Problems Chapter 1

Quantum Mechanics
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Problem 1

Form all possible quantities having the dimension of a length, by using m_e, $\hbar$, e^2. Try the same by allowing also the velocity of light, c.

Solution

Having at our disposal m_e, $\hbar$ and e^2 only, the unique combination possible is the Bohr radius

$$a = \frac{\hbar^2}{m e^2} \simeq 5.29\, 10^{-9}\, \text{cm}. \tag{1.1}$$

e^2 has dimension Energy × Length = m v^2L; $\hbar$ has dimensions p L= m v L, then

$$\frac{\hbar^2}{m e^2} = \frac{m^2 v^2 L^2}{m\, m\, v^2 L} = L\,.$$

When c is also allowed, other quantities can be formed. One is the "classical eletron radius":

$$r_e = \frac{e^2}{m c^2} \simeq 2.8\, 10^{-13}\, \text{cm}\,; \tag{1.2}$$

another is the "Compton length" of the electron

$$r_c = \frac{\hbar}{m c} \simeq 3.8\, 10^{-11}\, \text{cm} \quad \text{or } \lambda_c = \frac{h}{m c} \simeq 2.43\, 10^{-10}\, \text{cm} \tag{1.3}$$

Note that r_c is tipically 100 times the standard nuclear radius, 10^{-13}cm. This was one of the reasons why the electron was considered to be absent in the nucleus.

As the combination

$$\alpha = \frac{\hbar^2}{e c} \simeq \frac{1}{137} \tag{1.4}$$

("fine structure constant") is dimensionless, actually we can contruct an infinite number of quantities by using m_e, $\hbar$, e^2 and c.

Problem 2

Consider the scattering of an X-ray on an electron at rest. The X ray with wavelength λ is regarded as a flux of photons, each with energy $h\nu$ and momentum $k = h\,\nu/c$, where $\nu = c/\lambda$. Let p and E the momentum and the energy of the electron in the final state and let ϕ and θ the angle between the directions of the final photon momentum and electron momentum, respectively, with respect to the incident photon direction. The final photon has energy and momentum $h\nu'$ and $h\,\nu'/c$. Use energy and momentum conservation to obtain the Compton formula

$$\lambda' - \lambda = \frac{2h}{m c} \sin^2\left[\frac{\phi}{2}\right] = \frac{h}{m c}(1 - \cos[\phi])\,. \tag{2.1}$$

Solution

The proof of equation (1) requires only energy and momentum conservation. We use momentum conservation to eliminate the electron scattering angle and energy conservation to eliminate its velocity.

Energy conservation implies

$$m c^2 + h\nu = h\nu' + E_e = h\nu' + \frac{m c^2}{\sqrt{1-\frac{v^2}{c^2}}}. \tag{2.2}$$

Let the x axis be in the direction of the incident X ray and y perpendicular to it, such that the momenta of the final particles are contained in the x-y plane. Using the notation of the text, momentum conservation reads

$$\begin{aligned} \frac{h\nu}{c} &= p_x + k'_x = p\,\mathrm{Cos}[\theta] + \frac{h\nu'}{c}\mathrm{Cos}[\phi] = \frac{m v}{\sqrt{1-\frac{v^2}{c^2}}}\mathrm{Cos}[\theta] + \frac{h\nu'}{c}\mathrm{Cos}[\phi] \\ 0 &= p_y + k'_y = \frac{m v}{\sqrt{1-\frac{v^2}{c^2}}}\mathrm{Sin}[\theta] + \frac{h\nu'}{c}\mathrm{Sin}[\phi] \end{aligned} \tag{2.3}$$

The electron angle θ can be eliminated from the two equations, giving

$$\frac{m^2 v^2}{1-\frac{v^2}{c^2}} = \left(\frac{h\nu}{c} - \frac{h\nu'}{c}\mathrm{Cos}[\phi]\right)^2 + \left(\frac{h\nu'}{c}\mathrm{Sin}[\phi]\right)^2 = \left(\frac{h\nu}{c}\right)^2 + \left(\frac{h\nu'}{c}\right)^2 - 2\frac{h\nu}{c}\frac{h\nu'}{c}\mathrm{Cos}[\phi]. \tag{2.4}$$

The left hand side can be written as

$$\frac{m^2 v^2}{1-\frac{v^2}{c^2}} = \frac{m^2 c^2 \left(v^2/c^2\right)}{1-\frac{v^2}{c^2}} = -m^2 c^2 + \frac{m^2 c^2}{1-\frac{v^2}{c^2}},$$

while energy conservation (2) yields

$$\frac{m c}{\sqrt{1-\frac{v^2}{c^2}}} = m c + \frac{h\nu}{c} - \frac{h\nu'}{c} \Rightarrow \frac{m^2 c^2}{1-\frac{v^2}{c^2}} = \left(m c + \frac{h\nu}{c} - \frac{h\nu'}{c}\right)^2.$$

The left hand side of eq.(4) is then

$$\left(m c + \frac{h\nu}{c} - \frac{h\nu'}{c}\right)^2 - m^2 c^2 = \left(\frac{h\nu}{c}\right)^2 + \left(\frac{h\nu'}{c}\right)^2 + 2 m (h\nu - h\nu') - 2\frac{h\nu}{c}\frac{h\nu'}{c}.$$

Substitution in equation (4) gives

$$2 m (h\nu - h\nu') - 2\frac{h\nu}{c}\frac{h\nu'}{c} = -2\frac{h\nu}{c}\frac{h\nu'}{c}\mathrm{Cos}[\phi].$$

Multiplication by $c/h\,\nu\, c\,/h\nu'$ finally gives eq.(1):

$$\frac{m c^2}{h\nu'} - \frac{m c^2}{h\nu} = 1 - \mathrm{Cos}[\phi] \quad \Rightarrow \quad \lambda' - \lambda = \frac{h}{m c}(1 - \mathrm{Cos}[\phi]). \tag{2.5}$$

■ **The use of invariants**

This derivation is **not** the easiest one; it is actually better to work directly with four - momentum. We have

$$k = \left(\frac{h\nu}{c}, \mathbf{k}\right); \quad p = \left(\frac{E}{c}, \mathbf{p}\right); \quad k^2 = 0; \quad p^2 = m^2 c^2. \tag{2.6}$$

The four - momentum conservation reads

$$k + p = k' + p' \Rightarrow (k - k')^2 = (p - p')^2.$$

Expanding this relation and using the invariants (6) one has immediately

$$\frac{h\nu}{c}\frac{h\nu'}{c}(1 - \mathrm{Cos}[\phi]) = \frac{E}{c} m c - m^2 c^2. \tag{2.7}$$

Using the energy conservation