Momento Angolare

$$\hat{L} = \hat{r} \times \hat{p} \Rightarrow \begin{cases} L_x = y\hat{p}_z - z\hat{p}_y \\ L_y = z\hat{p}_x - x\hat{p}_z \\ L_z = x\hat{p}_y - y\hat{p}_x \end{cases}$$

$\Rightarrow$ Relazioni di commutazione

$$[L_x, L_y] = [y\hat{p}_z - z\hat{p}_y, z\hat{p}_x - x\hat{p}_z]$$

$$= -i\hbar y p_x + i\hbar x p_y = i\hbar L_z ;$$

$$\Rightarrow [L_i, L_j] = i\hbar \epsilon_{ijk} L_k ;$$

$\hat{L}_i/\hbar \Rightarrow \hat{L}_i$ (adimensionale)

$$[L_i, L_j] = i \epsilon_{ijk} L_k .$$

$\mathbb{L}$, ℓ momenti angolari orbitali

$\mathbb{S}$, s, spin (mom. angolare intrinseco)

$\mathbb{J}$, j, momento angolare generico

- - -

2 particelle

$\mathbb{L}_1$, $\mathbb{L}_2$ $\Rightarrow$

$$\mathbb{L}_{tot} = \mathbb{L}_1 + \mathbb{L}_2 , \Leftarrow \begin{cases} \mathbb{L}_1 = \mathbb{r}_1 \times \mathbb{p}_1 ; \\ \mathbb{L}_2 = \mathbb{r}_2 \times \mathbb{p}_2 \end{cases}$$

$$= \mathbb{L}_1 \otimes \mathbb{1} + \mathbb{1} \otimes \mathbb{L}_2$$

Più generale,

$$\vec{J} = \vec{J}_1 + \vec{J}_2\,,$$

$$[J_{1i}, J_{1j}] = i\epsilon_{ijk} J_{1k}$$
$$\quad J_2 \quad\ J_2 \qquad\qquad\quad J_{2k}$$

$$\Rightarrow \quad [J_i, J_j] = i\epsilon_{ijk} J_k.$$

$$J_+ \equiv J_x + iJ_y = J_1 + iJ_2$$
$$J_- \equiv J_x - iJ_y = J_1 - iJ_2$$

- $[J_+, J_-] = 2J_3$
- $[J_+, J_3] = [J_1 + iJ_2, J_3] = -iJ_2 - J_1$
$$= -J_+$$
- $[J_-, J_3] = +J_-$

Infine $\quad \vec{J}^2 = J_1^2 + J_2^2 + J_3^2$ (mom. angolare quadrato)

- $\vec{J}^2 = J_+J_- + J_3^2 - J_3$
- $\vec{J}^2 = J_-J_+ + J_3^2 + J_3$

$\Rightarrow$ Quantizzazione del mom. angolare.

(universale)

- $[\vec{J}^2, J_i] = 0$, e.g. $[\vec{J}^2, J_3] = 0$.

$|T, m\rangle$

$J^2 |T, m\rangle = T\, |T, m\rangle$

$J_3 |T, m\rangle = j_3\, |T, m\rangle$

T? m?

$J_3 J_+ |T, m\rangle = J_+ J_3 |T, m\rangle + J_+ |T, m\rangle$

$= (m+1)\, J_+ |T, m\rangle$

$\therefore$ $|T, m\rangle$ è (se $|\ \rangle \neq 0$) autostato di J_3

con $m+1$

(e con lo stesso T di J^2) .

allora con applicazioni successive

$|T, m\rangle \xrightarrow{J_+} |T, m+1\rangle \xrightarrow{J_+} |T, m+2\rangle \to \ldots$?

$\langle T m | J^2 | T, m\rangle = \langle\ | J_3^2 + J_1^2 + J_2^2 |\ \rangle$

$\boxed{T \geqq m^2} \Rightarrow$ Max e Min di m.

Max $m \equiv j \Rightarrow T$?

$$\mathbb{J}^2 = J_- J_+ + J_3^2 + J_3$$

$$\langle T, j \,|\, \mathbb{J}^2 \,|\, T, j \rangle = T$$

$$\Rightarrow \quad j^2 + j + \underbrace{\langle Max | J_- J_+ | Max \rangle}_{= 0 \,!}$$

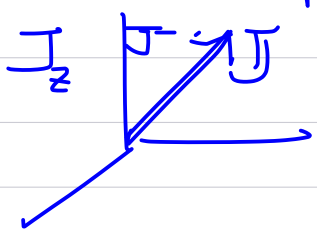
ma allora

$$T = j^2 + j = \underline{j(j+1)} \quad !$$

(N.B. In mecc. classica, $\mathbb{J} = \mathbb{r} \times \mathbb{p}$.

$$(J_z)_{Max} = J$$

just a vector !



Cambio notazione

$$|T, m\rangle \longrightarrow |j, m\rangle$$
$$\overset{\shortparallel}{j(j+1)}$$

Analogamente, $\quad m^2 \leq T \Rightarrow \underline{Min(m)}$

Lo stesso argomento,

$$J_3(J_-|j, m\rangle) = (m-1)\, J_-|j, m\rangle$$

$$\therefore \quad J_-|j, m\rangle \propto |j, m-1\rangle$$

$$|j, m\rangle \xrightarrow{J_-} |j, m-1\rangle \xrightarrow{J_-} |j, m-2\rangle \cdots \quad ?$$

$$\vec{J}^2 = J_+ J_- + J_3^2 - J_3$$

$$\text{Min } J_3 = \tilde{j}$$

$$\langle j, \tilde{j} | \vec{J}^2 | j, \tilde{j} \rangle = j(j+1)$$

$$= \tilde{j}^2 - \tilde{j} \quad \Rightarrow \quad \tilde{j} = -j \quad !$$

⇒ Torre di stati (multipletto)

$$J_+ |j, j\rangle = 0$$

$$J_- \downarrow \quad |j, j\rangle \uparrow J_+$$

$$|j, j-1\rangle$$

$$\vdots \qquad \qquad \vdots$$

$$J_- \downarrow \quad |j, -j\rangle \uparrow J_+$$

$$J_- |j, -j\rangle = 0$$

} multipletto con mom. ang. j $(\vec{J}^2 = j(j+1))$

Ma $\quad -j = j - n$!

ma $\quad n \in \mathbb{Z}_{>0}$

3 —
2 —
1 —
0 —
−1 —
−2 —
−3 —

$\therefore \quad j = \frac{n}{2} \qquad n =$ integer !

Quantizzazione del mom. angolare (indip. dalla natura di $\vec{J}$, $\vec{L}$, $\vec{S}$, $\vec{L}+\vec{S}$ ....)

Mom. angolare prende soltanto valori interi o semi-interi !!!!

In Natura,

mom. angolare orbitale,

$$j = L = 0, 1, 2 \dots$$

$$\left. L_z = m = L, L-1, \dots, -L \right) \text{ interi.}$$

$\Rightarrow$ dimostreremo

Spin $\quad j = 0, 1/2, 1, 3/2 \dots$

fatto empirico

$e, p, n \dots \quad : \quad s = 1/2$

$\pi^{\pm}, \pi^0, K \quad s = 0$

$H \quad s = 0$

$W^{\pm}, \gamma \quad s = 1.$

- Quantizzazione di J (J^2)
- Quantizzazione di

$$J_z = J,\ J-1,\ J-2,\ \cdots,\ -J.$$

(L'asse di quantizzazione è arbitrario!!)

Abbiamo scelto sopra
$\{J^2, J_z\}$ diagonal. (J_x, J_y non diagonal.)

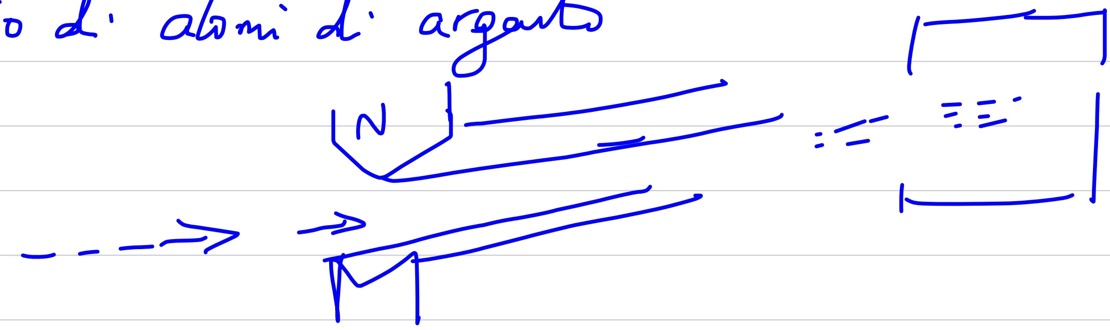
$\Rightarrow \{J^2, J_x\} \rightarrow J_y, J_z$ non diagonal.

$\dashrightarrow \{J^2, J\cdot n\}$ $\quad n^2 = 1$
n versore arbitrario!

$\dashrightarrow$ Quantizzazione direzional

Esperimento di Stern-Gerlach (1922)

Fascio di atomi di argento



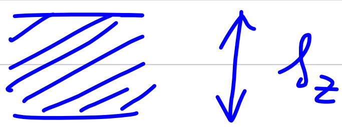
con $\frac{\partial B(z)}{\partial z} \neq 0$!

$$V = -\mu \cdot B \Rightarrow F = -\nabla V .$$

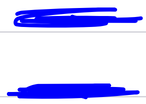
$$= -g\, s \cdot B = -g B(z)\, s_z$$

$$F_z \propto \frac{\partial B(z)}{\partial z} \cdot s_z$$

Mecc. classica $\Rightarrow$



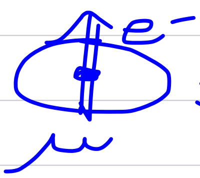
Osservato :



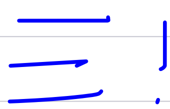
due strisce !

$\Rightarrow$ Quantizzazione $s_z = \pm 1/2$! $\boxed{s = 1/2}$

(Modello di Bohr $l = 1$?

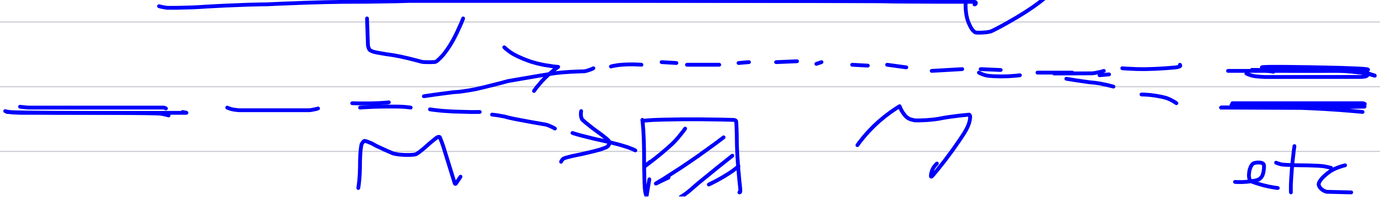


$\Rightarrow$



!)

$\Rightarrow$ "Quantizzazione direzionale"



etc.

Elementi di matrice di J_x, J_y, J_z

$$J_+ |j\, m\rangle = c\, |j, m+1\rangle$$

$$J_- |j, m\rangle = \tilde{c}\, |j, m-1\rangle$$

$$\vec{J}^2 = J_- J_+ + J_3^2 + J_3$$

$$\langle j\, m| \vec{J}^2 |j\, m\rangle = j(j+1) \qquad \left(\begin{array}{l} j^2 + j - m^2 - m \\ = (j-m)(j+m+1) \end{array} \right)$$

$$= \langle j\, m| J_- J_+ + J_3^2 + J_3 |j, m\rangle$$

$$= \langle j\, m| J_- J_+ |j, m\rangle + m^2 + m.$$

ma $\langle j\, m| J_- J_+ |j\, m\rangle = |\langle j, m+1| J_+ |j, m\rangle|^2$!

perciò $\langle j, m+1| J_+ |j, m\rangle = \sqrt{(j-m)(j+m+1)}$

$$\Leftrightarrow \quad J_+ |j, m\rangle = \sqrt{(j-m)(j+m+1)}\; |j, m+1\rangle$$

analogamente $\quad (J_+ |j, j\rangle = 0 \quad$ OK.)

$$\langle j, m-1| J_- |j, m\rangle = \sqrt{(j+m)(j-m+1)}$$

$$\Leftrightarrow \quad J_- |j, m\rangle = \sqrt{(j+m)(j-m+1)}\; |j, m-1\rangle$$

$$(J_- |j, -j\rangle = 0 \quad \text{OK.})$$

Ma $\quad J_+ \equiv J_x + i J_y$, $\quad J_- \equiv J_x - i J_y$ $\Big\}$ Elem. di matrice di $J_\pm$ $\Rightarrow$ " " di $J_{x,y}$

$J_x = \frac{1}{2}(J_+ + J_-) \Rightarrow$ El. matr. tra $|j\,m\rangle$ e $|j, m\pm1\rangle$

$J_y = \frac{1}{2i}(J_+ - J_-) \Rightarrow$ " " "

$$\langle j, m+1 | J_x | j, m\rangle = \frac{1}{2}\sqrt{(j-m)(j+m+1)}$$
$$\langle j, m-1 | J_x | j, m\rangle = \frac{1}{2}\sqrt{(j+m)(j-m+1)}$$
$$\langle j, m+1 | J_y | j, m\rangle = -\frac{i}{2}\sqrt{(j-m)(j+m+1)}$$
$$\langle j, m-1 | J_y | j, m\rangle = \frac{i}{2}\sqrt{(j+m)(j-m+1)}$$

e $\quad \langle j, m | J_z | j, m\rangle = m.$

Tutti gli elem. matrice di J_x, J_y, J_z (universali, dipendono solo da j)

$s = 1/2$ (spin $1/2$, $e, p, n, \ldots$)

$$s_z = \frac{1}{2}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

$$s_x = \frac{1}{2}\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$$

$$s_y = \frac{1}{2}\begin{pmatrix} & -i \\ i & \end{pmatrix}$$

$\Longleftrightarrow$

$$\mathbf{s} = \frac{1}{2}\boldsymbol{\sigma}$$

$$\sigma_x = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} & -i \\ i & \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

sono le matrici di Pauli

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stato di spin $1/2$, con $s_z = \pm 1/2$

$s_z|\uparrow\rangle = \frac{1}{2}|\uparrow\rangle$, $|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ spin up (su)

$s_z|\downarrow\rangle = -\frac{1}{2}|\downarrow\rangle$, $|\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ spin down (giù)

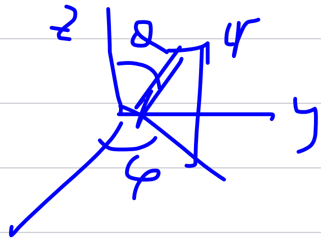
Spin in direzione $\mathbf{n} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$

$$\mathbf{n}\cdot\boldsymbol{\sigma}|+\rangle_n = |+\rangle_n \qquad |+\rangle_n = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$(\mathbf{n}\cdot\boldsymbol{\sigma}) = \begin{pmatrix} \cos\theta & \sin\theta e^{-i\varphi} \\ \sin\theta e^{i\varphi} & -\cos\theta \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \Longrightarrow \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} e^{-i\varphi/2}\cos\frac{\theta}{2} \\ e^{i\varphi/2}\sin\frac{\theta}{2} \end{pmatrix}$$

Momento angolare orbitale $\mathbb{L} = \mathbb{r} \times \mathbb{p}$

$$\begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \tan^{-1} \dfrac{\sqrt{x^2+y^2}}{z} \\ \varphi = \tan^{-1} \dfrac{y}{x} \end{cases} \qquad \begin{pmatrix} z = r\cos\theta \\ x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \end{pmatrix}$$



Preparo:

$$\frac{\partial}{\partial x} = \frac{r}{x}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial x}\frac{\partial}{\partial\theta} + \frac{\partial\varphi}{\partial x}\frac{\partial}{\partial\varphi}$$

$$= \frac{x}{r}\frac{\partial}{\partial r} + \frac{\frac{x}{\sqrt{x^2+y^2}}\frac{1}{z}}{1+\frac{x^2+y^2}{z^2}}\frac{\partial}{\partial\theta} - \frac{\frac{y}{x^2}}{1+\frac{y^2}{x^2}}\frac{\partial}{\partial\varphi}$$

$$\boxed{\frac{\partial}{\partial x} = \frac{x}{r}\frac{\partial}{\partial r} + \frac{xz}{r^2\sqrt{x^2+y^2}}\frac{\partial}{\partial\theta} - \frac{y}{x^2+y^2}\frac{\partial}{\partial\varphi}}$$

$$\boxed{\frac{\partial}{\partial y} = \frac{y}{r}\frac{\partial}{\partial r} + \frac{yz}{r^2\sqrt{x^2+y^2}}\frac{\partial}{\partial\theta} + \frac{x}{x^2+y^2}\frac{\partial}{\partial\varphi}}$$

$$\frac{\partial}{\partial z} = \frac{z}{r}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial z}\frac{\partial}{\partial\theta}$$

$$= \frac{z}{r}\frac{\partial}{\partial r} + \frac{-\frac{\sqrt{x^2+y^2}}{z^2}}{1+\frac{x^2+y^2}{z^2}}\frac{\partial}{\partial\theta}$$

$$\frac{\partial}{\partial z} = \boxed{\frac{z}{r}\frac{\partial}{\partial r} - \frac{\sqrt{x^2+y^2}}{r^2}\frac{\partial}{\partial\theta}}$$

$$L_z = -i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) = -i\frac{\partial}{\partial \varphi}$$

$$L_+ = L_x + iL_y = -i\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) + \left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)$$

$$= -(x+iy)\frac{\partial}{\partial z} + z\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

$$= -r\sin\theta\, e^{i\varphi}\left(\cancel{\frac{z}{r}\frac{\partial}{\partial r}} - \frac{\sqrt{x^2+y^2}}{r^2}\frac{\partial}{\partial \theta}\right)$$

$$+ z\left(\cancel{\frac{x}{r}\frac{\partial}{\partial r}} + \frac{xz}{r^2\sqrt{x^2+y^2}}\frac{\partial}{\partial \theta} - \frac{y}{x^2+y^2}\frac{\partial}{\partial \varphi}\right)$$

$$+ iz\left(\cancel{\frac{y}{r}\frac{\partial}{\partial r}} + \frac{yz}{r^2\sqrt{x^2+y^2}}\frac{\partial}{\partial \theta} + \frac{x}{x^2+y^2}\frac{\partial}{\partial \varphi}\right)$$

$$= +(x+iy)\left(\frac{\sqrt{x^2+y^2}}{r^2} + \frac{z^2}{r^2\sqrt{x^2+y^2}}\right)\frac{\partial}{\partial \theta}$$

$$+ i(x+iy)\frac{z}{\sqrt{x^2+y^2}}\frac{\partial}{\partial \varphi}$$

$$L_+ = e^{i\varphi}\left(\frac{\partial}{\partial \theta} + i\cot\theta\frac{\partial}{\partial \varphi}\right)$$

$$L_- = e^{-i\varphi}\left(-\frac{\partial}{\partial \theta} + i\cot\theta\frac{\partial}{\partial \varphi}\right).$$

$$L_+ = e^{i\varphi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right)$$

$$L_- = e^{-i\varphi}\left(-\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right).$$

$$L_+L_- = e^{i\varphi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right)e^{-i\varphi}\left(-\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right)$$

$$= \cot\theta\left(-\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right)$$

$$+ \left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right)\left(-\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right) \qquad \left(\partial_\theta \cot\theta = -\frac{1}{s^2\theta}\right)$$

$$= -\frac{\partial^2}{\partial\theta^2} - \frac{i}{s^2\theta}\frac{\partial}{\partial\varphi} - \cot\theta\frac{\partial}{\partial\theta} + i\frac{\cos^2\theta}{s^2\theta}\frac{\partial}{\partial\varphi}$$

$$= -i\frac{\partial}{\partial\varphi} - \frac{\partial^2}{\partial\theta^2} - \cot\theta\frac{\partial}{\partial\theta}$$

$$\Rightarrow \mathbb{L}^2 = L_+ L_- + L_3^2 - L_3$$

$$= -\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

$$\Delta = \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} - \frac{\hat{\mathbb{L}}^2}{r^2}$$

$$-\frac{\hbar^2}{2m}\nabla^2 = H_0 =$$

$$= -\frac{\hbar^2}{2m}\left(\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}\right) + \underbrace{\frac{\hbar^2}{2mr^2}\hat{\mathbb{L}}^2}_{\text{potenziale centrifugo}}$$

Secondo la quantizzazione generale del mom. angolare

$$\mathbb{L}^2 \to \ell(\ell+1) \qquad \ell =$$

$$\left.\begin{aligned} \mathbb{L}^2|\ell, m\rangle &= \ell(\ell+1)|\ell, m\rangle \\ L_z|\ell, m\rangle &= m\,|\ell, m\rangle \end{aligned}\right\} (*)$$

$$L_z = -i\frac{\partial}{\partial\varphi}.$$

Eq. $\quad -i\frac{\partial}{\partial\varphi}\psi_m(\varphi) = m\,\psi_m(\varphi)$

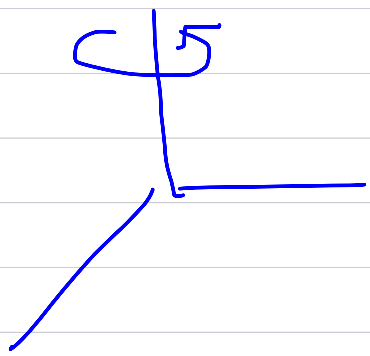
$$\psi_m(\varphi) = e^{im\varphi}.$$

ma $\psi_m(\varphi+2\pi) = \psi_m(\varphi) \Rightarrow \quad m = 0, \pm 1, \pm 2, \ldots \quad !$

Note:

$$\psi(\varphi + 2\pi) = \psi(\varphi).$$

perché $\psi(\varphi+2\pi) = e^{i\alpha}\psi(\varphi)$?!



$\pi_1(\mathbb{R}^3) = \mathbb{1}$!

cfr. $s = 0, 1/2, 1, \dots$

Visto che

$m = l, l-1, \dots, -l,$

$\Rightarrow l = 0, 1, 2, \dots \quad \Rightarrow$ mom. angolare orbitale quantizzato integralmente.

$(\times \hbar)$.

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mentre per gli spin,

$s = 0, \frac{1}{2}, 1, \frac{3}{2} \; \dots$

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autofunzioni $(*)$

$\Psi_{l,m}(\theta, \varphi)$

$$\left[-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]\Psi_{lm}(\theta\varphi)$$

$$= l(l+1)\Psi_{lm}(\theta,\varphi)$$

$$\Psi_{lm}(\theta,\varphi) = \Phi_{lm}(\theta)\, e^{im\varphi}$$

$$\Rightarrow \left[-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{m^2}{\sin^2\theta}\right]\Phi_{lm} = l(l+1)\Phi_{lm}$$

$$\left[-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{m^2}{\sin^2\theta}\right]\Phi_{\ell m}=\ell(\ell+1)\Phi_{\ell m}$$

$$\cos\theta \equiv x \qquad dx=-\sin\theta\, d\theta$$

$$\left\{-\frac{d}{dx}(1-x^2)\frac{d}{dx}+\frac{m^2}{1-x^2}\right\}\Phi_{\ell,m}=\ell(\ell+1)\Phi_{\ell,m}$$

$$\left\{\frac{d}{dx}(1-x^2)\frac{d}{dx}-\frac{m^2}{1-x^2}+\ell(\ell+1)\right\}\Phi_{\ell,m}(\theta)=0.$$

soluzione polinomiale (regolare $-1\leq x\leq 1$)

nota come

polinomi associati di Legendre

$$P_\ell^m(\theta)$$

N.B. Caso $m=0$ $\qquad P_\ell^0(\theta)\equiv P_\ell(\theta)$

$$\left\{\frac{d}{dx}(1-x^2)\frac{d}{dx}+\ell(\ell+1)\right\}P_\ell(x)=0 \qquad (*)$$

$P_\ell(x)=P_\ell(\cos\theta)=$ polinomi di Legendre.

Polinomi di Legendre (funzione generatrice)

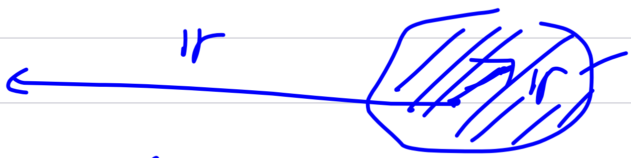
$$\frac{1}{\sqrt{1-2xs+s^2}} = \sum_{n=0}^{\infty} s^n P_\ell(x) \qquad (0<s<1)$$

N.B.
$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{\sqrt{\mathbf{r}^2+\mathbf{r}'^2-2\mathbf{r}\cdot\mathbf{r}'}}$$

$$\underset{r>r'}{=} \frac{1}{r}\frac{1}{\sqrt{1+\frac{r'^2}{r^2}-2\frac{r'}{r}\cos\theta}}$$

$$= \frac{1}{r}\sum_{\ell=0}^{\infty}\left(\frac{r'}{r}\right)^\ell P_\ell(\cos\theta)$$

$$\left(\underset{r'>r}{=} \frac{1}{r'}\sum_{\ell=0}^{\infty}\left(\frac{r}{r'}\right)^\ell P_\ell(\cos\theta)\right)$$



$\ell=0$ carico totale,

$\ell=1$ dipolo,

$\ell=2$ quadrupolo, ...

$$P_0(x)=1\ ; \quad P_1(x)=x\ ; \quad P_2(x)=\frac{1}{2}(3x^2-1)$$

...

$$P_\ell(-x) = (-)^\ell P_\ell(x)$$

$\frac{\partial}{\partial x}, \frac{\partial}{\partial s}$ (Funz. generatrice) $\Rightarrow$ (*)

$$\boxed{P_\ell^m(x) = (1-x^2)^{m/2} \frac{d^m}{dx^m} P_\ell(x)}$$

polinomi associati di Legendre.

$$P_\ell(1) = 1, \quad P_\ell(-1) = (-)^\ell.$$

$$\int_{-1}^{1} dx\, P_\ell(x)\, P_{\ell'}(x) = \frac{2}{2\ell+1}\, \delta_{\ell,\ell'}$$

$$\Phi_{\ell m}(\theta,\varphi) \equiv Y_{\ell,m}(\theta,\varphi)$$

$$= \boxed{C_{\ell,m} P_\ell^m(x)\, e^{im\varphi}}$$

Funzioni armoniche sferiche.

$$\equiv \langle \theta,\varphi | \ell, m \rangle$$

Normaliz.

$$\int d\theta \sin\theta\, d\varphi\; Y^*_{\ell',m'}(\theta,\varphi)\, Y_{\ell,m}(\theta,\varphi) = \delta_{\ell',\ell}\, \delta_{m',m}$$

Esempi:

$$Y_{0,0}(\theta,\varphi) = \frac{1}{\sqrt{4\pi}}$$

$$\begin{cases} Y_{1,1}(\theta,\varphi) = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\varphi} \\ Y_{1,-1}(\theta,\varphi) = \sqrt{\frac{3}{8\pi}}\sin\theta\, e^{-i\varphi} \\ Y_{1,0}(\theta,\varphi) = \sqrt{\frac{3}{4\pi}}\cos\theta \end{cases}, \quad \text{etc.}$$

Stati di momento angolare orbitale:

l, L	ONDA
0	S
1	P
2	D
3	F
	⋮

<u>N.B.</u>

$$Y_{1,1} \propto -\frac{x+iy}{\sqrt{2}}\frac{1}{r}$$

$$Y_{1,-1} \propto \frac{x-iy}{\sqrt{2}}\frac{1}{r}$$

$$Y_{1,0} \propto \frac{z}{r}$$

$$\begin{cases} Y_{l,m}(-\hat{r}) = (-)^l Y_{l,m}(\hat{r}) \\ Y^*_{l,m}(\hat{r}) = (-)^m Y_{l,-m}(\hat{r}) \end{cases} \Leftarrow \hat{r} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$$

Parità

Funzione d'onda di particelle con spin.

$s = 0 \qquad \psi = \psi(\mathbf{r})$

$s = 1/2 \qquad \psi = \begin{pmatrix} \psi_1(\mathbf{r}) \\ \psi_2(\mathbf{r}) \end{pmatrix} = \psi_1(\mathbf{r})|\uparrow\rangle + \psi_2(\mathbf{r})|\downarrow\rangle$

...

s generico $\qquad \psi = \begin{pmatrix} \psi_1(\mathbf{r}) \\ \vdots \\ \psi_{2s+1}(\mathbf{r}) \end{pmatrix} \Big\} 2s+1.$

---

e.g. $\quad H = \dfrac{p^2}{2m} + \dfrac{m\omega^2}{4}(\sigma_z + \mathbb{1})x^2$

autostat.:

$\psi_{1,n}(x) = |\uparrow\rangle \cdot \psi_n^{H.O.}(x)$

$\psi_{2,p}(x) = |\downarrow\rangle \; e^{ipx/\hbar}$

misura $\quad A = \lambda\sigma_x \Rightarrow$

$t=0 \quad A = \lambda \Rightarrow \quad \psi(0) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-\frac{x^2}{2\Delta}}$

e.g. quando $\mathbf{L} \cdot \mathbf{S}$ (interaz. spin-orbita).

$\begin{pmatrix} \psi_i(\mathbf{r}) \end{pmatrix} \Big\} 2s+1 \Rightarrow (\hat{\mathbf{L}} \cdot \hat{\mathbf{S}}) \begin{pmatrix} \psi_1(\mathbf{r}) \\ \psi_2(\mathbf{r}) \\ \vdots \end{pmatrix}$

La funzion d'onda di particella con spin.

$\Longleftrightarrow$ Legge di trasform. per rotazione $\underline{3D}$

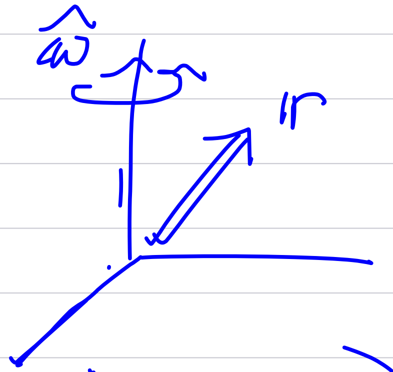
N.B. $\hat{\mathbf{p}} = -i\hbar\nabla$: operatore di traslaz. infinit.

$$e^{i\hat{\mathbf{p}}\cdot\mathbf{r}_0/\hbar}\,\psi(\mathbf{r}) = \psi(\mathbf{r}+\mathbf{r}_0).$$

analogamente $\hat{\mathbf{L}} = \hat{\mathbf{r}}\times\hat{\mathbf{p}}$: operatore di rotaz. infinitesimale.

<u>Particella</u> <u>senza</u> spin <u>$s=0$</u>.

Rot. di angolo $|\omega|$ nella direz. di $\hat{\omega}$ $\Longrightarrow$ degli assi di coordinate



$\Rightarrow$ $\mathbf{r} \longrightarrow$

$$\mathbf{r}' = \mathbf{r} - \omega\times\mathbf{r} + \cdots$$

"posizione apparente"

oppure

$$\mathbf{r} = \mathbf{r}' + \omega\times\mathbf{r}' + \cdots$$

vecchie coord. — nuove coordinate

dello stesso punto spaziale, P.

$$\psi = \psi(P)$$
$$= \psi(\mathbf{r}) \longrightarrow \psi(\mathbf{r}) = \psi(\mathbf{r}(\mathbf{r}'))$$
$$= \psi(\mathbf{r}' + \omega \times \mathbf{r}' + \cdots) \equiv \psi'(\mathbf{r}')$$

Cioè, la forma funzionale della funz. d'onda

$$\psi(\mathbf{r}) \to \psi'(\mathbf{r}) = \psi(\mathbf{r} + \omega \times \mathbf{r} + \cdots)$$
$$= \psi(\mathbf{r}) + \epsilon_{ijk}\, \omega_j\, r_k \cdot \frac{\partial}{\partial x_i} \psi(\mathbf{r}) + \cdots$$
$$= \psi(\mathbf{r}) + \epsilon_{kij}\, r_k \cdot \nabla_i\, \omega_j\, \psi$$
$$= \psi(\mathbf{r}) + i(\mathbf{r} \times \mathbf{p}) \cdot \omega\, \psi(\mathbf{r})$$
$$= \psi(\mathbf{r}) + i\, \mathbb{L} \cdot \omega\, \psi(\mathbf{r}) + \cdots$$
$$= e^{i\, \mathbb{L} \cdot \omega}\, \psi(\mathbf{r}) \quad !$$

Dunque la funz. d'onda si trasforma con l'operatore di rotaz.

$$e^{i\, \mathbb{L} \cdot \omega}$$

Una particella con spin s (funz. d'onda con $(2s+1)$ componenti)

$$\begin{pmatrix} \psi_s(\mathbf{r}) \\ \psi_{s-1} \\ \vdots \\ \psi_{-s} \end{pmatrix} \longrightarrow e^{i(\mathbb{L} + \mathbb{S}) \cdot \omega} \begin{pmatrix} \\ \\ \end{pmatrix}$$

$$\psi(r) \to e^{i\mathbb{L}\cdot\omega}\,\psi(r)$$ $\left(\dfrac{\text{Rappresentazione}}{\text{del gruppo } SO(3)}\right)$.

Legge semplice, e universale, ma implicita (in generale).

$$Y_{lm} \to e^{i\omega\cdot L}\, Y_{l,m}$$

$$= \sum_{m=-l}^{l} D^{l}_{m'm}(\hat{\omega})\, Y_{l,m'}(\theta,\varphi)$$ Def. $(D^{l}_{mm'})$

$D^{l}_{m'm}$ = matrice di rotazione

$$= \langle l\, m' |\, e^{i\omega\cdot L}\, | l, m \rangle$$

$$= \int d\theta \sin\theta\, d\varphi\; Y^{*}_{l m'}\, e^{i\omega\cdot L}\, Y_{lm}$$

Funz. d'onda di part. con spin S

$$\psi = \begin{pmatrix} \psi_s \\ \vdots \\ \psi_{-s} \end{pmatrix} \to e^{i(\mathbb{L}+\mathbb{S})\cdot\tilde{\omega}} \begin{pmatrix} \psi_s \\ \vdots \\ \psi_{-s} \end{pmatrix}$$

$$\psi_\sigma \to \left(e^{i\mathbb{S}\cdot\omega}\right)_{\sigma\sigma'}\, \psi_{\sigma'}$$

N.B. $\psi \to e^{iL\omega}\,\psi$

$$\sum_{l,m} R_{l,m} \, |l,m\rangle$$

$$\Rightarrow \; e^{-i\mathbb{L}\cdot\omega} \sum_{l,m} R_{l,m} \, |l,m\rangle$$

$$= \sum_{l,m} R_{lm} \, e^{i\mathbb{L}\cdot\omega} \, |l,m\rangle$$

$$= \quad '' \quad '' \quad \sum_{m'} D^{l}_{m'm} \, |l,m'\rangle$$

$$= \sum_{l,m'} \sum_{m} \left(D^{l}_{m'm} \cdot R_{l,m} \right) |l,m'\rangle$$

$$= \sum_{m'} R'_{l,m'} \, |l,m'\rangle$$

$$\therefore \; R'_{l\,m'} = \sum_{m} D^{l}_{m'm} \, \underbrace{R^{l}_{m}}$$

$$\Leftrightarrow \; \psi'_{\sigma} = \left(e^{iS\cdot\omega}\right)_{\sigma\sigma'} \psi_{\sigma'}$$

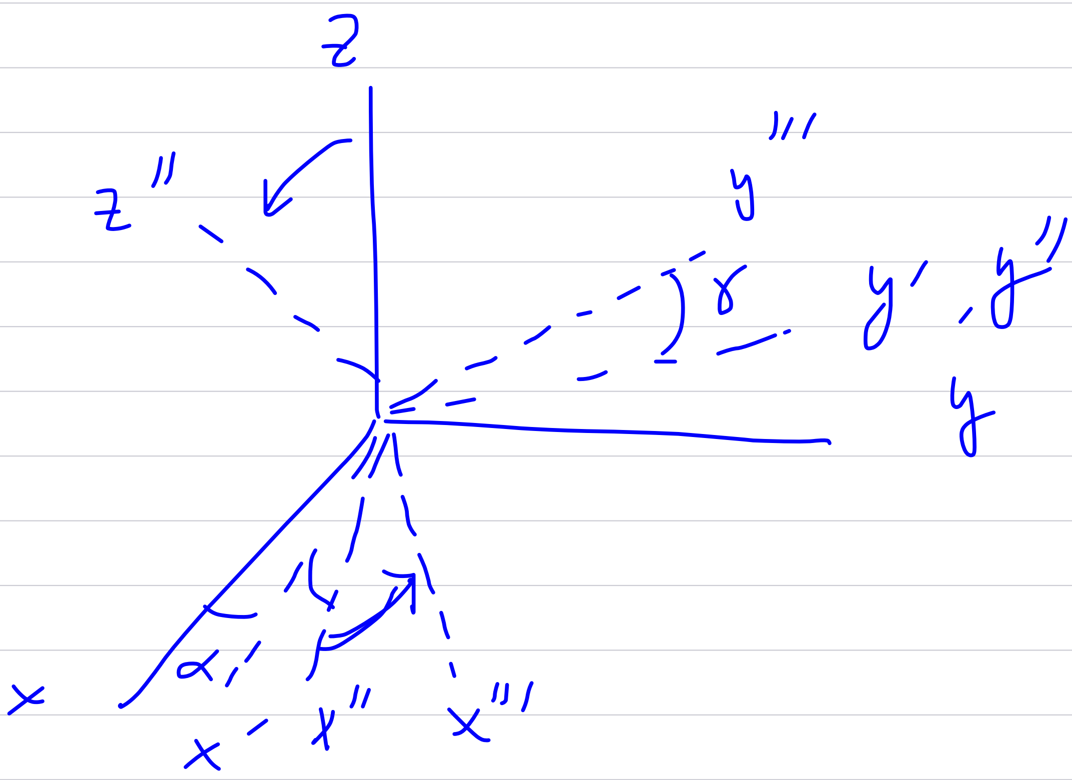
$$\hookrightarrow \; \psi = \begin{pmatrix} \psi_{\frac{1}{2}} \\ \vdots \\ \vdots \end{pmatrix} = \psi_{\frac{1}{2}} |1, 1\rangle + \psi_{\frac{1}{2}} |1, 1-1\rangle$$

$$+ \cdots$$

Matrici di rotazione per $j = s = \frac{1}{2}$.

Rotazione 3D generale $\sim (\alpha, \beta, \gamma)$
(angoli di Eulero)

(i) Rotazione attorno a $\hat{z}$ di angolo α
(ii) " " $\hat{y}'$ " " β
(iii) " " $\hat{z}''$ " " γ



$$D^{1/2}_{m'm} = U_z(\gamma)\, U_y(\beta)\, U_z(\alpha)$$

dove

$$U_z(\alpha) = e^{i\frac{1}{2}\alpha} = e^{\frac{i}{2}\sigma_z\alpha} = \begin{pmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2} \end{pmatrix}$$

$$U_y(\beta) = e^{i\frac{1}{2}\beta} \underline{\underline{\ }}\ e^{\frac{i}{2}\sigma_y\beta} \underline{\underline{\ }} \begin{pmatrix} \cos\beta/2 & \sin\beta/2 \\ -\sin\beta/2 & \cos\beta/2 \end{pmatrix}$$

$$U_z(\gamma) = \ldots$$

<u>N.B.</u> $$e^{i\boldsymbol{\alpha}\cdot\boldsymbol{\sigma}} = \cos|\boldsymbol{\alpha}| + i\,\frac{\boldsymbol{\alpha}\cdot\boldsymbol{\sigma}}{|\boldsymbol{\alpha}|}\sin|\boldsymbol{\alpha}|$$

$$\therefore\ D^{1/2}_{m'm}(\alpha,\beta,\gamma) = U_z(\gamma)\,U_y(\beta)\,U_z(\alpha)$$

$$= \begin{pmatrix} e^{+i\gamma/2} & 0 \\ 0 & e^{-i\gamma/2} \end{pmatrix}\begin{pmatrix} \cos\beta/2 & \sin\beta/2 \\ -\sin\beta/2 & \cos\beta/2 \end{pmatrix}\begin{pmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2} \end{pmatrix}$$

$$= \begin{pmatrix} & \\ & \end{pmatrix}\begin{pmatrix} e^{i\alpha/2}\cos\beta/2 & e^{-i\alpha/2}\sin\beta/2 \\ -e^{i\alpha/2}\sin\beta/2 & e^{-i\alpha/2}\cos\beta/2 \end{pmatrix}$$

$$= \begin{pmatrix} e^{i\frac{\alpha+\gamma}{2}}\cos\beta/2 & e^{-i\frac{\alpha-\gamma}{2}}\sin\beta/2 \\ -e^{i\frac{\alpha-\gamma}{2}}\sin\beta/2 & e^{-i\frac{\alpha+\gamma}{2}}\cos\beta/2 \end{pmatrix}$$

<u>N.B.</u> $$U_z(2\pi) = -\mathbb{1}$$

$$U_x(2\pi) = -\mathbb{1},\quad U_y(2\pi) = -\mathbb{1} \quad !$$

La funz. d'onda di $s=1/2$ ~ $\boxed{\text{spinori}}$.

Composizione di momenti angolari.

$$J = J_1 + J_2 = J_1 \otimes \mathbb{1} + \mathbb{1} \otimes J_2 .$$

operatori compatibili massimali. $\rightarrow \underbrace{J_z = J_{1z} + J_{2z}}$

$$\Rightarrow \{ \underline{J_1^2, J_{1z}\ J_2^2, J_{2z}} \} \Rightarrow \underline{|j_1 m_1 ; j_2 m_2 \rangle}$$

oppure

$$\Rightarrow \{ \underline{J^2, J_z\ , \ J_1^2, J_2^2} \} \Rightarrow \underline{|J M ; j_1 j_2 \rangle}$$

$$[J_z , J_1^2] = [J_{1z} + \cancel{J_{2z}} , J_{1x}^2 + J_{1y}^2 + J_{1z}^2]$$

$$= J_{1x}\, i J_{1y} + i J_{1y} J_{1x} + J_{1y}(-i J_{1x}) + (-i J_{1x}) J_{1y}$$

$$= 0$$

$$\Rightarrow [J^2, J_1^2] = 0 .$$

Vice versa

$$[J^2, J_{1z}] = [(J_{1x} + J_{2x})^2 + (J_{1y} + J_{2y})^2, J_{1z}] \neq 0$$

$\Rightarrow$ 2 basi di stati:

(I) $|j_1, m_1\rangle |j_2, m_2\rangle \equiv |j_1 m_1 ; j_2 m_2\rangle$

(II) $|J M ; j_1 j_2\rangle$.

Q $\left\{ \begin{array}{l} j_1, j_2 \longrightarrow J \ ? \\ \text{stati di (1)} \Longleftrightarrow \text{stati di (2)} \end{array} \right\}$ strettamente collegati

1° Relazione

Lo stato più alto nella base (J_i)

$$\sim \underset{m_1}{|j_1\, j_1\rangle}\,\underset{m_2}{|j_2\, j_2\rangle} = \text{Lo stato di Max } J_z \quad (\underbrace{J_z = J_{1z} + J_{2z}})$$

$\Rightarrow$ Max $J = j_1 + j_2$!

& $|j_1\, j_1; j_2\, j_2\rangle = |\underbrace{j_1+j_2}_{J}, \underbrace{j_1+j_2}_{M}, j_1, j_2\rangle$ (*)

$|j_1\, j_1\rangle$ ——
$|j_1\, j_1-1\rangle$ ——
$\vdots$
$|j_1\, -j_1\rangle$ ——

$\otimes$

$|j_2\, j_2\rangle$ ——
——
$\vdots$
$|j_2\, -j_2\rangle$ ——

$\Rightarrow$

$|J_1\, J_1\rangle$ ——
——
$\vdots$
$|J_1\, -J_1\rangle$ ——
$\} J_1$

$|J_2\, J_2\rangle$ ——
$\vdots$
——
$\} J_2$

$\vdots$

$J_- = J_{1-} + J_{2-}$ su $(*)$

(base I) $(J_{1-} + J_{2-}) |j_1 \, j_1\rangle |j_2 \, j_2\rangle$

$$= \sqrt{2j_1}\, |j_1, j_1 - 1\rangle |j_2 \, j_2\rangle + \sqrt{2j_2}\, |j_1 \, j_1\rangle |j_2 \, j_2 - 1\rangle \quad))$$

(base II) $$J_- \, |\underbrace{j_1 + j_2}_{J}, \underbrace{j_1 + j_2}_{M}\rangle = \sqrt{2(j_1 + j_2)}\, |\underbrace{j_1 + j_2}_{J}; \underbrace{j_1 + j_2 - 1}_{M}\rangle$$

Segue che:

$$|j_1 + j_2, j_1 + j_2 - 1\rangle = \sqrt{\frac{j_1}{j_1 + j_2}}\, |j_1 \, j_1 - 1\rangle |j_2 \, j_2\rangle + \sqrt{\frac{j_2}{j_1 + j_2}}\, |j_1 \, j_1\rangle |j_2 \, j_2 - 1\rangle \quad (**)$$

Abbiamo trovato una seconda relazione tra (I) e (II) !

Ora, ci sono due stati con $J_z = j_1 + j_2 - 1$.

Nella base (I), sono $|j_1 \, j_1 - 1\rangle |j_2 \, j_2\rangle$ e $|j_1 \, j_1\rangle |j_2 \, j_2 - 1\rangle$

Nella base (II) uno è proprio $(*)$. $\Rightarrow$ L'altro?

L'altro stato deve essere

$$|\underbrace{j_1 + j_2 - 1}_{J}, \underbrace{j_1 + j_2 - 1}_{M}\rangle, \qquad \text{poiché}$$

$J \geq M$ ma $J = j_1 + j_2$ è stato già considerato $(*)$

D'altronde $J \not< j_1 + j_2 - 1$!

$\therefore J = j_1 + j_2 - 1, \; M = j_1 + j_2 - 1$

$$|j_1+j_2-1, j_1+j_2-1\rangle = \text{?}$$

$$= \sqrt{\frac{j_2}{j_1+j_2}}\,|j_1\, j_1-1, j_2\, j_2\rangle - \sqrt{\frac{j_1}{j_1+j_2}}\,|j_1\, j_1; j_2\, j_2-1\rangle \quad (\ast\ast)$$

Abbiamo trovato una terza relazione tra (I) e (II)

Ora $J_- = J_{1-} + J_{2-}$ su

$J_-(\ast\ast) \Rightarrow$

$$|j_1+j_2, j_1+j_2-2\rangle = \circ\, |j_1\, j_1-2\rangle|j_2\, j_2\rangle + \circ\, |j_1\, j_1-1\rangle|j_2\, j_2-1\rangle + \circ\, |j_1\, j_1\rangle|j_2\, j_2-2\rangle$$

$$|j_1+j_2-1, j_1+j_2-2\rangle = \circ' \quad '' \quad \circ' \quad '' \quad + \circ' \quad ''$$

ma ci deve essere un altro stato con $M = j_1+j_2-2$ nella base (II). Qual'è ?

$\Rightarrow |j_1+j_2-2, j_1+j_2-2\rangle$!

$\Rightarrow$ Fino a quando ?

- a certo punto J_{1-} o J_{2-} invece di generare uno stato nuovo, annichila

per es. $J_{2-}|j_2, -j_2\rangle = 0$!

Questo avviene al $n = 2j_2+1$ ($j_2 \leq j_1$) se simo passo. ...

$$J_{min} = j_1 - j_2 \quad !!$$

Verifico:

\# di stati totale:

Nella base (I) $\Rightarrow (2j_1+1)(2j_2+1)$

Nella base (II) $\Rightarrow \sum_{j_1-j_2}^{j_1+j_2}(2j+1)$

$$= \left(\sum_{1}^{j_1+j_2} 2j - \sum_{1}^{j_1-j_2-1} 2j\right) + 2j_2+1$$

$$= (j_1+j_2)(j_1+j_2+1) - (j_1-j_2-1)(j_1-j_2) + 2j_2+1$$

$$= (j_1+j_2)^2 + j_1+j_2 - (j_1-j_2)^2 + j_1 - j_2 + 2j_2 + 1$$

$$= 4j_1j_2 + 2j_1 + 2j_2 + 1 = (2j_1+1)(2j_2+1) \quad \text{ok.}$$

$$\underset{\downarrow}{j_1} \otimes \underset{\downarrow}{j_2} = \underline{j_1+j_2} \oplus \underline{j_1+j_2-1} \oplus \cdots \oplus |j_1-j_2|$$

multipletti

$j_1, j_2 \Rightarrow J = j_1+j_2, j_1+j_2-1, \ldots |j_1-j_2|$

$\frac{1}{2} \otimes \frac{1}{2} = \underline{1} \oplus \underline{0} \quad \Rightarrow$