Problems Chapter 16

Quantum Mechanics
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Problem 1

Write in the Born approximation the amplitude for two identical particles, subject to a spin-independent interaction. Consider in particular the Coulomb interaction.

Solution

Let us consider the system in the center of mass frame. We will consider a central potential.

The problem is formally equivalent to a scattering of a single particle of reduced mass μ = m/2 in a potential V[r]. The amplitude in Born approximation is (ignoring the problem of the identity of particles)

$$f[\theta] = -\frac{\mu}{2\pi\hbar^2}\langle \mathbf{p}' | V | \mathbf{p}\rangle = -\frac{\mu}{2\pi\hbar^2}\int d^3\mathbf{r}\, V[r]\, e^{\frac{i}{\hbar}(\mathbf{p}-\mathbf{p}')\mathbf{r}}.$$
$$\text{Coulomb}: \quad f_c[\theta] = -\frac{\mu}{2\pi\hbar^2}\, e^2\, \frac{4\pi\hbar^2}{|\mathbf{p}-\mathbf{p}'|^2} \tag{1.1}$$

θ is the scattering angle, $\mathbf{p}\cdot\mathbf{p}' = p^2 \, \mathrm{Cos}[\theta]$. We have

$$|\mathbf{p}-\mathbf{p}'|^2 = 2p^2(1-\mathrm{Cos}[\theta]) = 4\mu^2 v^2 \,\mathrm{Sin}\left[\frac{\theta}{2}\right]^2; \quad v = \text{velocity in the c.m.}$$

In the Coulomb case

$$f_c[\theta] = -\frac{e^2}{2\mu v^2}\frac{1}{\mathrm{Sin}\left[\frac{\theta}{2}\right]^2} = -\frac{e^2}{m v^2}\frac{1}{\mathrm{Sin}\left[\frac{\theta}{2}\right]^2}.$$

Identical particles

Let us now consider identical particles, electrons as an instance. The admitted states satisfying Pauli principle are

$$\frac{1}{\sqrt{2}}(|\mathbf{p},-\mathbf{p}\rangle + |-\mathbf{p},\mathbf{p}\rangle); \quad \text{spin singlet};$$
$$\frac{1}{\sqrt{2}}(|\mathbf{p},-\mathbf{p}\rangle - |-\mathbf{p},\mathbf{p}\rangle); \quad \text{spin triplet}. \tag{1.2}$$

We have to compute matrix elements of the Hamiltonian interaction between these states.

For the singlet state

$$\mathcal{F}_s = -\frac{\mu}{2\pi\hbar^2}\frac{1}{2}\,2\,(\langle\mathbf{p},-\mathbf{p}|V|\mathbf{p},-\mathbf{p}\rangle + \langle\mathbf{p},-\mathbf{p}|V|-\mathbf{p},+\mathbf{p}\rangle)$$

The matrix elements depend only on p and θ. In the second matrix element the final momentum is opposite to the first one, then $\theta \to \pi - \theta$. For the amplitude we get

$$\mathcal{F}_s = f[\theta] + f[\pi-\theta].$$

In the same manner for the triplet

$$\mathcal{F}_t = f[\theta] - f[\pi-\theta].$$

The corresponding differential cross section are

$$d\sigma_s = |\mathcal{F}_s|^2 d\Omega = |f[\theta] + f[\pi-\theta]|^2; \quad d\sigma_t = |\mathcal{F}_t|^2 d\Omega = |f[\theta] - f[\pi-\theta]|^2.$$

For an unpolarized beam, with 3/4 probability to be in a triplet state and 1/4 to be in a singlet state

$$d\bar{\sigma} = \frac{3}{4} d\sigma_t + \frac{1}{4} d\sigma_s . \tag{1.3}$$

Problem 2

Derive the eikonal approximation from the asymptotic form of Legendre polynomials (for large L and small θ):

$$P_L[\mathrm{Cos}[\theta]] \sim J_0[L\,\theta] . \tag{2.1}$$

Solution

The validity of eq.(1) has been established in the text, see Chapter on WKB approximation.

Let b the impact parameter. The angular momentum is given by L = k b.

At high energy the amplitude is the sum of a very large number of partial waves. Approximating the sum on L with an integral and using (1)

$$f[\theta] = \frac{1}{2\,i\,k}\sum_L (2\,L+1)\,P_L[\mathrm{Cos}[\theta]]\,(\mathrm{Exp}[2\,i\,\delta_L]-1) \simeq \frac{1}{2\,i\,k}\int_0^\infty k\,db\,(2\,k\,b)\,J_0[k\,b\,\theta]\,(\mathrm{Exp}[2\,i\,\delta]-1) =$$
$$-i\,k\int_0^\infty b\,db\,J_0[k\,b\,\theta]\left(e^{2\,i\,\delta}-1\right)$$

Let us now consider the phase scattering. For large L the semiclassical approximation works. The semiclassical phase, computed along the classical trajectory, is

$$\varphi = \frac{1}{\hbar}\int p[\mathbf{x}]\,d\mathbf{x} .$$

At high energy the trajectory canbe approximated by a straight line parallel to the z axis, with an impact parameter b. Subtracting the free phase we get the scattering phase shift

$$\delta = \frac{1}{\hbar}\int_{-\infty}^{+\infty} dz\,\sqrt{2\,m\,(E - V[r])} - \frac{1}{\hbar}\int_{-\infty}^{+\infty} dz\,\sqrt{2\,mE} \sim -\frac{\sqrt{2\,mE}}{\hbar}\,\frac{1}{2\,E}\int_{-\infty}^{+\infty} dz\,V[r] = -\frac{m}{2\,\hbar^2\,k}\int_{-\infty}^{+\infty} V\left[\sqrt{z^2+b^2}\right] dz.$$

and

$$f[\theta] = -i\,k\int_0^\infty b\,db\,J_0[k\,b\,\theta]\left(e^{2\,i\,\delta}-1\right) . \tag{2.2}$$

which coincides with the formula given in the text.

Problem 3

Compute the scattering amplitude for an impenetrable sphere of radius R in the Kirchhoff approximation.

Solution

This is a standard problem in optics or acoustics. The solution given below is similar to the one presented in ref.[1], except for some adaptations for the Schrödinger equation boundary conditions.

The Green' s formula

The Schrödinger equation reads

$$\triangle\,\psi[\mathbf{x}] + k^2\,\psi[\mathbf{x}] = 0, \qquad |\,\mathbf{x}\,| > R,\ \ \psi[\mathbf{x}] = 0 \text{ for } |\,\mathbf{x}\,| = R. \tag{3.1}$$

The solution of the scattering problem can be written as

$$\psi[\mathbf{x}] = \psi_i[\mathbf{x}] + \varphi[\mathbf{x}], \tag{3.2}$$

where $\psi_i$is the incident wave and φ the scattered wave, which behaves asymptotically as a divergent spherical wave. As ψ_i satisfies the free wave equation, also φ is a solution of (1), with the boundary condition $\varphi = -\psi_i$ on the sphere.

Let us consider the Green's function

$$G[\mathbf{x},\mathbf{y}] = \frac{1}{|\mathbf{x}-\mathbf{y}|} \operatorname{Exp}[i\, k\, |\mathbf{x}-\mathbf{y}|], \tag{3.3}$$

This function is a solution of the free wave equation except at x=y, where it is singular:

$$\triangle_y G[\mathbf{x},\mathbf{y}] + k^2 G[\mathbf{x},\mathbf{y}] = -4\pi\,\delta[\mathbf{x}-\mathbf{y}]. \tag{3.4}$$

It follows

$$\varphi[\mathbf{y}]\,\triangle_y G[\mathbf{x},\mathbf{y}] - G[\mathbf{x},\mathbf{y}]\,\triangle_y \varphi[\mathbf{y}] = -4\pi\,\delta[\mathbf{x}-\mathbf{y}]\,\varphi[\mathbf{y}]. \tag{3.5}$$

Integration on the region R external to the sphere gives

$$\int_R \varphi[\mathbf{y}]\,\triangle_y G[\mathbf{x},\mathbf{y}] - G[\mathbf{x},\mathbf{y}]\,\triangle_y \varphi[\mathbf{y}] = -4\pi\,\varphi[\mathbf{x}]. \tag{3.6}$$

The left hand side can be written as

$$\varphi[\mathbf{y}]\,\triangle_y G[\mathbf{x},\mathbf{y}] - G[\mathbf{x},\mathbf{y}]\,\triangle_y \varphi[\mathbf{y}] = \frac{\partial}{\partial \mathbf{y}}\left\{\varphi[\mathbf{y}]\frac{\partial}{\partial \mathbf{y}} G[\mathbf{x},\mathbf{y}] - G[\mathbf{x},\mathbf{y}]\frac{\partial}{\partial \mathbf{y}}\varphi[\mathbf{y}]\right\}. \tag{3.7}$$

Using Gauss theorem in (6) and neglecting contribution at infinity (this is correct as the integrand vanishes more rapidly then $1/r^2$) we get

$$4\pi\,\varphi[\mathbf{x}] = \int_{\text{sphere}} \left\{\varphi[\mathbf{y}]\frac{\partial}{\partial \mathbf{y}} G[\mathbf{x},\mathbf{y}] - G[\mathbf{x},\mathbf{y}]\frac{\partial}{\partial \mathbf{y}}\varphi[\mathbf{y}]\right\} d\mathbf{A}. \tag{3.8}$$

The additional minus sign in (8) is due to the fact that Gauss theorem involve the normal direction *outside* the region R, while in (8) we used the usual convention considering d**A** directed toward the outside of the sphere, then inside R.

Formula (8) is the well known Green's formula, here re-derived just for convenience of the reader.

■ The Kirchhoff approximation

Let z be the direction of the incident particle, z=0 the plane through the origin of the sphere and x-z the scattering plane.

The boundary condition $\psi = 0$ in the surface of the sphere imply $\varphi = -\psi_i$.

The hemisphere at z<0 can be considered the illuminated zone (I) in optics language while the hemisphere for z > 0 is the shadow (S) zone. On I the wave is reflected and approximately we can put there

$$\frac{\partial \varphi}{\partial n} = \frac{\partial \psi_i}{\partial n}, \tag{3.9}$$

n is the normal direction. This approximation is easily understood by considering a reflection against a plane, let say z=0, the vanishing of ψ and the existence of the reflected wave gives

$$\psi \simeq e^{i k z} + \varphi = e^{i k z} - e^{-i k z} \Rightarrow \left(\frac{\partial \varphi}{\partial z}\right)_{z=0} = i\,k = \left(\frac{\partial \psi_i}{\partial z}\right)_{z=0}.$$

In the shadow zone in a first approximation $\psi = 0$ then

$$\frac{\partial \varphi}{\partial n} = -\frac{\partial \psi_i}{\partial n},$$

i.e. in the shadow zone both φ and its normal derivative are (approximately) opposite to the incident wave.

The scattered wave (8) can then be written as a sum of the integrals over the two zones

$$\varphi[\mathbf{x}] = -\frac{1}{4\pi}\int_I \left\{\psi_i[y]\frac{\partial}{\partial \mathbf{y}} G[\mathbf{x},\mathbf{y}] + G[\mathbf{x},\mathbf{y}]\frac{\partial}{\partial \mathbf{y}}\psi_i[y]\right\} d\mathbf{A} - \frac{1}{4\pi}\int_S \left\{\psi_i[y]\frac{\partial}{\partial \mathbf{y}} G[\mathbf{x},\mathbf{y}] - G[\mathbf{x},\mathbf{y}]\frac{\partial}{\partial \mathbf{y}}\psi_i[y]\right\} d\mathbf{A} \tag{3.10}$$

In the wave region, $|\mathbf{x}| \gg R$ we can approximate the Green's function with

$$G[\mathbf{x},\mathbf{y}] \simeq \frac{1}{r} \operatorname{Exp}[i\, k\, r]\, \operatorname{Exp}[-i k\, \mathbf{n}\cdot\mathbf{y}]. \tag{3.11}$$

with

$$r = |\mathbf{x}|;\quad \mathbf{n} = \mathbf{x}/r = (\operatorname{Sin}[\theta], 0, \operatorname{Cos}[\theta]).$$

■ The shadow integral

Let us first consider the integral over the hemisphere S. As |x|>R inside the sphere

$$\frac{\partial}{\partial \mathbf{y}}\left\{\psi_i[y]\,\frac{\partial}{\partial \mathbf{y}}G[\mathbf{x},\mathbf{y}] - G[\mathbf{x},\mathbf{y}]\,\frac{\partial}{\partial \mathbf{y}}\psi_i[y]\right\} \equiv \frac{\partial}{\partial \mathbf{y}}\mathbf{H} = 0,$$

then the flux of the vector **H** on a closed surface is zero, this means that the flux through S is equal to the flux through the disk D, intersection of the plane z=0 with the sphere, easier to compute.

With

$$x = r\,(\mathrm{Sin}[\theta],\,0,\,\mathrm{Cos}[\theta]);\quad \mathbf{y} = (\xi,\,\eta,\,\zeta);\quad (\xi,\,\eta) = b\,(\mathrm{Cos}[\varphi_0],\,\mathrm{Sin}[\varphi_0]);$$

$$\mathbf{k} = k\,(0,\,0,\,1);\qquad \mathbf{h} = \text{normal to D (z axis)} = (0,\,0,\,1);$$

we have, from (11) and using that the normal to D is directed along z:

$$\mathbf{h}\cdot\frac{\partial}{\partial \mathbf{y}}G[\mathbf{x},\mathbf{y}] = -i\,\frac{1}{r}\,e^{ikr}\,e^{-i\,k\,\mathbf{n}\cdot\mathbf{y}}\,k\,\mathbf{n}\cdot\mathbf{h} = -i\,\frac{1}{r}\,e^{ikr}\,e^{-i\,k\,b\,\mathrm{Sin}[\theta]\,\mathrm{Cos}[\varphi_0]}\,k\,\mathrm{Cos}[\theta]$$

$$\mathbf{h}\cdot\frac{\partial}{\partial \mathbf{y}}\psi_i[\mathbf{y}] = i\,\mathbf{h}\cdot\mathbf{k}\,e^{i\,\mathbf{k}\cdot\mathbf{y}} = i\,k\,e^{i\,k\,\zeta} \to (\text{on S}) \to i\,k$$

Then the integral on S is

$$\varphi[x] = +\frac{1}{4\pi}\int_0^R b\,db\int_0^{2\pi}d\varphi_0\,\frac{1}{r}\,e^{ikr}\,e^{-i\,k\,b\,\mathrm{Sin}[\theta]\,\mathrm{Cos}[\varphi_0]}\,i\,k\,(1+\mathrm{Cos}[\theta]) =$$

$$\frac{i\,k}{4\pi}\,(1+\mathrm{Cos}[\theta])\,\frac{1}{r}\,e^{ikr}\int_0^R b\,db\int_0^{2\pi}d\varphi_0\,e^{-i\,k\,b\,\mathrm{Sin}[\theta]\,\mathrm{Cos}[\varphi_0]} =$$

$$\frac{i\,k}{4\pi}\,(1+\mathrm{Cos}[\theta])\,\frac{1}{r}\,e^{ikr}\int_0^R b\,db\,2\pi\,J_0[k\,b\,\mathrm{Sin}[\theta]] = \frac{i\,k}{4\pi}\,(1+\mathrm{Cos}[\theta])\,\frac{1}{r}\,e^{ikr}\,2\pi\,\frac{R}{k\,\mathrm{Sin}[\theta]}\,J_1[k\,R\,\mathrm{Sin}[\theta]].$$

Then the contribution to scattering amplitude from the shadow zone is

$$f_S = \frac{i}{2}\,R\,\frac{(1+\mathrm{Cos}[\theta])}{\mathrm{Sin}[\theta]}\,J_1[k\,R\,\mathrm{Sin}[\theta]]. \qquad (3.12)$$

■ The illuminated zone

Here the integral has to be done directly on the hemisphere. The normal is

$$\mathbf{h} = (\mathrm{Sin}[\theta_0]\,\mathrm{Cos}[\varphi_0],\,\mathrm{Sin}[\theta_0]\,\mathrm{Sin}[\varphi_0],\,\mathrm{Cos}[\theta_0]). \qquad (3.13)$$

On I, with γ the angle between **x** and **h**:

$$\mathbf{h}\cdot\frac{\partial}{\partial \mathbf{y}}G[\mathbf{x},\mathbf{y}] = -i\,\frac{1}{r}\,e^{ikr}\,e^{-i\,k\,\mathbf{n}\cdot\mathbf{y}}\,k\,\mathbf{n}\cdot\mathbf{h} = -i\,\frac{1}{r}\,e^{ikr}\,e^{-i\,k\,R\,\mathrm{Cos}[\gamma]}\,k\,\mathrm{Cos}[\gamma];$$

$$\mathbf{h}\cdot\frac{\partial}{\partial \mathbf{y}}\psi_i[\mathbf{y}] = i\,\mathbf{h}\cdot\mathbf{k}\,e^{i\,\mathbf{k}\cdot\mathbf{y}} = i\,k\,\mathrm{Cos}[\theta_0]\,e^{i\,k\,R\,\mathrm{Cos}[\theta_0]};$$

$$\mathrm{Cos}[\gamma] = \mathrm{Cos}[\theta]\,\mathrm{Cos}[\theta_0] + \mathrm{Sin}[\theta]\,\mathrm{Sin}[\theta_0]\,\mathrm{Cos}[\varphi]$$

$$\varphi[\mathbf{x}] = -\frac{1}{4\pi}\int_I\left\{\psi_i[y]\,\frac{\partial}{\partial \mathbf{y}}G[\mathbf{x},\mathbf{y}] + G[\mathbf{x},\mathbf{y}]\,\frac{\partial}{\partial \mathbf{y}}\psi_i[y]\right\}d\mathbf{A} - \frac{1}{4\pi}\int_S\left\{\psi_i[y]\,\frac{\partial}{\partial \mathbf{y}}G[\mathbf{x},\mathbf{y}] - G[\mathbf{x},\mathbf{y}]\,\frac{\partial}{\partial \mathbf{y}}\psi_i[y]\right\}d\mathbf{A} \qquad (3.14)$$

The integral over I is

$$\varphi[\mathbf{x}] = -\frac{R^2}{4\pi}\int_{\pi/2}^{\pi}\mathrm{Sin}[\theta_0]\,d\theta_0\int_0^{2\pi}d\varphi_0\,\frac{e^{ikr}}{r}\,\mathrm{Exp}[i\,k\,R\,\mathrm{Cos}[\theta_0] - i\,k\,R\,\mathrm{Cos}[\gamma]]\,i\,k\,(\mathrm{Cos}[\theta_0] - \mathrm{Cos}[\gamma]) \qquad (3.15)$$

The integral can be estimated by saddle point method. In φ_0 the saddle point is clearly $\varphi_0 = 0$. In θ_0 we have the condition of stationary phase (with $\varphi_0 = 0$)

$$0 = \frac{\partial}{\partial\theta_0}(\mathrm{Cos}[\theta_0] - \mathrm{Cos}[\gamma]) = -\mathrm{Sin}[\theta_0] + \mathrm{Cos}[\theta]\,\mathrm{Sin}[\theta_0] - \mathrm{Sin}[\theta]\,\mathrm{Cos}[\theta_0] = -\mathrm{Sin}[\theta_0] + \mathrm{Sin}[\theta_0 - \theta],$$

with solution

$$\theta_0 = \frac{\pi}{2} + \frac{\theta}{2}. \qquad (3.16)$$

The phase at the stationary point is

$$i\, k\, R\, (\mathrm{Cos}[\theta_0] - \mathrm{Cos}[\gamma])_{\theta_0=\frac{\pi}{2}+\frac{\theta}{2}} = -2\, i\, k\, R\, \mathrm{Sin}\left[\frac{\theta}{2}\right];$$

The quadratic fluctuations are, with $\theta_0 = = \frac{\pi}{2} + \frac{\theta}{2} + u$

$$i\, k\, R\, (\mathrm{Cos}[\theta_0] - \mathrm{Cos}[\gamma]) \simeq -2\, i\, k\, R\, \mathrm{Sin}\left[\frac{\theta}{2}\right] \mathrm{Cos}[u] - i\, k\, R\, \mathrm{Cos}\left[\frac{\theta}{2}\right] \mathrm{Sin}[\theta] \left(\frac{-1}{2}\varphi^2\right) \simeq$$

$$i\, k\, R \left(\mathrm{Sin}\left[\frac{\theta}{2}\right] u^2 + \frac{1}{2}\varphi^2\, \mathrm{Cos}\left[\frac{\theta}{2}\right] \mathrm{Sin}[\theta]\right)$$

Using the Fresnel integrals

$$\int_{-\infty}^{\infty} \mathrm{Exp}[i\, \alpha\, x^2]\, dx = \frac{\sqrt{\pi}}{\sqrt{-i\, \alpha}},$$

the integral over quadratic fluctuations gives

$$\frac{1}{-i}\, \frac{1}{kR}\, \frac{\pi}{\sqrt{\frac{1}{2}\mathrm{Sin}\left[\frac{\theta}{2}\right] \mathrm{Cos}\left[\frac{\theta}{2}\right] \mathrm{Sin}[\theta]}} = i\, \frac{2}{k\, R}\, \frac{\pi}{\mathrm{Sin}[\theta]}.$$

Finally the saddle point value of the non exponential factors is

$$\mathrm{Sin}[\theta_0]\, (\mathrm{Cos}[\theta_0] - \mathrm{Cos}[\gamma]) \to \mathrm{Cos}\left[\frac{\theta}{2}\right] \left(-2\, \mathrm{Sin}\left[\frac{\theta}{2}\right]\right) = -\mathrm{Sin}[\theta].$$

Then from (15) the contribution of the illuminated zone to the scattering amplitude is

$$f_I = -\frac{R^2}{4\pi}\, (i\, k)\, (-\mathrm{Sin}[\theta])\, \frac{2}{k\, R}\, \frac{i\, \pi}{\mathrm{Sin}[\theta]}\, \mathrm{Exp}\left[-2\, i\, k\, R\, \mathrm{Sin}\left[\frac{\theta}{2}\right]\right] = -\frac{R}{2}\, \mathrm{Exp}\left[-2\, i\, k\, R\, \mathrm{Sin}\left[\frac{\theta}{2}\right]\right]$$

The full scattering amplitude is then

$$f[\theta] = -\frac{R}{2}\, \mathrm{Exp}\left[-2\, i\, k\, R\, \mathrm{Sin}\left[\frac{\theta}{2}\right]\right] + \frac{i}{2}\, R\, \frac{(1 + \mathrm{Cos}[\theta])}{\mathrm{Sin}[\theta]}\, J_1[k\, R\, \mathrm{Sin}[\theta]]. \qquad (3.17)$$

The first term, with constant amplitude, represents the reflected wave.

References

1. P.M. Morse, H. Feshbach: *Methods of Theoretical Physics*, McGraw-Hill Book Company, Inc. (1953) .