Onde sferiche.

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V(r)\right)\psi(\mathbf{r}) = E\psi(\mathbf{r})$$

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} - \frac{\hat{L}^2}{r^2}$$

$$\Psi(\mathbf{r}) = R_\ell(r)Y_{\ell m}(\theta,\varphi)\,; \quad L^2 Y_{\ell m} = \ell(\ell+1)Y_{\ell m}$$

$$\rightarrow \left\{\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr} + \frac{2m}{\hbar^2}\left(E - V_{eff}(r)\right)\right\}R_\ell = 0$$

con $$V_{eff}(r) = V(r) + \underbrace{\frac{\ell(\ell+1)\hbar^2}{2mr^2}}_{\text{forza centrifuga}}$$

$$F_r = mr\dot{\theta}^2 \sim \frac{L^2}{mr^3}$$

$$\dashrightarrow V_r \sim \frac{L^2}{mr^2} \quad \text{o.k.}$$

$R(r) \equiv \frac{\chi(r)}{r}$ Def $\chi(r)$, $\chi(0) = 0$

$$-\frac{\hbar^2}{2m}\frac{d^2}{dr^2}\chi + V_{eff}(r)\chi = E\chi\,.$$

Eq. Schrödinger 1D con $V = V_{eff}$ $r > 0$
$= \infty$ $r < 0$

Particella libera. $V(r) = 0$

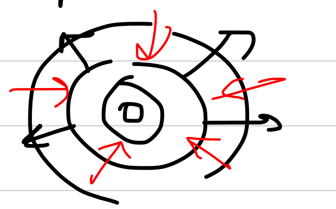
$$\left[-\frac{\hbar^2}{2m}\left(\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr}\right) + \frac{l(l+1)\hbar^2}{2mr^2}\right]R_l = E R_l$$

$$\Leftrightarrow \quad -\frac{\hbar^2}{2m}\chi_l'' + \frac{l(l+1)\hbar^2}{2mr^2}\chi_l = E\chi_l$$

N.B. $\quad H = \frac{\mathbf{p}^2}{2m}$.

$[H, \mathbf{p}] = 0 \Rightarrow H, \mathbf{p}$ diag. $\Rightarrow$ Onde piane $\sim e^{i\mathbf{p}\cdot\mathbf{r}/\hbar}$

$[H, \mathbf{L}] = 0 \Rightarrow H, \mathbf{L}^2, L_z$ diagonal. $\Longrightarrow$ Onde sferiche



(i) Sol. $l = 0$

$$-\frac{\hbar^2}{2m}\chi'' = E\chi \qquad k = \frac{\sqrt{2mE}}{\hbar}$$

$\chi = \sin kr \Rightarrow R_0(r) = \frac{\sin kr}{r}$ Sol. regolare a $r=0$

$ = \cos kr \Rightarrow Q_0(r) = \frac{\cos kr}{r}$ Sol. singolare a $r=0$.

(ii) $l \neq 0$.

$$R_l'' + \frac{2}{r} R_l' - \frac{l(l+1)}{r^2} R_l + k^2 R_l = 0 \quad (*)$$

$r \simeq 0$ $\quad R_l'' + \frac{2}{r} R_l' - \frac{l(l+1)}{r^2} R_l \simeq 0$

$$R_l \sim r^A \Rightarrow$$

$$A(A-1) + 2A = l(l+1)$$

$$A^2 - l^2 + A - l = 0$$

$$(A-l)(A+l+1) = 0$$

$$\Rightarrow \quad A = l, \quad A = -l-1$$

$\therefore \quad R_l(r) \sim r^l$ $\qquad$ sol. regolare,

$\qquad\qquad \sim r^{-l-1}$ $\qquad$ sol. singolare.

$$\boxed{R_l(r) \equiv r^l \eta_l(r)} \quad \text{Def } \eta_l(r) \Rightarrow (*)$$

$$R' = \left(\frac{l}{r}\eta_l + \eta_l'\right) r^l$$

$$R'' = r^l\left(\eta_l'' + \frac{2l}{r}\eta_l' + \frac{l(l-1)}{r^2}\eta_l\right)$$

$$\Rightarrow \eta_l'' + \frac{2l+2}{r}\eta_l' + k^2\eta_l = 0 \quad (**)$$

$$\eta_\ell'' + \frac{2\ell+2}{r}\eta_\ell' + k^2\eta_\ell = 0 \qquad (**)$$

$$\zeta_\ell \equiv \frac{1}{r}\frac{d}{dr}\eta_\ell = \frac{1}{r}\eta_\ell' \rightarrow \eta_\ell' = r\zeta_\ell$$

$$\frac{d}{dr}(*) \Rightarrow \eta_\ell''' + \frac{2\ell+2}{r}\eta_\ell'' - \frac{(2\ell+2)}{r^2}\eta_\ell' + k^2\eta_\ell' = 0$$

$$(r\zeta_\ell)'' + \frac{2\ell+2}{r}(r\zeta_\ell)' - \frac{2\ell+2}{r^2}r\zeta_\ell + k^2 r\zeta_\ell = 0$$

$$r\zeta_\ell'' + 2\zeta_\ell' + \frac{2\ell+2}{r}(\zeta_\ell + r\zeta_\ell') - \frac{2\ell+2}{r}\zeta_\ell + k^2 r\zeta_\ell = 0$$

$$\zeta_\ell'' + \frac{2(\ell+2)}{r}\zeta_\ell' + k^2\zeta_\ell = 0$$

$\Rightarrow \quad \zeta_\ell = \eta_{\ell+1}$!! In altre parole, abbiamo trovato l'op ricorrenza:

$$\eta_{\ell+1} = \frac{1}{r}\frac{d}{dr}\eta_\ell \qquad !!$$

$$R_\ell(r) = r^\ell \eta_\ell(r)$$

$$R_0(r) = \eta_0(r) = \frac{\sin kr}{r}$$

$$Q_0(r) = \frac{\cos kr}{r}$$

$$R_\ell(r) = N_\ell\, r^\ell \left(\frac{1}{r}\frac{d}{dr}\right)^\ell \cdot \frac{\sin kr}{r}$$

$$Q_\ell(r) = \tilde{N}_\ell \cdot r^\ell \left(\frac{1}{r}\frac{d}{dr}\right)^\ell \frac{\cos kr}{r}$$

Funzioni di Bessel sferiche, j_ℓ, n_ℓ

$$R_{k,\ell}(r) = 2k\, j_\ell(kr) = \sqrt{\frac{2\pi k}{r}}\, J_{\ell+1/2}(kr)$$

$$Q_{k,\ell}(r) = 2k\, n_\ell(kr) = \sqrt{\frac{2\pi k}{r}}\, N_{\ell+1/2}(kr),$$

J_ℓ, N_ℓ sono funzioni di Bessel (e Neumann): soluzioni dell'eq. di Bessel,

$$Z_\nu'' + \frac{1}{z} Z_\nu' + \left(1 - \frac{\nu^2}{z^2}\right) Z_\nu = 0,$$

di cui J_ν è la sol. regolare a $z=0$.

Le funz. Bessel sferiche $j_\ell, n_\ell \sim$ sol. eq. radiale

$$R'' + \frac{2}{r} R' + \left(k^2 - \frac{\ell(\ell+1)}{r^2}\right) R = 0,$$

$kr \equiv x$

$$j_\ell''(x) + \frac{2}{x} j_\ell'(x) + \left(1 - \frac{\ell(\ell+1)}{x^2}\right) j_\ell = 0$$

$$n_\ell \qquad n_\ell \qquad n_\ell$$

$$j_\ell(x) = (-)^\ell x^\ell \left(\frac{1}{x}\frac{d}{dx}\right)^\ell \frac{\sin x}{x}$$

$$n_\ell(x) = -(-)^\ell x^\ell \left(\frac{1}{x}\frac{d}{dx}\right)^\ell \frac{\cos x}{x}$$

e.g. $j_0(x) = \frac{\sin x}{x}$, $j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x}$, ...

$n_0(x) = -\frac{\cos x}{x}$, $n_1(x) = -\frac{\sin x}{x} - \frac{\cos x}{x^2}$, ...

Andamenti a $x \to \infty$ e a $x \to 0$

$x \to \infty$:

$$\begin{cases} j_\ell(x) \sim (-)^\ell \frac{1}{x}\left(\frac{d^\ell}{dx^\ell}\sin x\right) \sim \boxed{\frac{1}{x}\cos\left(x - \frac{\ell+1}{2}\pi\right)} \\ n_\ell(x) \sim (-)^{\ell+1}\frac{1}{x}\frac{d^\ell}{dx^\ell}\cos x \sim \boxed{\frac{1}{x}\sin\left(x - \frac{\ell+1}{2}\pi\right)} \end{cases}$$

$x \to 0$: $\quad \xi = x^2, \quad \frac{1}{x}\frac{d}{dx} = 2\frac{d}{d\xi}$.

$$\frac{\sin x}{x} = \sum_{n=0}^{\infty} \frac{(-)^n}{(2n+1)!} x^{2n} = \sum_n \frac{(-)^n}{(2n+1)!}\xi^n$$

$$\therefore \; j_\ell(x) \simeq \frac{2^\ell}{(2\ell+1)!} x^\ell \sim \frac{1}{(2\ell+1)!!} x^\ell$$

$$n_\ell(x) \simeq -\frac{(2\ell-1)!!}{x^{\ell+1}}$$

$$j_\ell(x) = (-)^\ell x^\ell \left(\frac{1}{x}\frac{d}{dx}\right)^\ell \frac{\sin x}{x}$$

$$n_\ell(x) = -(-)^\ell x^\ell \left(\frac{1}{x}\frac{d}{dx}\right)^\ell \frac{\cos x}{x}$$

Funzioni di Hankel sferiche.

$$h_\ell^{(1)}(x) \equiv j_\ell(x) + i\, n_\ell(x)$$

$$h_\ell^{(2)}(x) \equiv j_\ell(x) - i\, n_\ell(x)$$

con andamento asintotico:

$$h^{(1)}(x) \sim \frac{1}{x} e^{i\left(x - \frac{\ell+1}{2}\pi\right)} ;$$

$$h^{(2)}(x) \sim \frac{1}{x} e^{-i\left(x - \frac{\ell+1}{2}\pi\right)} ;$$

Ricapitolando!

$$R_{k,\ell}(r) = 2k\, j_\ell(kr)$$ sol. reg. a $r=0$

$$Q_{k,\ell}(r) = 2k\, n_\ell(kr)$$ sol. singolare a $r=0$

$$R^{(1)}_{k,\ell}(r) = 2k\, h^{(1)}_\ell(kr)$$ Funz. Hankel sferica tipo 1.

$$R^{(2)}_{k,\ell}(r) = 2k\, h^{(2)}_\ell(kr)$$ Funz. Hankel tipo II.

Particella libera. ($V \equiv 0$)

Sol. $R = R_{k,\ell}(r) = 2k\, j_\ell(kr)$

$\Psi = R_{k,\ell}(r)\, Y_{\ell,m}(\theta, \varphi) \qquad \{H, \mathbb{L}^2, L_z\}$

$\Longleftrightarrow \quad e^{i\mathbf{k}\cdot\mathbf{r}} \qquad (\{H, \mathbf{p}\}\ L_{x,y,z})$

$$e^{ikz} = e^{ikr\cos\theta} = \sum_\ell (2\ell+1)\, i^{\ell}\, j_\ell(kr)\, P_\ell(\cos\theta)$$

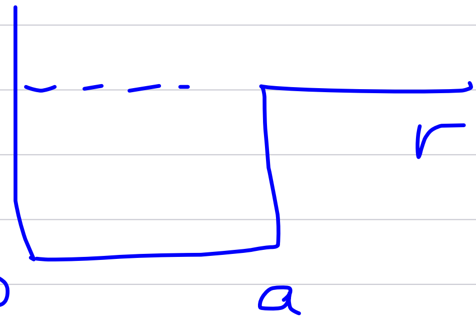
$$e^{i\mathbf{k}\cdot\mathbf{r}} = 4\pi \sum_{\ell=0}^{\infty} i^{\ell} j_\ell(kr) \sum_{m=-\ell}^{\ell} Y_{\ell,m}(\underset{\hat{\mathbf{r}}}{\Omega_1})\, Y^*_{\ell,m}(\underset{\hat{\mathbf{k}}}{\Omega_2})$$

dove $$P_\ell(\cos\theta) = \frac{4\pi}{2\ell+1} \sum_m Y_{\ell,m}(\hat{\mathbf{r}})\, Y^*_{\ell,m}(\hat{\mathbf{k}})$$

$$\hat{\mathbf{r}} \cdot \hat{\mathbf{k}} = \cos\theta\,.$$

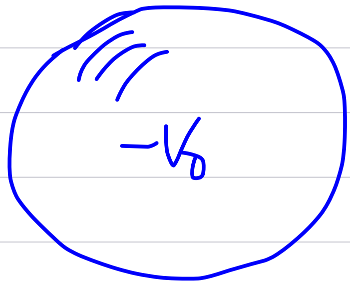
Buca di potenziale 3D.

$$V(r) = -V_0 \qquad 0<r<a$$
$$ = 0 \qquad r \geq a$$



Sol. $0 > E > -V_0$?

$V=0$



Sol. interna

$$k' = \frac{\sqrt{2m(E+V_0)}}{\hbar}$$

$$R^{int} = R_{k'l}(r) = A\, j_l(k'r)$$

Sol. esterna:

$$k = \frac{\sqrt{2mE}}{\hbar} = i\kappa\,, \qquad \kappa = \frac{\sqrt{-2mE}}{\hbar}$$

$$\dashrightarrow \quad h_l^{(1)}(kr) \sim \frac{e^{ikr}}{r} \sim \frac{e^{-\kappa r}}{r} \quad \text{ok},$$

N.B.

$$j_l(kr) \sim \frac{\cos\left(kr - (l+1)\frac{\pi}{2}\right)}{r} \sim \frac{e^{\kappa r}}{r}$$

$$n_l(kr) \propto \sin(\quad) \sim \cancel{\frac{e^{\kappa r}}{r}}$$

$$h^{(2)}(kr) \sim \cancel{\frac{e^{-ikr}}{r}} \sim \cancel{\frac{e^{\kappa r}}{r}}.$$

$R^{ext} = B\, h^{(1)}(ikr)$

Cond. raccordo a $r=a$.

$$\boxed{\frac{ik\, h_\ell^{(1)\prime}(ika)}{h_\ell^{(1)}(ika)} = \frac{k'\, j_\ell'(k'a)}{j_\ell(k'a)}} \quad (*)$$

Sol. con $\ell = 0$

$$\begin{cases} j_0(x) = \dfrac{\sin x}{x}, \\ h_0^{(1)}(x) = j_0(x) + i\, n_0(x) = \dfrac{\sin x - i\cos x}{x} \end{cases}$$

$$= -i\,\frac{e^{ix}}{x}$$

$$\Rightarrow j_0'(x) = \frac{\cos x}{x} - \frac{\sin x}{x^2} \Rightarrow \frac{j_0'}{j_0} = \cot x - \frac{1}{x}$$

2^0 membro $(*)$ $= k' \cdot \left(\cot k'a - \dfrac{1}{k'a}\right)$

$$= k' \cot k'a - \frac{1}{a} \; ;$$

$$h_0^{(1)}(x) = -i\frac{e^{ix}}{x},$$

$$h_0^{(1)\prime}(x) = (-i)\left(\frac{i e^{ix}}{x} - \frac{e^{ix}}{x^2}\right)$$

$$\frac{h_0^{(1)\prime}}{h_0^{(1)}} = i - \frac{1}{x}$$

1^0 membro (*)

$$= ik\left(i - \frac{1}{ika}\right) = -k - \frac{1}{a}$$

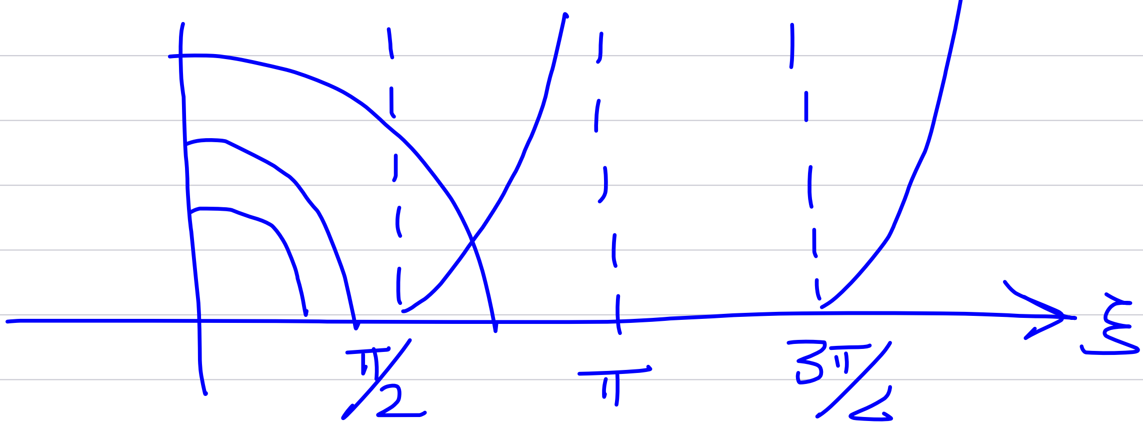
$\therefore$ (*) $\Rightarrow$

$$-k = k' \cot k'a .$$

$ka \equiv \eta$; $k'a \equiv \xi$,

$$\boxed{\eta = -\xi \cot \xi}$$

$$\boxed{\xi^2 + \eta^2 = \frac{2mV_0 a^2}{\hbar^2}}$$



$\Longrightarrow$

(i) $\sqrt{\dfrac{2mV_0 a^2}{\hbar^2}} \leq \pi/2 \Rightarrow$ No soluzioni

(ii) $\pi/2 < \sqrt{} \leq 3\pi/2$ 1 soluzione

(iii) $3\pi/2 < \sqrt{} \leq 5\pi/2$ 2 soluzioni

...

(iv) $\dfrac{(2n-1)\pi}{2} < \sqrt{} \leq \dfrac{(2n+1)\pi}{2}$, n soluzioni

Esercizi ...

1. Trovare tutte le soluz. discreti x
$V = 300\,\text{MeV}$, $a = 3\,\text{fm}$, $\text{fm} = 10^{-13}\,\text{cm}$
$\hbar c \sim 200\,\text{MeV}\cdot\text{fm}$
usando <u>Mathematica</u>

2. Buca 3D

$$V(r) = 0 \qquad r < a$$
$$= \infty \qquad r > a.$$

3.
$$V(r) = \infty \qquad 0 < r < R$$
$$= 0 \qquad R < r < R + a$$
$$= \infty \qquad r > R + a$$