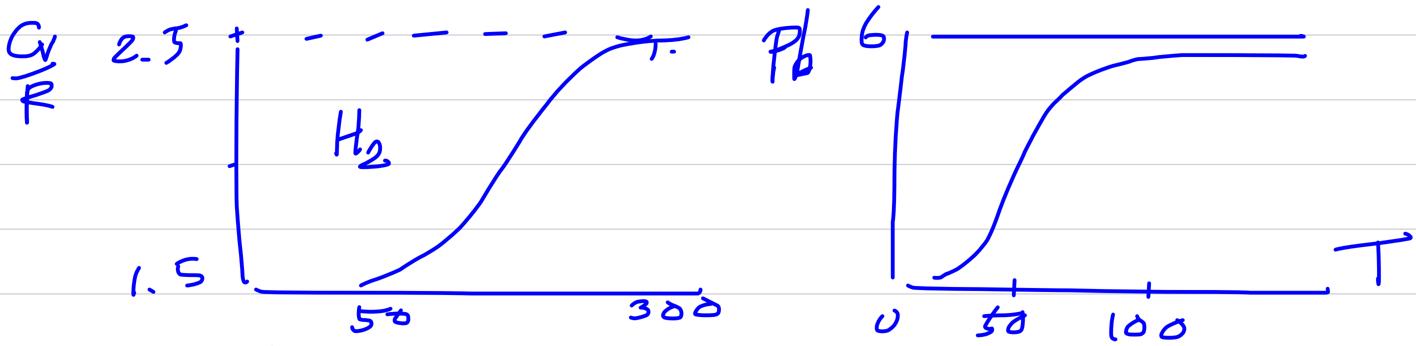
$$\rightsquigarrow C = 3R \simeq 5.9 \text{ Cal/mol/K}$$

(Legge di Dulong-Petit).

N.B. $C_p = C_V + R$, $pV = NkT$ (gas)

	Elem.	C_V	C_V/R	Teo cl. (C_V/R)
1-at	Ar	2.96	1.50	1.50
	He	2.96	1.50	1.5
	Ne	2.96	1.5	⋮
	Kr	2.96	1.5	⋮
2-at	H_2	4.82	2.43	2.5
	O_2	5.01	2.54	
	N_2	4.97	2.49	
Solidi	Al	5.81		C_V: 5.90
	Au	6.07		
	Pb	6.30		

Ma C_V dipende da T



Osservazioni

(i) Teo OK per $T \simeq T_{amb.}$

(ii) H_2 $\quad C_V \sim \frac{5}{2} R \xrightarrow[\text{basse } T]{} \frac{3}{2} R$

(iii) $T \sim T_{ambiente}$: i gradi di libertà di elettroni ?

(iv) Gas bi-atomici : i modi radiali ?

(v) $C_V \xrightarrow[T \to 0]{} 0$

Morale :

a data T certi gradi di libertà sono "attivi", mentre certi altri non contano, "inattivi", "congelati" ... ??

Enigma della stessa natura si presentava nel cosiddetto problema del <u>corpo nero</u>.

Calore specifico del "vuoto".

$$H = \frac{1}{8\pi}\int dv\,(\mathbb{E}^2 + \mathbb{B}^2)$$

$$L = \frac{1}{8\pi}\int dv\,(\mathbb{E}^2 - \mathbb{B}^2)$$

Eq. d. Maxwell.

$$\begin{cases} \nabla \times \mathbb{E} + \frac{1}{c}\frac{\partial \mathbb{B}}{\partial t} = 0 & (*) \quad \text{Farady} \\ \nabla \cdot \mathbb{B} = 0 & (**) \end{cases}$$

$$\begin{cases} \partial \times \mathbb{B} - \frac{1}{c}\frac{\partial \mathbb{E}}{\partial t} = 0 & (\vec{j}\,) \\ \partial \cdot \mathbb{E} = 0 & (\rho) \end{cases}$$

$\epsilon_{ijk} B^k$

$$\partial_\mu F^{\mu\nu} = 0 \qquad \partial_0 F^{0j} + \partial_i F^{ij} = 0$$

$$\partial_\mu \tilde{F}^{\mu\nu} = 0 \qquad \partial_i F^{i0} = 0$$

$$\partial_\mu \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} = 0 \qquad \begin{aligned} F^{0j} &= E^j \\ F^{ij} &= \epsilon^{ijk} B_k \end{aligned}$$

$$\frac{1}{c}\frac{\partial \mathbb{E}}{\partial t} - \partial \times \mathbb{B} = 0 \qquad \text{Biot-Savart}$$

$$\frac{1}{c}\frac{\partial \mathbb{B}}{\partial t} + \partial \times \mathbb{E} = 0 \qquad \text{Faraday!}$$

Le prime due equazioni (Bianchi) si risolvono ponendo

$$\left.\begin{aligned} \mathbf{B} &= \nabla \times \mathbf{A} \\ \mathbf{E} &= -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \nabla \phi \end{aligned}\right\}$$

inv. per $\mathbf{A} \to \mathbf{A}' - \nabla f$

$$\phi \to \phi + \frac{1}{c}\frac{\partial f}{\partial t}$$

Nel vuoto

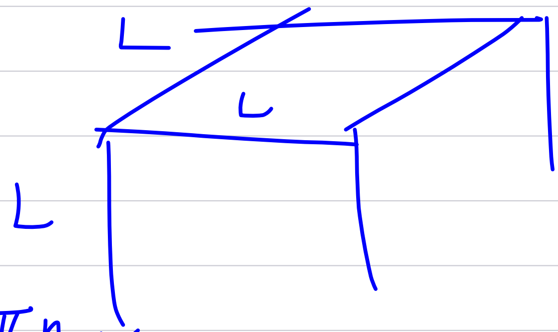
$$\phi = 0, \quad \nabla \cdot \mathbf{A} = 0.$$

(gauge di radiazione) o di Coulomb.

Eq. di Maxwell

$$\Rightarrow \Delta \mathbf{A} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{A} = 0, \quad \Box \mathbf{A} = 0$$

con $\nabla \cdot \mathbf{A} = 0$.



Soluzione:

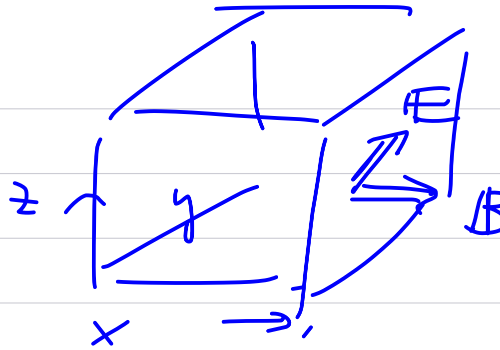
$$A_1 = \sum_{\{n\}} q_1(\{n\}, t) \sin\frac{\pi n_1 x}{L} \cos\frac{\pi n_2 y}{L} \cos\frac{\pi n_3 z}{L}$$

$$A_2 = \quad c \quad s \quad c$$

$$A_3 = \quad c \quad c \quad s$$

$$\nabla \cdot \mathbf{A} = 0 \Rightarrow \mathbf{q}(\{n\}, t) \cdot \mathbf{n} = 0$$

∴ $E \propto \frac{\partial A}{\partial t} = \begin{pmatrix} 0 \\ * \\ * \end{pmatrix}$



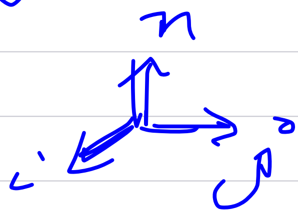
$B = \nabla \times A = \begin{pmatrix} * \\ 0 \\ 0 \end{pmatrix}$

$P = E \times B = \begin{pmatrix} 0 \\ * \\ * \end{pmatrix}$ OK.

$L = \frac{1}{8\pi} \int d^3x \, (E^2 - B^2)$

$\nabla \cdot A = 0$ $\quad$ $\underbrace{n \cdot q = 0}$

→ Sol. $q = Q_{1n}\,\hat{i} + Q_{2n}\,\hat{j}$



$E = -\frac{1}{c}\frac{\partial A}{\partial t} \Rightarrow E_1 = \sum_n \dot{q}_1 \sin \cos \cos.$

$E_2 = \sum_n \dot{q}_2 \cos \sin \cos$

...

$\int_0^L dx \sin^2 \frac{\pi n_1 x}{L} = \frac{L}{2}$

$\int dx \cos \frac{\pi n_1 x}{L} \sin \frac{\pi n_1 x}{L} = 0,$

$\int dy \cos^2 \frac{\pi n_2 y}{L} = \frac{L}{2}$ etc.

$$E^2 = \frac{L}{2}\sum_n (\dot q_1^2 + \dot q_2^2 + \dot q_3^2) = \frac{L}{2}\sum_{n} (\dot Q_{1n}^2 + \dot Q_{2n}^2)$$

$$\mathbb{B} = \sum (\mathbb{Q} \times \mathbb{A})$$

$$|\mathbb{n} \times \mathbb{q}|^2 = |\mathbb{n}|^2 (Q_{1n}^2 + Q_{2n}^2)$$

$$B_1 = \partial_y A_z - \partial_z A_y = (\mathbb{n} \times \mathbb{q})_1$$

$$= \sum_n \left(\frac{\pi}{L}\right)(n_2 q_3 - n_3 q_2) \cdot \cos \sin \sin$$

$$L = \frac{1}{8\pi}\left(\frac{L}{2}\right)^2 \sum_n \left(\frac{1}{c^2}\dot Q_{1n}^2 - k_n^2 Q_{1n}^2\right) + \sum_n \left(\frac{1}{c^2}\dot Q_{2n}^2 - k_n^2 Q_{2n}^2\right)$$

$$\mathbb{k}_n^2 = \left(\frac{\pi}{L}\right)^2 (n_1^2 + n_2^2 + n_3^2) \qquad \mathbb{k}_n = \frac{\pi \mathbb{n}}{L}$$

$$k_n = |\mathbb{k}_n| = \frac{\pi}{L}|\mathbb{n}|.$$

$L \to H$

$$P_{1n} = \frac{\partial L}{\partial \dot Q_{1n}} = \frac{2}{c^2}\dot Q_{1n} \Rightarrow \dot Q_{1n} = \frac{c^2}{2} P_{1n}$$

etc

$$H = \sum P_{1n}\dot Q_{1n} + P_{2n}\dot Q_{2n} - L \Rightarrow$$

$$\boxed{H = \sum_n \left(\frac{c^2}{4} P_{1n}^2 + k_n^2 Q_{1n}^2\right) + 1 \leftrightarrow 2}$$

$$\frac{c^2}{4} \rightarrow \frac{1}{2m} \qquad k_n^2 = \frac{1}{2} m \omega^2$$

$$\omega^2 = \frac{2k_n^2}{m} = \frac{4k_n^2}{2m} = \boxed{k_n^2 c^2}$$

$$\boxed{\omega = k_n c}$$

$$L = \frac{1}{c^2}\dot{Q}^2 - k_n^2 Q^2$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{Q}} - \frac{\partial L}{\partial Q} = 0$$

$$\frac{1}{c^2}\ddot{Q} = -k_n^2 Q$$

$$\ddot{Q} = -\underline{k_n^2 c^2} Q = -\omega^2 Q$$

$$H = \frac{c^2}{4}P^2 + k^2 Q^2$$

$$\dot{Q} = \frac{\partial H}{\partial P} = \frac{c^2}{2}P \rightarrow P = \frac{2}{c^2}\dot{Q}$$

$$\dot{P} = -\frac{\partial H}{\partial Q} = -2k^2 Q$$

$$\therefore \boxed{\ddot{Q} = -k^2 c^2 Q}$$

$\Rightarrow$ 2 somme infinite di oscillatori disaccoppiate !!

$\Rightarrow$ $U = (\#\text{ di oscillatori}) \cdot kT$

$C = \frac{\partial U}{\partial T} = (\quad) \cdot k = \infty$!

cfr $U = \sigma T^4$ Stefan-Boltzmann

$\sigma = 7.64 \cdot 10^{-15} \text{ erg } K^{-4} \text{ cm}^{-3}$

Disastro ?

In realtà NON tutto è perduto. :-

Consideriamo

$u(\nu)\,d\nu =$ l'energia EM dovuta ai modi $[\nu, \nu + d\nu]$.

$\omega = 2\pi\nu = k_n c$ $\quad k_n = \frac{\pi n}{L}$

$\nu = \frac{k_n c}{2\pi} = \frac{c\,n}{2L}$ $\quad d\nu = \frac{c}{2L}\,dn$

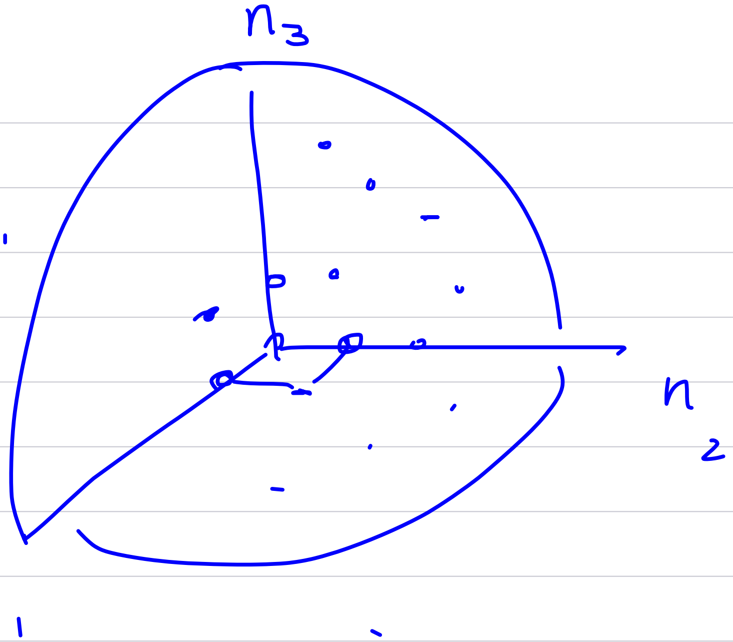
$(\nu, \nu + d\nu) \Rightarrow (n, n + dn)$ $\quad \boxed{n = \frac{2L}{c}\nu}$

dei modi $(n, n+dn)$

$$= \frac{4\pi n^2}{8} \cdot dn \times 2 \rightarrow \text{polarizz.}$$

$$= \frac{\pi}{2} \nu^2 d\nu \cdot \left(\frac{2L}{c}\right)^3 \times 2$$

$$= \frac{8\pi}{c^3} L^3 \nu^2 d\nu$$



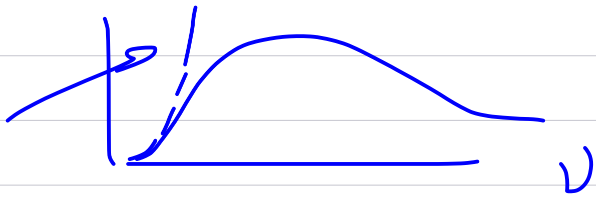
ogni oscillatore!

$$\frac{1}{L^3} \Rightarrow u\, d\nu = \frac{8\pi}{c^3} \nu^2 d\nu \cdot kT$$

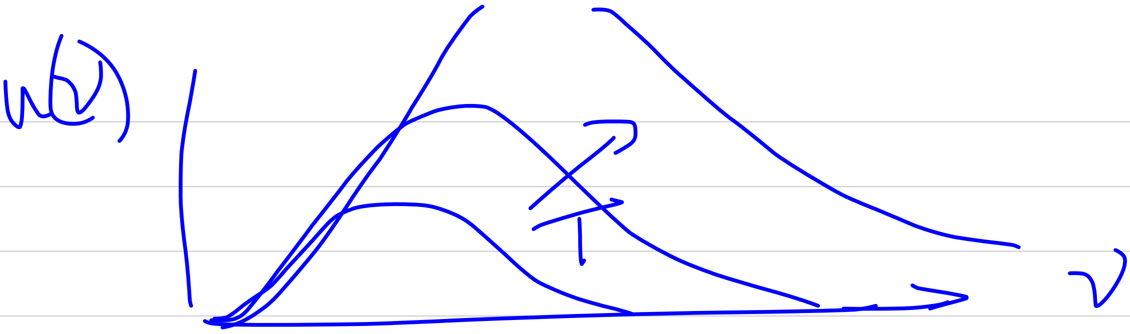
Formula di Reyleigh-Jeans.

$$\int_0^\infty u_{R-J}\, d\nu = \infty \quad \text{infatti!}$$

(i) Ma i dati sperim. OK con u_{RJ} a basse frequenze



(ii) ad alte ν, invece, i dati sono compatibili con $\propto e^{-\text{cost}\,\nu}$. (Wien)



(1) a data T, la regione di basse ν OK con RJ.

(2) Ad aumentare di T, gli intervalli di ν dove la formula classica è valida, aumenta.

(3) a fisso T, i modi con ν alti non contribiscono $\propto kT$

Legge di scaling (Wien) ~1893

$$u(\nu)\,d\nu = \frac{8\pi}{c^3} F\left(\frac{\nu}{T}\right) \nu^3\, d\nu$$

$$\int d\nu\, u(\nu) \propto T^4 . \qquad \text{OK.}$$

ad alte ν la forma empirica è:

$$F(x) = k\beta\, e^{-\beta x} = \qquad \beta = h/k$$

$$\boxed{F\left(\frac{\nu}{T}\right) = k\beta\, e^{-\beta\nu/T}} = h\, e^{-h\nu/kT}$$

$$\therefore\ u_W(\nu)\,d\nu = \frac{8\pi}{c^3} \underbrace{h\nu\, e^{-h\nu/kT}} \cdot \nu^2\, d\nu$$

$$\Longleftrightarrow$$

$$u_{RJ}\, d\nu = \frac{8\pi}{c^3} kT\, \nu^2 d\nu$$

Quindi:

$$kT \xleftarrow[\nu\to 0]{} (\ ?\ ) \xrightarrow[\nu\to\infty]{} h\nu\, e^{-h\nu/kT}$$

$\Longrightarrow$ FORMULA DI PLANCK

$$(?) = \frac{h\nu}{e^{h\nu/kT} - 1}$$

$$u(\nu)_{\text{Planck}}\, d\nu = \boxed{\frac{8\pi}{c^3} \frac{h\nu}{e^{h\nu/kT} - 1} \nu^2 d\nu} \qquad (*)$$

- OK con tutti i dati
- OK con la legge di Stefan-Boltzmann
- $\frac{h\nu}{kT} \ll 1 \rightarrow$ RJ (equipartizione Fisica classica)
- $\frac{h\nu}{kT} \gg 1$ i modi sono soppressi.

Ma qual'è il significato di $(*)$?

$\Rightarrow$ Quanto di energia

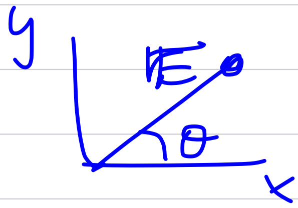
$$E_\nu = \underbrace{h\nu}_{} \cdot n \qquad n = 1, 2, 3 \dots$$

Calcolo

$$H = aq^2 + bp^2$$

$$\langle E \rangle = \frac{\int dq\,dp\; H \cdot e^{-H/kT}}{\int dq\,dp\; e^{-H/kT}}$$

$$= \frac{\int dx\,dy\,(x^2+y^2)\, e^{-(x^2+y^2)/kT}}{\int dx\,dy\; e^{-(x^2+y^2)/kT}}$$



$$x^2 + y^2 = E^2$$

$$dx\,dy = J\, dE\, d\theta$$

$$x = \sqrt{E}\cos\theta \qquad \frac{\partial x}{\partial E} = \frac{\cos\theta}{2\sqrt{E}} \qquad \frac{\partial x}{\partial \theta} = -\sqrt{E}\sin\theta$$

$$y = \sqrt{E}\sin\theta \qquad \frac{\partial y}{\partial E} = \frac{\sin\theta}{2\sqrt{E}}, \qquad \frac{\partial y}{\partial \theta} = \sqrt{E}\cos\theta$$

$$J = \left| \qquad \right| = 1/2$$

$$\langle E \rangle = \frac{\int_0^\infty dE\; E\; e^{-E/kT}}{\int_0^\infty dE\; e^{-E/kT}} = kT \,.$$

OK!

but $\Rightarrow$ $\underline{\underline{\cancel{R}\ J}}$

Ora assumiamo che $E = \varepsilon \cdot n$

$n = 0, 1, 2 \dots$

$$\int dE \longrightarrow \varepsilon \sum_n$$

$$\frac{\int dE\, E\, e^{-E/kT}}{\int dE\, e^{-E/kT}} \longrightarrow \frac{\sum_n \varepsilon n\, e^{-\varepsilon n/kT}}{\sum_n e^{-\varepsilon n/kT}}$$

$\beta = \frac{1}{kT}$

$$= -\frac{\partial}{\partial \beta} \sum_n e^{-\beta \varepsilon n} \Big/ \sum_n e^{-\varepsilon n \beta} \qquad \sum_n e^{-\varepsilon n \beta} \to \frac{1}{1 - e^{-\varepsilon\beta}}$$

$$= (-)(-)\frac{\frac{1}{(1-e^{-\beta\varepsilon})^2} \cdot e^{-\beta\varepsilon} \cdot \varepsilon}{1/1-e^{-\beta\varepsilon}}$$

$$= \frac{\varepsilon\, e^{-\beta\varepsilon}}{1-e^{-\beta\varepsilon}} = \frac{\varepsilon}{e^{\beta\varepsilon}-1} \quad !$$

$\varepsilon = h\nu \Rightarrow$ Formula di Planck!!!