Problems Chapter 3

Quantum Mechanics
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Problem 1

Show that the condition

$$\psi'_{+}[0] - \psi'_{-}[0] = -\frac{2\,m\,g}{\hbar^2}\,\psi[0],$$

for a potential V[x] = - g δ[x], *is compatible with the continuity of the current density.*

Solution

The current density is

$$j = \frac{i\,\hbar}{2\,m}\,(\psi\,\partial_x\psi^* - \psi^*\,\partial_x\psi)\,.$$

At the discontinuity of the potential, x=0, ψ is continuous while for the derivative of ψ we have

$$\triangle(\partial_x\psi) = -\frac{2\,m\,g}{\hbar^2}\,\psi[0];\quad \triangle(\partial_x\psi^*) = -\frac{2\,m\,g}{\hbar^2}\,\psi^*[0];\qquad \text{with}\ \ \triangle F = \lim_{\epsilon\to 0}(F[\epsilon] - F[-\epsilon])\,.$$

For the current :

$$\triangle j = \frac{i\,\hbar}{2\,m}\,(\psi[0]\,\triangle(\partial_x\psi^*) - \psi^*[0]\,\triangle(\partial_x\psi)) = -\frac{i\,g}{\hbar^2}\,(\psi[0]\,\psi^*[0] - \psi^*[0]\,\psi[0]) = 0.$$

Problem 2

A particle moves in a three dimensional well

$$V[r] = 0;\quad (0 < x < a;\ \ 0 < y < b;\ \ 0 < z < c);\quad V[r] = \infty,\quad \text{otherwise}.$$

Find the eigenvalues and the eigenstates. Discuss the degeneracy for the low-lying levels, in particular for the cubic box a=b=c.

Solution

The Hamiltonian is the sum of three independent one dimensional operators

$$H = \frac{1}{2\,m}\left(p_x^2 + p_y^2 + p_z^2\right) \equiv H_x + H_y + H_z\,. \tag{2.1}$$

The operators H_x. H_y, H_z commute between themselves, then can be simultaneously diagonalized. Each of them has the spectrum of a one dimensional well:

$$H_x:\ \frac{\hbar^2\,\pi^2}{2\,m}\,\frac{n_x^2}{a^2};\quad H_y:\ \frac{\hbar^2\,\pi^2}{2\,m}\,\frac{n_y^2}{b^2};\quad H_z:\ \frac{\hbar^2\,\pi^2}{2\,m}\,\frac{n_z^2}{c^2};\qquad n_x,\ n_y,\ n_z = 1,\ 2\ldots \tag{2.2}$$

The spectrum of H is

$$E_{n_x\,n_y\,n_z} = \frac{\hbar^2\,\pi^2}{2\,m}\left(\frac{n_x^2}{a^2} + \frac{n_y^2}{b^2} + \frac{n_z^2}{c^2}\right)\,. \tag{2.3}$$

The corresponding, normalized, eigenstates are

$$\psi_{n_x n_y n_z}[x, y, z] = \sqrt{\frac{8}{a\, b\, c}}\, \mathrm{Sin}\left[\frac{\pi\, n_x\, x}{a}\right] \mathrm{Sin}\left[\frac{\pi\, n_y\, y}{b}\right] \mathrm{Sin}\left[\frac{\pi\, n_z\, z}{c}\right]. \qquad (2.4)$$

For generic (non relative rational) a^2, b^2, c^2 the state are non degenerate. If the ratios of the edges are rational some levels can be degenerate. The total number of states up to an energy E is approximately (the integral is for one sector of a sphere, with volume 1/8 of the whole ball):

$$N[E] = \int dn_x\, dn_y\, dn_y\, \Theta\left[E - E_{n_x n_y n_z}\right] = \left(\frac{2\, m}{\hbar^2\, \pi^2}\right)^{3/2} a\, b\, c \int_{x,y,z>0} dx\, dy\, dz\ \Theta\left[E > x^2 + y^2 + z^2\right] =$$
$$\left(\frac{2\, m}{\hbar^2\, \pi^2}\right)^{3/2} a\, b\, c\, \frac{1}{8}\, \frac{4}{3}\, \pi\, E^{3/2} = \frac{(2\, m\, E)^{3/2}}{\hbar^3}\, \frac{2}{3}\, \frac{V}{4\, \pi^2}. \qquad (2.5)$$

V = a b c is the volume. In the problems of chap. 13 it is shown that a more accurate formula is

$$N[E] = \frac{(2\, m\, E)^{3/2}}{\hbar^3}\, \frac{2}{3}\, \frac{V}{4\, \pi^2} - \left(\frac{2\, m}{\hbar^2}\right) \frac{S}{16\, \pi}; \qquad (2.6)$$

S = 2 (a b + a c + b c) is the surface of the box.

- **Note on the degeneracy**

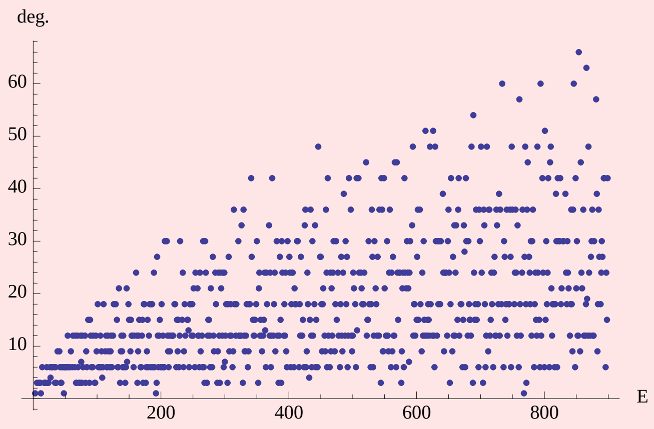
Let us consider the case a = b = c:

$$E = \frac{\hbar^2\, \pi^2}{2\, m\, a^2} \left(n_x^2 + n_y^2 + n_z^2\right).$$

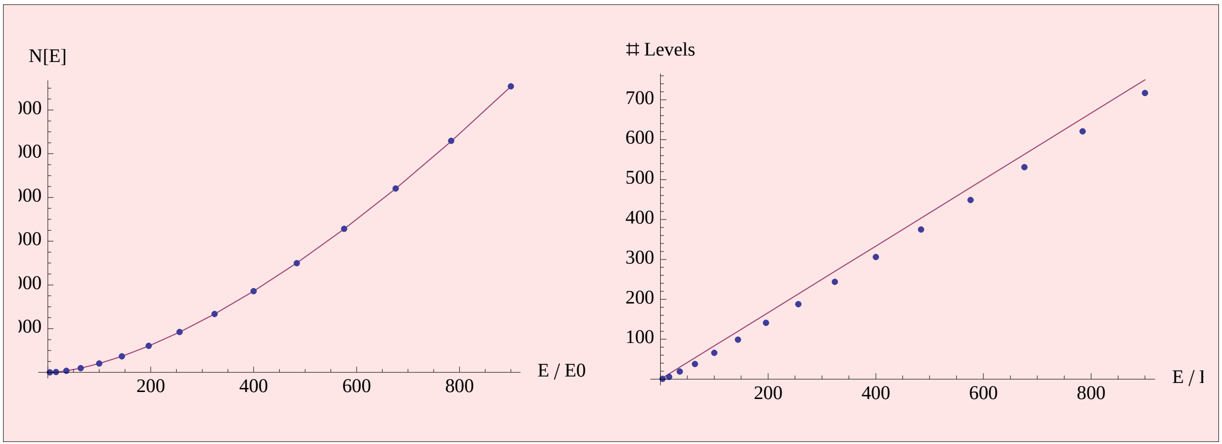
It is easy to compute energies and degeneracies for the first few levels

E/E0	3	6	9	11	12	14	17	18	19	21
deg.	1	3	3	3	1	6	3	3	3	6

While for low energy it is quite clear the appearance of degeneracy 6 due to the permutations of 3 (different) numbers n_x, n_y, n_z for higher energies things are not so simple. Here is an example of the degeneracy distribution



The total number of states N[E] follow very accurately the prediction (6), the number of *levels* is proportional to E, and very close to 5/6 E/E0:



The problem of levels and of their number is closely related to a classic problem in number theory: find the numbers that can be written as a sum of 3 squares. In number theory usually the value 0 for n_i is included, and the problem has been solved by Legendre and Gauss. We quote a result, the numbers which cannot be written as a sum of 3 squares have the form $4^a(8\,m + 7)$. These are certainly excluded from the spectrum, other numbers, built with the value 0 for n_i are included in number theory classification and possibly excluded in the spectrum (essentially the number sum of two squares). These are the first numbers of these series

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nExcluded = {7, 15, 23, 28, 31, 39, 47, 55, 60, 63, 71, 79}
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confronting with the first E/E0 levels we can check the statement:

```
E / E0 = {3, 6, 9, 11, 12, 14, 17, 18, 19, 21, 22, 24, 26, 27, 29, 30, 33, 34, 35, 36, 38, 41, 42, 43,
   44, 45, 46, 48, 49, 50, 51, 53, 54, 56, 57, 59, 61, 62, 65, 66, 67, 68, 69, 70, 72, 73, 74, 75, 76, 77};
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Problem 3

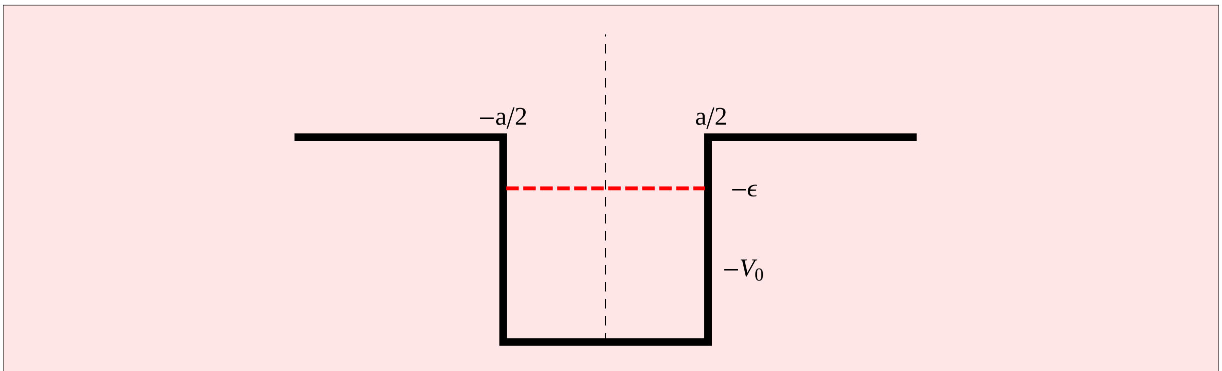
If the one-dimensional square well has the parameters such that

$$\sqrt{\frac{m\,a^2\,V_0}{2\,\hbar^2}} = \delta \ll 1\,,$$

it has a unique bound state. Compute the energy of this state approximately, in the first non trivial order in δ, as a function of V_0.

Solution

With the notations of the figure



the (even) solution of Schrödinger equation is

$$\psi[x] = A\,\mathrm{Cos}[q\,x]; \quad |x| < a/2; \qquad \psi[x] = B\,\mathrm{Exp}[-\alpha\,|x|];$$

with

$$\hbar\,q = \sqrt{2\,m\,(V_0 - \epsilon)}\;; \quad \hbar\,\alpha = \sqrt{2\,m\,\epsilon}\;.$$

Matching the logarithmic derivative at x = a/2 gives

$$q\,\mathrm{Tan}\left[q\,\frac{a}{2}\right] = \alpha \Rightarrow q^2\,\frac{a}{2} \simeq \alpha$$

For small ϵ

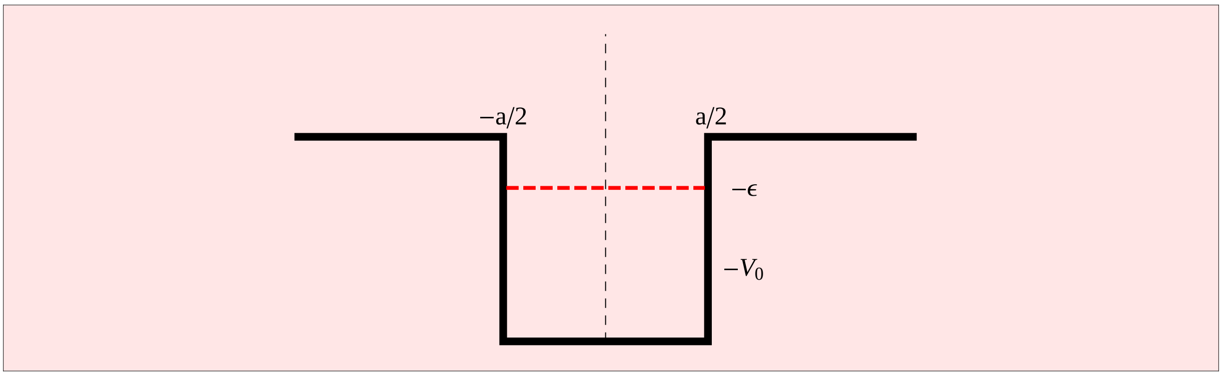
$$\frac{2\,m\,V_0}{\hbar}\,\frac{a}{2} \simeq \sqrt{2\,m\,\epsilon} \quad \Rightarrow \quad \epsilon = \frac{m\,V_0^2\,a^2}{2\,\hbar^2} \Rightarrow E = -\epsilon = -\frac{m\,V_0^2\,a^2}{2\,\hbar^2} = -V_0\,\delta^2.$$

Problem 4

Consider the limit $a \to 0$, $V_0 \to \infty$ with $aV_0 = f$ fixed, in the problem of a square well potential. Compare the result with the $\delta[x]$ potential.

Solution

The notations are given in the figure below



The potential being even, we can limit the study of solutions to the region x > 0.

The (even, odd) solutions of Schrödinger equation are (for x > 0)

$$\psi[x] = A\,\mathrm{Cos}[q\,x];\ |x| < a/2; \qquad \psi[x] = B\,\mathrm{Exp}[-\alpha\,x];$$

$$\psi[x] = A\,\mathrm{Sin}[q\,x];\ |x| < a/2; \qquad \psi[x] = B\,\mathrm{Exp}[-\alpha\,x];$$

with

$$\epsilon = |E|;\ \hbar\,q = \sqrt{2\,m\,(V_0 - \epsilon)}\ ;\ \hbar\,\alpha = \sqrt{2\,m\,\epsilon}\ .$$

The match of logarithmic derivative at x = a/2 gives

$$q\,\mathrm{Tan}\left[q\,\frac{a}{2}\right] = \alpha\ ; \quad q\,\mathrm{Cot}\left[q\,\frac{a}{2}\right] = \alpha\ . \tag{4.1}$$

In the limit $a \to 0$, $V_0 \to \infty$ with $aV_0 = f$ fixed,

$$q \to \frac{1}{\hbar}\sqrt{2\,m\,V_0} = \frac{1}{\hbar}\sqrt{2\,\frac{m}{a}\,f} \to \infty; \quad q\,a \to 0.$$

Only the first of the equations (1) can have a limit

$$q\,\mathrm{Tan}\left[q\,\frac{a}{2}\right] \to \frac{q^2\,a}{2} \to \frac{m\,f}{\hbar^2}.$$

The only eigenvalue is, from (1)

$$E = -\epsilon = -\left(\frac{m\,f}{\hbar^2}\right)^2\,\frac{\hbar^2}{2\,m} = -\frac{m\,f^2}{2\,\hbar^2}. \tag{4.2}$$

In a δ potential - g δ[x] there is only one bound state (see text) with energy

$$E_\delta = -\frac{m\,g^2}{2\,\hbar^2} \tag{4.3}$$

The "strength" of the δ potential is defined by

$$-g = -\int dx\, V[x] = -\int dx\, g\, \delta[x].$$

For a deep well the integration over a region including the well gives

$$\int dx\, V[x] = -\int dx\, V_0 = -V_0\, a \to -f.$$

We see that the well goes into a δ potential with strength f and its energy eigenvalue (2) gives the correct result (3).

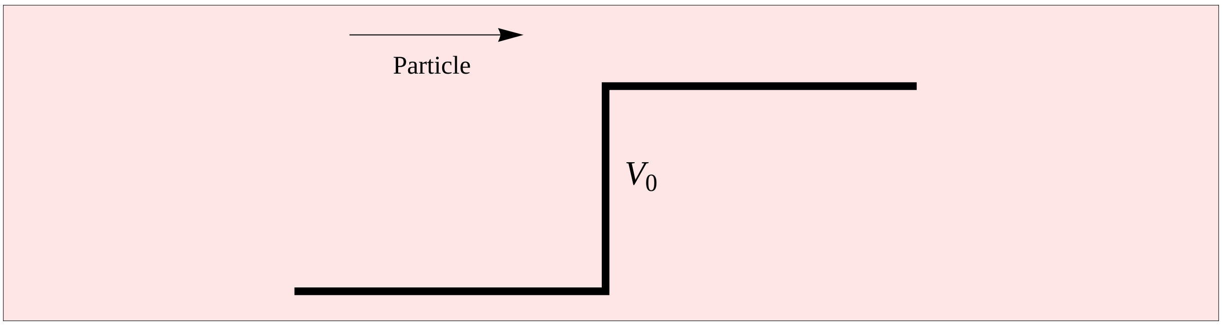
Problem 5

Find the transmission coefficient for the step potential

$$V = 0,\ x < 0; \quad V = V_0 > 0,\ x \geq 0,$$

for $E < V_0$ and $E > V_0$.

Solution



For $E < V_0$ there is not transmitted wave as for x > 0 the solution is exponentially depressed.

For $E > V_0$ the solution giving a transmitted wave is

$$\psi[x] = e^{i k x} + A\, e^{-i k x}; \quad x < 0; \qquad \psi[x] = B\, e^{i q x}; \quad x > 0. \tag{5.1}$$

Where

$$\hbar k = \sqrt{2 m E}; \quad \hbar q = \sqrt{2 m (E - V_0)}.$$

Matching ψ'/ψ at x = 0 gives

$$k \frac{1 - A}{1 + A} = q \Rightarrow A = \frac{k - q}{k + q}.$$

The reflection and transmission coefficients, are fixed by the ratio of the currents

$$R = |A|^2; \quad T = \frac{q}{k} |B|^2 = 1 - R.$$

We have :

$$R = \left(\frac{k - q}{k + q}\right)^2; \quad T = \frac{4 k q}{(k + q)^2}. \tag{5.2}$$

To check this result we compute B from the continuity of ψ at x=0:

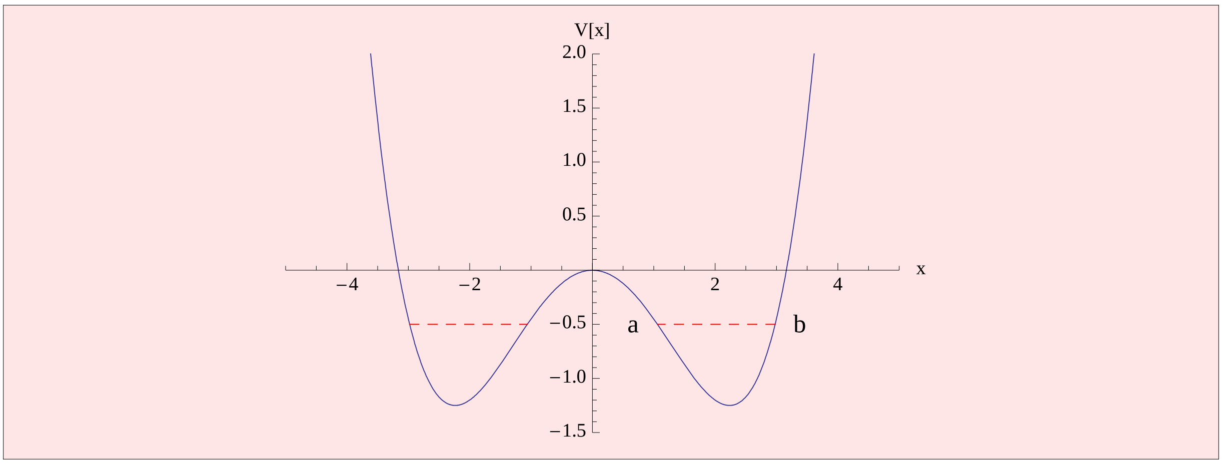
$$B = 1 + A = \frac{2 k}{k + q}; \quad \Rightarrow \quad T = \frac{q}{k} \frac{|B|^2}{|A|^2} = \frac{4 k q}{(k + q)^2}; \quad \text{c.v.d.}$$

Problem 6

Discuss the qualitative features of the low-lying energy levels in a deep double well potential, by using the symmetry argument (parity) and the non degeneration theorem.

Solution

The situation is outlined below:



In a first approximation we have two separate wells, exactly degenerate. This cannot represents the true ground state of the model as we know that the ground state is non degenerate and even. A good candidate for an approximate wave function is a superposition of the two separate localized solutions describing the particle in each well:

$$\psi = \frac{\psi_L[x] + \psi_R[x]}{\sqrt{2}}. \qquad (6.1)$$

Parity invariance impose $\psi_L[-x] = \psi_R[x]$ (a better way to say this is that parity invariance allow to choose the phases in such a way that $\psi_L[-x] = \psi_R[x]$).

The physical mechanism which allow the superposition is the tunneling between the two wells. As an effective Hamiltonian in the basis ψ_L, ψ_R one can consider

$$H = \begin{pmatrix} E_0 & -\epsilon \\ -\epsilon & E_0 \end{pmatrix}.$$

ϵ is proportional to the tunnel probability. The ground state of H is (1). The reader can verify that H commute with

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

which exchange ψ_L, ψ_R and acts as parity operator in this system. The state (1) is even.

Problem 7

A particle of mass m is confined in a potential well V[x], with

$$V[x] = \begin{cases} 0; & 0 \le x \le a \\ \infty; & x < 0 \text{ and } x > a \end{cases}$$

The particle is in in the ground state.

1. Compute the force exerted on the walls by the the particle.
2. At time t=0 the right wall (at x=a) moves suddenly to x=2a. Compute the probability to find the particle in each stationary state of the new well.
3. If the wall moves adiabatically from x=a to x=2a which is the final state? Compute the work done during this process.

Solution

1

The force can be found equating the work done in an adiabatic expansion to the energy variation of the system:

$$F\,\delta a = -\delta E;\quad F = -\frac{\partial E}{\partial a}.$$

For the ground state

$$E = \frac{\pi^2 \hbar^2}{2\,m\,a^2};\qquad F = \frac{\pi^2 \hbar^2}{m\,a^3}. \tag{7.1}$$

2

In a sudden expansion the state is frozen, i.e. remains, in the short interval of the expansion,

$$\psi_0[x] = \sqrt{\frac{2}{a}}\,\mathrm{Sin}\left[\pi\frac{x}{a}\right]\theta[x]\,\theta[a-x]$$

After the expansion energy eigenstates are given by

$$\varphi_n[x] = \sqrt{\frac{2}{2a}}\,\mathrm{Sin}\left[n\,\pi\frac{x}{2a}\right] \tag{7.2}$$

The requested probabilities are

$$P_n = \left|\int_0^a dx\,\psi_0[x]\,\varphi_n[x]\right|^2 = \begin{cases} 1/2; & n=2 \\ 0\;; & n = 2L;\ L>1 \\ \frac{32}{\pi^2}\frac{1}{(4-n^2)^2} & n = 2L+1. \end{cases} \tag{7.3}$$

It is easy to check (try for example with *Mathematica*) that $\sum_n P_n = 1$.

3

In an adabatic expansion the state "follows" the expansion and continues to be the ground state of the expanding well. The total energy change is then

$$\triangle E = \frac{\pi^2\hbar^2}{2\,m\,a^2} - \frac{\pi^2\hbar^2}{2\,m\,4\,a^2} = \frac{3}{8}\frac{\pi^2\hbar^2}{m\,a^2}.$$

As expected from the definition of force, equation (1),

$$\triangle E = \int_a^{2a} F[a]\,da\,.$$

Problem 8

A particle of mass m is subject to the one dimensional potential

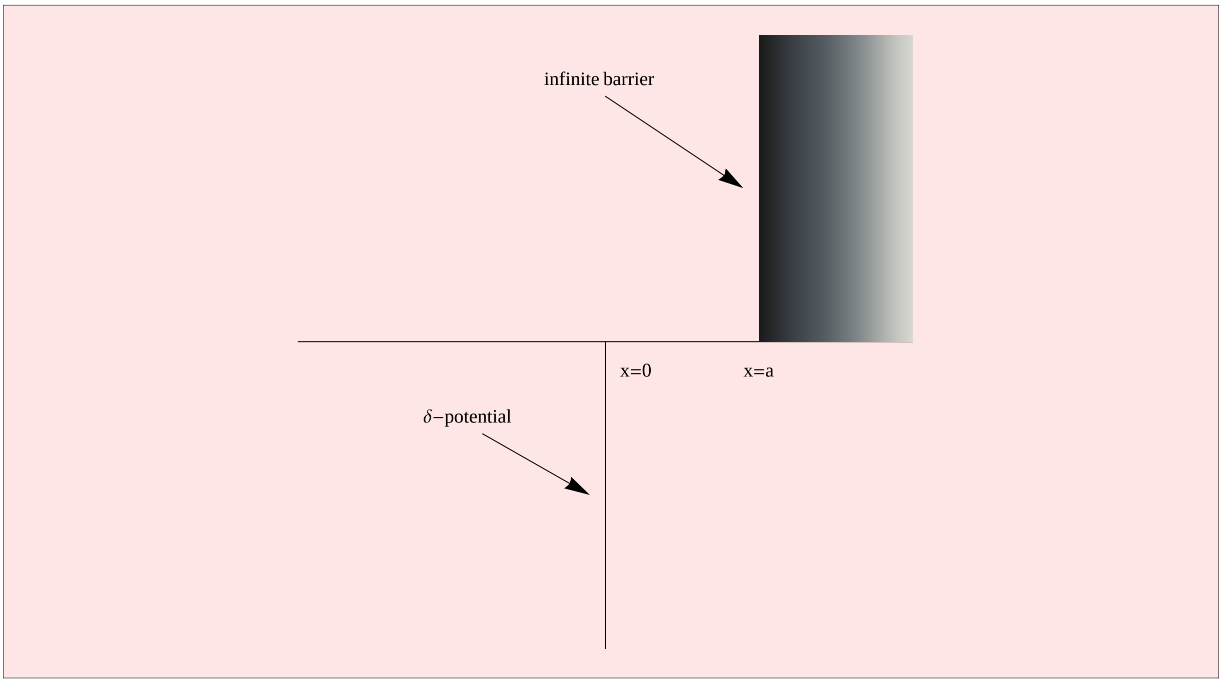
$$V[x] = \begin{cases} -g\,\delta[x]; & x \le a \\ \infty; & x \ge a \end{cases}$$

1. *Find the implicit equation for finding a bound state with E < 0 and the conditions on parameters (m,g) for the existence of such a state.*
2. *Discuss the limit a → ∞.*
3. *Is there a degeneracy in the continuum spectrum?*

Solution

1

The potential is sketched in the following figure



For E < 0 the general form of the solution of the Schrödinger equation is

$$\psi[x] = \begin{cases} A\,\text{Exp}[\alpha\, x], & x < 0 \\ B\,\text{Exp}[\alpha\, x] + C\,\text{Exp}[-\alpha\, x], & x > 0 \end{cases} \tag{8.1}$$

With a δ potential in x_0, ψ must be continuous and the discontinuity of its first derivative is fixed by Schrödinger equation:

$$\psi'[x_0^+] - \psi'[x_0^-] = \frac{2\,m}{\hbar^2} \underset{\varepsilon\to 0}{\text{Lim}} \int_{x_0-\varepsilon}^{x_0+\varepsilon} V[x]\,\psi[x]\,dx\,. \tag{8.2}$$

In the case at hand the constraint on ψ and its derivative are

$$A = B + C; \quad \alpha\,(B - C) - \alpha\,A = -\frac{2\,m}{\hbar^2}\,g\,A. \tag{8.3}$$

i.e.

$$B = \left(1 - \frac{m\,g}{\alpha\,\hbar^2}\right); \quad C = \frac{m\,g}{\alpha\,\hbar^2}\,A\,. \tag{8.4}$$

At point x = a the function must vanish

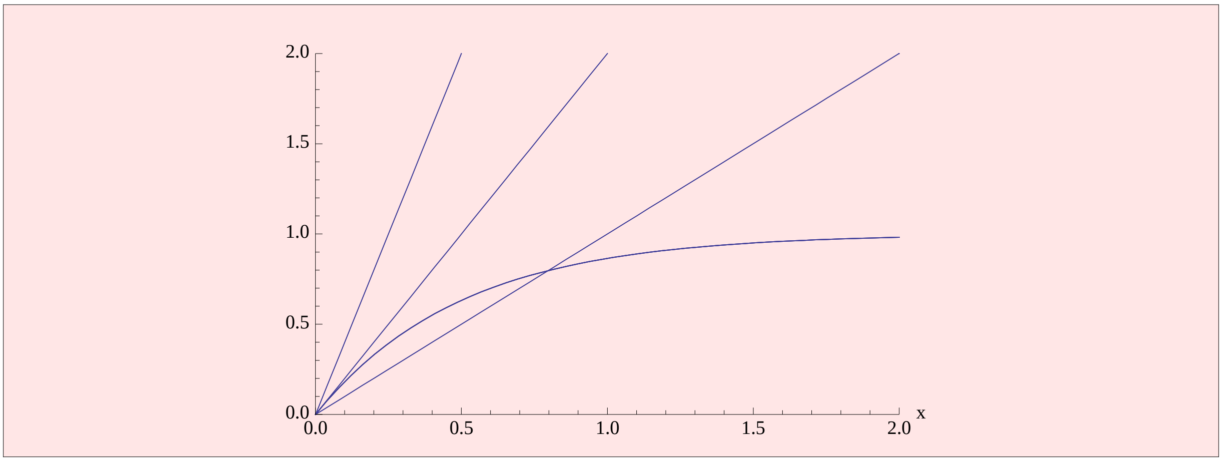
$$\psi[a] = B\,\text{Exp}[\alpha\,a] + C\,\text{Exp}[-\alpha\,a] = 0$$

Using eq (4)

$$1 - e^{-2\,a\,\alpha} = \frac{\alpha\,\hbar^2}{g\,m} \tag{8.5}$$

From the plots of the functions

$$f_1[x] = 1 - \text{Exp}[-2\,a\,x]; \quad f_2[x] = x\,\frac{\hbar^2}{g\,m}$$



it is evident that the possibility on a non trivial solution depends on derivative at the origin, i.e. we have an intercept if

$$\frac{\hbar^2}{g\,m} \leq 2\,a, \quad \Rightarrow \quad g \geq \frac{\hbar^2}{2\,m\,a}. \tag{8.6}$$

Only in this range we can have a bound state.

■ **2**

As a → ∞ condition (5) gives

$$\alpha = \frac{g\,m}{\hbar^2}$$

which is the known result for a simple δ - potential.

■ **3**

There is no degeneracy in the continuum levels. To show this let us write the form of a stationary wave function for positive energy. k is a positive number and $E = \hbar^2\,k^2 / 2\,m$

$$\psi = \begin{cases} A\,e^{i\,k\,x} + B\,e^{-i\,k\,x}, & x < 0 \\ C\,e^{i\,k\,x} + D\,e^{-i\,k\,x}, & 0 < x < a \\ 0, & x \geq a \end{cases}$$

The condition $\psi[a] = 0$ gives a relation between C and D. Continuity and gap in first derivative, eq.(2), give two more condition. Three equations for 4 coefficients fix the solution up to an overall normalization factor.

Problem 9

A particle of mass m *is subjected to the one dimensional potential*

$$V[x] = \begin{matrix} \infty; & x \leq -a \\ g\,\delta[x];; & -a \leq x \leq 2\,a\,. \\ \infty & x > 2\,a \end{matrix}$$

We want to study the first two energy levels. Let us call $\psi_-[x]$, $\psi_+[x]$ *the wave functions for negative and positive x respectively.*

1. *Write the matching conditions for connecting* ψ_+ *and* ψ_-. *Show that one of these condition gives exact results for a set of levels, those for which* $\psi[0] = 0$.
2. *There are another set of states, for which* $\psi[0]\neq 0$. *Using a graph show that the ground state of the system belongs to this set, and compute its energy for* g→0 *and* g→∞.

Solution

▫ **1**

The vanishing of ψ at the external boundaries imply

$$\psi_-[x] = A\,\mathrm{Sin}[k\,(x + a)]\,; \qquad \psi_+[x] = B\,\mathrm{Sin}[k\,(x - 2\,a)]$$

where $E = \hbar^2\,k^2/2m$. At x = 0 we have to match the functions and the derivative's gap

$$\psi_+[0] = \psi_-[0] \quad \Rightarrow \quad A\,\mathrm{Sin}[k\,a] = -B\,\mathrm{Sin}[2\,k\,a]; \tag{9.1}$$

$$\psi_+'[0] - \psi_-'[0] = \frac{2\,m\,g}{\hbar^2}\,\psi[0] \quad \Rightarrow \quad k\,(B\,\mathrm{Cos}[2\,k\,a] - A\,\mathrm{Cos}[k\,a]) = \frac{2\,m\,g}{\hbar^2}\,A\,\mathrm{Sin}[k\,a]. \tag{9.2}$$

From first conditions it follows

$$\mathrm{Sin}[k\,a]\,(A + 2\,B\,\mathrm{Cos}[k\,a]) = 0. \tag{9.3}$$

The first possibility is Sin[k a] = 0. In this case

$$k = \frac{\pi\,n}{a}; \qquad E_n = \frac{\hbar^2}{2\,m}\,\frac{\pi^2\,n^2}{a^2}; \quad n = 0,\,1,\,2\,\ldots \tag{9.4}$$

A and B coefficients are determined, up to an overall normalization by (2)

$$B\,\mathrm{Cos}[2\,\pi\,n] - A\,\mathrm{Cos}[\pi\,n] = 0, \quad \Rightarrow \quad B = (-1)^n\,A.$$

As Sin[k a] = 0 in this class of solutions $\psi[0] = 0$.

▫ **2**

The second possibility allowed by eq.(3) is

$$A = -2\,B\,\mathrm{Cos}[k\,a]. \tag{9.5}$$

Substitution in (2) gives

$$k\,\mathrm{Cos}[2\,k\,a] + 2\,k\,\mathrm{Cos}[k\,a]^2 = -\frac{4\,m\,g}{\hbar^2}\,\mathrm{Cos}[k\,a]\,\mathrm{Sin}[k\,a],$$

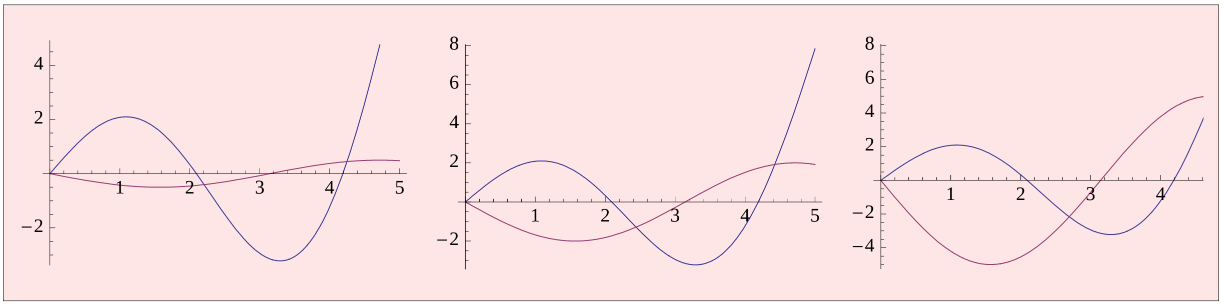
i.e. with $\xi = 2\,k\,a$

$$\xi\,(2\,\mathrm{Cos}[\xi] + 1) = -\Lambda\,\mathrm{Sin}[\xi]; \qquad \Lambda = \frac{4\,m\,g\,a}{\hbar^2} > 0. \tag{9.6}$$

We can plot the two curves

$$f_1[\xi] = \xi\,(2\,\mathrm{Cos}[\xi] + 1); \quad f_2[\xi] = -\Lambda\,\mathrm{Sin}[\xi],$$

for different values of Λ. The graph below show curves for $\Lambda = 0.5,\,2,\,5$.



One easily convince oneself that the first intersection, giving the first state in this group, corresponds to a value of $\xi = 2ka$ lying in the region

$$\frac{2\,\pi}{3} < \xi < \pi; \quad \frac{\pi}{3\,a} < k < \frac{\pi}{2\,a}. \tag{9.7}$$

Comparison with (4) show that ground state of the system is the first state of this second group, while n=1 state in (4) is the first excited state.

In the two limits $g \to 0$ and $g \to \infty$ also $\Lambda \to 0, \infty$. from the graph it is clear that as $\Lambda \to 0$ the intersection tends to the first zero of the curve $f_1[\xi]$, i.e. $\xi = 2\pi/3$. while for $\Lambda \to \infty$ the intersection approach the zero of the second curve, Sin[ξ], i.e. $\xi = \pi$. In the two limits then

$$\text{as } g \to 0: \quad k \to \frac{\pi}{3\,a}; \quad \text{as } g \to \infty: \quad k \to \frac{\pi}{2\,a}.$$

Problem 10

A particle of mass m *and energy* $E < 0$ *moves in a one-dimensional potential*

$$V[x] = f\,\delta[x] - g\,\delta[x-a]; \quad g > 0.$$

1. For a single δ-well (f = 0) compute the ground state energy and its normalized eigenfunction.

2. Write a complete set of conditions for the wave function with f≠0 and g > 0.

3. Write the conditions for the existence of a bound state.

4. Show that for |f| sufficiently small only one bound state exists and that for $(2\,g\,m\,a)/\hbar^2 < 1$ and for f > 0 and sufficiently large, no bound state can exist.

Solution

1

For a single δ potential the solution for E < 0 has the form (Θ is the Heavside theta function, or step function, in *Mathematica* is UnitStep[x]):

$$\psi[x] = A\, e^{\beta\,(x-a)}\,\Theta[a-x] + B\, e^{-\beta\,(x-a)}\,\Theta[x-a]; \quad E = -\frac{\beta^2\,\hbar^2}{2\,m} \tag{10.1}$$

For a δ potential matching conditions require continuity of ψ and assigned gap for derivatives :

$$\psi[a^+] = \psi[a^-]; \qquad \psi'[a^+] - \psi'[a^-] = \frac{2\,m}{\hbar^2} \lim_{\varepsilon\to 0} \int_{a-\varepsilon}^{a+\varepsilon} V[x]\,\psi[x]\,dx. \tag{10.2}$$

In our case

$$A = B; \quad -\beta\,B - \beta\,A = \frac{2\,m}{\hbar^2}\,(-g\,A) \quad \Rightarrow \quad \beta = \frac{m\,g}{\hbar^2}. \tag{10.3}$$

2

Negative energy, normalizable, solutions must decrease at x → ± ∞. The most general form of this solution is, with above notation (1) for E:

$$\psi[x] = A_1\,e^{\beta\,x}\,\Theta[-x] + \left(A_2\,e^{\beta\,x} + B_2\,e^{-\beta\,x}\right)\Theta[x]\,\Theta[a-x] + B_3\,e^{-\beta\,x}\,\Theta[x-a]. \tag{10.4}$$

We have two set of equations of type (2), i.e. four matching conditions, two for functions:

$$A_1 = A_2 + B_2; \qquad A_2\,e^{\beta\,a} + B_2\,e^{-\beta\,a} = B_3\,e^{-\beta\,a}; \tag{10.5}$$

and two for derivatives

$$\beta\,(-A_1 + (A_2 - B_2)) = \frac{2\,m\,f}{\hbar^2}\,A_1; \quad -\beta\left(B_3\,e^{-\beta\,a} + A_2\,e^{\beta\,a} - B_2\,e^{-\beta\,a}\right) = -\frac{2\,m\,g}{\hbar^2}\,B_3\,e^{-\beta\,a}. \tag{10.6}$$

3

We have a system of four homogenous equations in four unknown quantities A_1, A_2, B_2, B_3. The system has a nontrivial solution only if the determinant of the coefficients is zero, and this will determine one (or more) values for β, i.e. energies for the bound states.
Instead of writing te determinant let us proceed by elimination of variables. let us solve for $A_2$and B_2. From (5)

$$A_2 = \left(1 + \frac{m\,f}{\beta\,\hbar^2}\right)A_1; \quad B_2 = -\frac{m\,f}{\beta\,\hbar^2}\,A_1;$$

while from (6)

$$A_2\,e^{\beta\,a} = \frac{m\,g}{\beta\,\hbar^2}\,B_3\,e^{-\beta\,a}; \quad B_2\,e^{-\beta\,a} = \left(1 - \frac{m\,g}{\beta\,\hbar^2}\right)B_3\,e^{-\beta\,a};$$

Eliinating the unknown $A_i$and $B_i$we arrive at

$$e^{2\,a\,\beta}\left(g\,m - \beta\,\hbar^2\right)\left(f\,m + \beta\,\hbar^2\right) = f\,g\,m^2.$$

With

$$x = \frac{\beta\,\hbar^2}{m\,g}; \quad A = \frac{2\,m\,g\,a}{\hbar^2};$$

the previous condition take the form

$$\left(x + \frac{f}{g}\right)(x-1) = -\frac{f}{g}\,\text{Exp}[-A\,x] \tag{10.7}$$

4

We look for solutions with x > 0.

- For f = 0 we recover x=1, which is the solution (3).
- For f = -g, i.e. a symmetric δ potential, the condition becomes

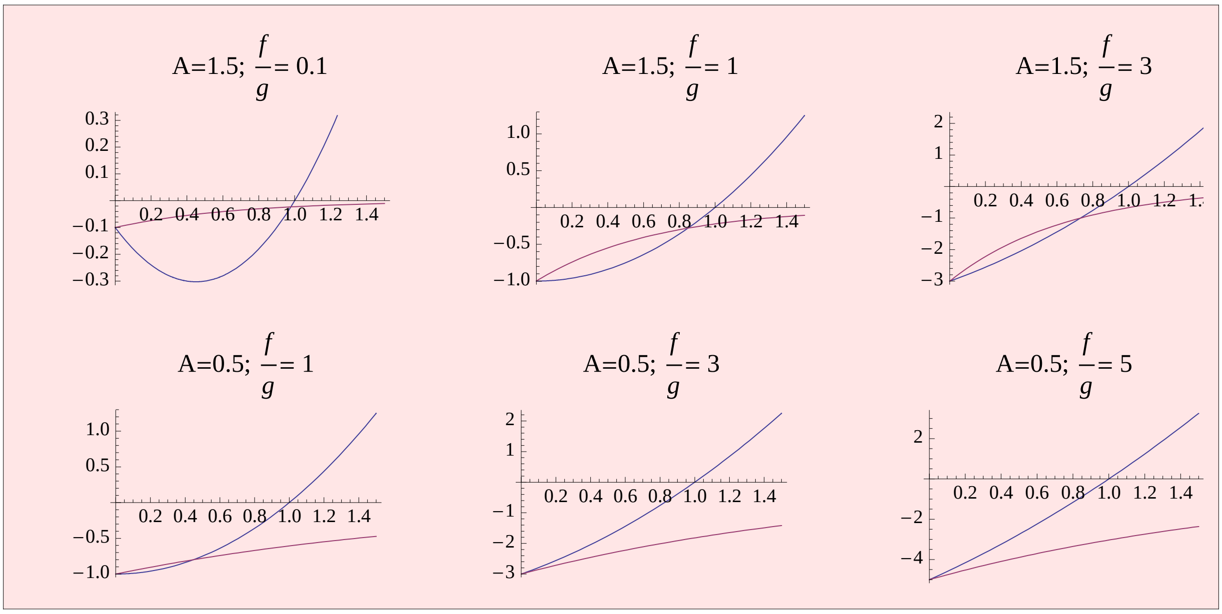
$$1 - \frac{m g}{\beta \hbar^2} = \pm \frac{m g}{\beta \hbar^2} \text{Exp}[-\beta a].$$

which has been found in [**].

- For small |f| the solution is unique :

$$x \simeq 1 - \frac{f}{g} e^{-A}; \quad \beta = \frac{m g}{\hbar^2}\left(1 - \frac{f}{g} e^{-A}\right); \quad E \simeq -\frac{m g^2}{\hbar^2} + \frac{m f g}{\hbar^2} e^{-\frac{2 m g a}{\hbar^2}}.$$

For the general case let us look at graphs of the to sides of equation (7):



It is apparent that if derivative in x = 0 of the first curve is bigger than derivative of the second curve then there is no solution

$$\frac{f}{g} - 1 > \frac{f}{g} A$$

This happens for A < 1 and suficiently large f:

$$f > \frac{g}{1 - A} = \frac{g}{1 - \frac{2 m g a}{\hbar^2}}.$$

Problem 11

A particle of mass m *moves in the one dimensional potential* (V_0> 0)

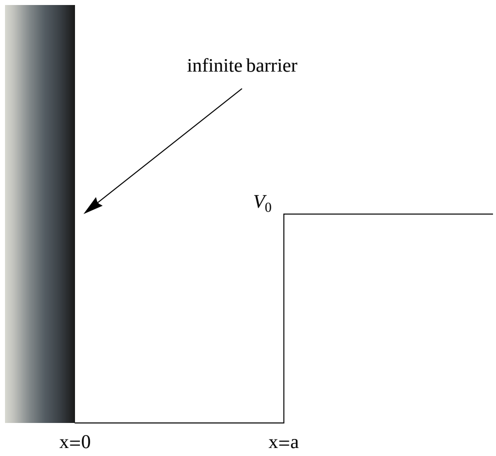
$$V[x] = \begin{cases} \infty; & x < 0 \\ 0; & 0 \le x \le a \\ V_0; & x > a \end{cases}.$$

1. Find the condition for having only one bound state.
2. Find approximatively the ground state wave function for large V_0.
3. Suppose that the particle is confined in the lowest bound state of the system; at time t=0 the part x > 2a of the potential gets modified to 0. Compute the rate of decay of the system i.e. the probability for unit of time to find the particle outside the potential well.

Solution

1

The potential is sketched in the figure. The region x < 0 is forbidden.



The solution for a bound state has the form

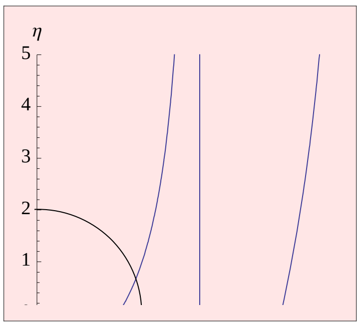
$$\psi[x] = \begin{cases} A\,\mathrm{Sin}[k\,x], & 0 \le x \le a \\ B\,\mathrm{Exp}[-\beta\,x], & x > a \end{cases}; \quad \text{with} \quad k = \sqrt{\frac{2\,m\,E}{\hbar^2}}; \ \beta = \sqrt{\frac{2\,m\,(V_0 - E)}{\hbar^2}}.$$

Continuity of ψ'/ψ at x = a gives the constraint

$$\beta = -k\,\mathrm{Cot}[k\,a].$$

Defining ξ = ka; $\eta = \beta$a the eigenvalue can be found by the intersection of the two curves

$$\eta = -\xi\,\mathrm{Cot}[\xi]; \quad \xi^2 + \eta^2 = \frac{2\,m\,V_0\,a^2}{\hbar^2} \qquad (11.1)$$



As is evident from the graph the condition to have only one bound state is

$$\frac{\pi}{2} < \sqrt{\frac{2\,m\,V_0\,a^2}{\hbar^2}} < \frac{3}{2}\pi.$$

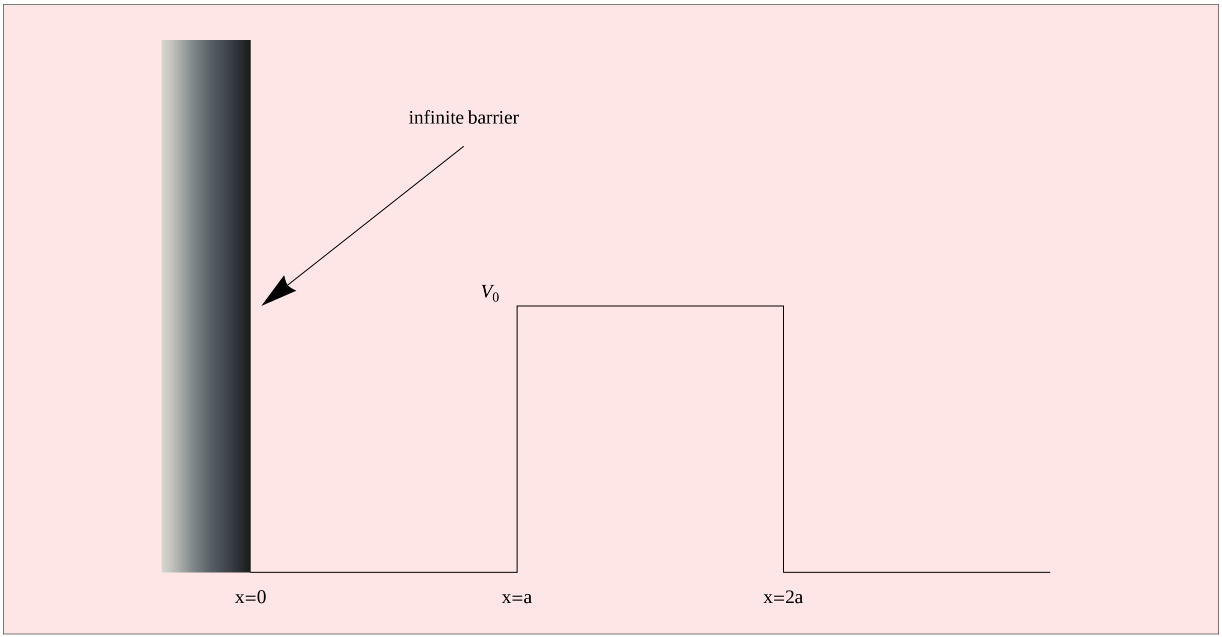
2

For large V_0 the wave function for x>a canbe neglected and we have

$$\psi[x] \simeq \sqrt{\frac{2}{a}}\,\mathrm{Sin}\left[\frac{\pi\,x}{a}\right]; \qquad k = \frac{\pi}{a}; \ E = \frac{\hbar^2\,\pi^2}{2\,m\,a^2} \qquad (11.2)$$

3

If the barrier behind x = 2 a is dropped we have the following potential



A particle which at t = 0 is confined in the region $0 \leq x \leq a$ can tunnel below the barrier and escape. The penetration factor for a rectangular barrier has been computed in the text:

$$D = \frac{4 k^2 \beta^2}{4 k^2 \beta^2 + (k^2 + \beta^2)^2 \text{Sinh}[\beta a]^2} \simeq \frac{16 k^2}{\beta^2} \text{Exp}[-2 \beta a]; \quad k \simeq \frac{\pi}{a}; \quad \beta \simeq \sqrt{\frac{2 m V_0}{\hbar^2}}. \qquad (11.3)$$

The number of hits per second against the barrier at x=a for a particle of velocity v is v/(2 a) the the probability of decay per unit time is

$$W = \frac{dP}{dt} = \frac{v}{2 a} D \simeq \frac{\hbar k}{m} \frac{1}{2 a} \frac{16 k^2}{\beta^2} \text{Exp}[-2 \beta a]. \qquad (11.4)$$

The number of hits per second can also be computed from the current inside the interval [0, a]. The wave function is stationary and superposition of two opposite traveling waves:

$$\psi[x] = \sqrt{\frac{2}{a}} \frac{e^{i k x} - e^{-i k x}}{2 i} = \psi_+ + \psi_-$$

The current relative to right movers, the one which hits the barrier, is the result we used in our previous formula:

$$j = \frac{\hbar}{2 m i} \left(\psi_+^* \frac{d \psi_+}{dx} - \frac{d \psi_+^*}{dx} \psi_+ \right) = \frac{\hbar}{2 m i} \frac{1}{2 a} 2 i k = \frac{\hbar k}{2 a m}.$$

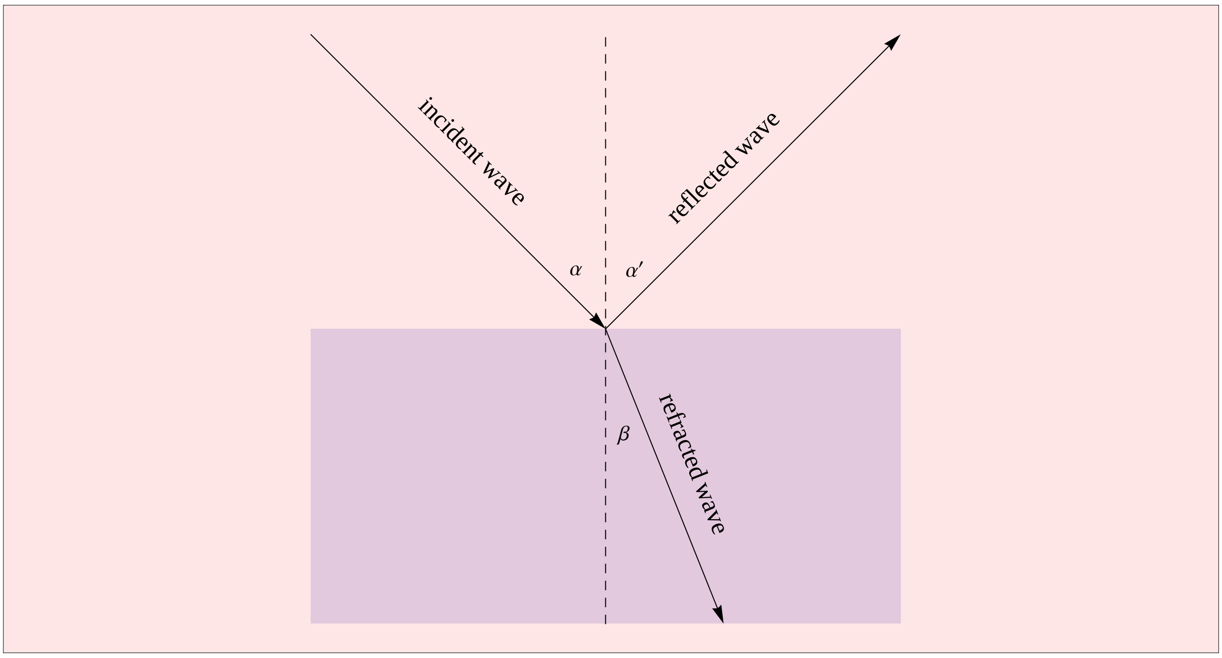
Problem 12

Consider the space divided in two regions separate by a plane. The potential energy being 0 and $-U_0$ in the two half spaces. A plane wave impinge on the plane with an incidence angle α. Describe the reflection and the transmission (refracted wave) of the wave. Write and verify the condition for conservation of the number of particles.

Solution

The Schrödinger in the two region is

$$\begin{aligned} &\Delta\psi + k^2 \psi = 0; \quad k^2 = 2 m E / \hbar^2 \\ &\Delta\psi + q^2 \psi = 0; \quad q^2 = k^2 + \frac{2 m U_0}{\hbar^2} \equiv k^2 + \beta^2; \end{aligned} \qquad (12.1)$$



With the notations shown in the figure we look for a solution in the form

$$y > 0: \ \psi = \mathrm{Exp}[i\, \mathbf{k_1}\, \mathbf{r}] + A_R\, \mathrm{Exp}[i\, \mathbf{k_2}\, \mathbf{r}]; \ \mathbf{k_1} = k\,(\mathrm{Sin}[\alpha], -\mathrm{Cos}[\alpha]); \ \mathbf{k_2} = k\,(\mathrm{Sin}[\alpha'], \mathrm{Cos}[\alpha']);$$
$$y < 0: \ \psi = A_T\, \mathrm{Exp}[i\, \mathbf{q}\, \mathbf{r}]; \ \mathbf{q} = q\,(\mathrm{Sin}[\beta], -\mathrm{Cos}[\beta]). \qquad (12.2)$$

Continuity for ψ, for the separation plane y=0 give

$$\mathrm{Exp}[i\, k\, \mathrm{Sin}[\alpha]\, x] + A_R\, \mathrm{Exp}[i\, k\, \mathrm{Sin}[\alpha']\, x] = A_T\, \mathrm{Exp}[i\, q\, \mathrm{Sin}[\beta]\, x] \qquad (12.3)$$

As the relation must hold for all x the phases must be equal:

$$k\, \mathrm{Sin}[\alpha] = k\, \mathrm{Sin}[\alpha'] = q\, \mathrm{Sin}[\beta] \ \Rightarrow \ \alpha' = \alpha; \ \frac{q}{k} = \frac{\mathrm{Sin}[\alpha']}{\mathrm{Sin}[\beta]}.$$

With q/k = n we recognize the usual Snell reflection and refraction laws:

$$\alpha' = \alpha; \quad n = \frac{\mathrm{Sin}[\alpha]}{\mathrm{Sin}[\beta]}. \qquad (12.4)$$

As the phases are equal from eq.(3) and continuity of ψ it follows

$$1 + A_R = A_T. \qquad (12.5)$$

The continuity of $\partial_x \psi$, using the equality of phases and (4) , gives

$$k\, \mathrm{Sin}[\alpha] + k\, A_R\, \mathrm{Sin}[\alpha] = q\, A_T\, \mathrm{Sin}[\beta]$$

which is automatically satisfied with (4) and (5): this is obvious because we imposed (3) for all x.

The continuity of $\partial_y \psi$ gives

$$-k\, \mathrm{Cos}[\alpha] + k\, A_R\, \mathrm{Cos}[\alpha] = -q\, A_T\, \mathrm{Cos}[\beta] \ \Rightarrow \ \mathrm{Cos}[\alpha]\,(1 - A_R) = n\, A_T\, \mathrm{Cos}[\beta]. \qquad (12.6)$$

Solving (5)and (6) for $A_R$and A_T gives

$$A_R = \frac{\mathrm{Cos}[\alpha] - n\, \mathrm{Cos}[\beta]}{\mathrm{Cos}[\alpha] + n\, \mathrm{Cos}[\beta]}; \quad A_T = \frac{2\, \mathrm{Cos}[\alpha]}{\mathrm{Cos}[\alpha] + n\, \mathrm{Cos}[\beta]}; \qquad (12.7)$$

the Fresnel reflection laws adapted to this case.

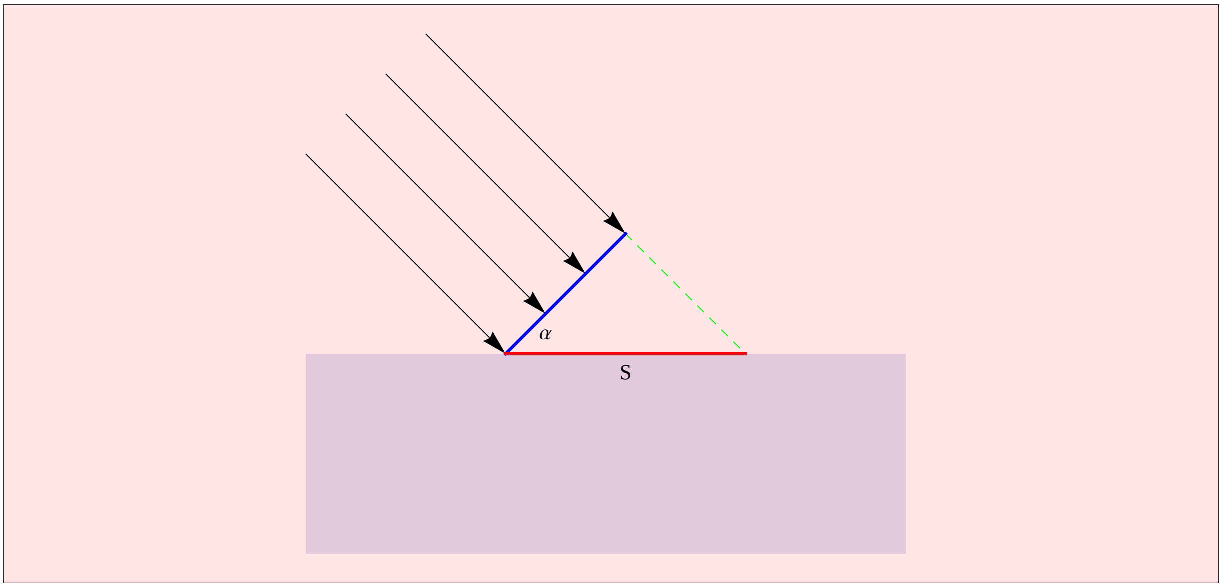
Let us note that it is **not** true the usual one dimensional relation $1 = A_R^2 + n\, A_T^2$. The unitarity instead imply

$$k\, \mathrm{Cos}[\alpha] = k\, A_R^2\, \mathrm{Cos}[\alpha] + q\, A_T^2\, \mathrm{Cos}[\beta] \ \Rightarrow \ \mathrm{Cos}[\alpha] = A_R^2\, \mathrm{Cos}[\alpha] + n\, A_T^2\, \mathrm{Cos}[\beta] \qquad (12.8)$$

which is indeed satisfied by (7).

In fact, consider an area Sin the separation plane. The number of the incident particle per second is

$$j_{inc}\, S \, Cos[\alpha] = k\, S\, Cos[\alpha].$$



The number of particles reflected and refracted per second

$$j_{refl}\, S\, Cos[\alpha] = k\, A_R^2\, S\, Cos[\alpha]; \quad j_T\, S\, Cos[\beta] = q\, A_T^2\, S\, Cos[\beta].$$

The conservation of particle's number gives (8).

Problem 13

A particle of mass m moves in one dimension and is subject to a harmonic force - k x. At time $t = 0$ the center of the force is suddenly shifted by x_0. The particle is assumed to be in the ground state for $t < 0$.

1. Compute the mean value of the energy at $t > 0$.
2. Compute the probability of finding the particle in the ground state and in the first excited state of the new system at $t > 0$.
3. Compute the behavior at $t \to 0^{\pm}$ for the mean values of Heisenberg operators x, p, $\dot{x}$, $\dot{p}$.

Solution

For $t < 0$ and $t > 0$ the Hamiltonian is

$$H_0 = \frac{p^2}{2\, m} + \frac{1}{2} m\, \omega^2\, x^2 \; ; \quad H_1 = \frac{p^2}{2\, m} + \frac{1}{2} m\, \omega^2\, (x - x_0)^2 \; ; \quad \omega = \sqrt{k/m} \; . \tag{13.1}$$

At negative times the system is in the state $\psi_0$of H_0. Just after the translation the state is unchanged.

■ **1**

The mean value of H_1do not change for $t > 0$, it is then sufficient to do the computation in the limit $t \to 0^+$.

$$\langle H_1 \rangle = \langle \psi_0 \mid H_0 + \frac{1}{2} m\, \omega^2\, x_0^2 - m\, \omega^2\, x\, x_0 \mid \psi_0 \rangle = \frac{1}{2} \hbar\, \omega + \frac{1}{2} m\, \omega^2\, x_0^2 \; .$$

We used $\langle \psi_0 \mid x \mid \psi_0 \rangle = 0$.

■ **2**

Let φ_n the eigenstates of H_1. We have for the initial state

$$\psi[x, 0] = \psi_0[x] = \sum_{n=0}^{\infty} a_n\, \varphi_n[x] \; . \tag{13.2}$$

The time evolution is

$$\psi[x, t] = \sum_{n=0}^{\infty} a_n\, \varphi_n[x]\, e^{-i\, E_n\, t/\hbar} \; . \tag{13.3}$$

The requested probabilities are time independent:

$$P_n[t] = \left| a_n e^{-i E_n t/\hbar} \right|^2 = a_n^2 .$$

and can be computed on the state at $t \to 0^+$.

The wave functions required to answer the question are

$$\psi_0[x] = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \text{Exp}\left[-\frac{m\omega}{2\hbar} x^2\right];$$

$$\varphi_0[x] = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \text{Exp}\left[-\frac{m\omega}{2\hbar}(x-x_0)^2\right]; \quad \varphi_1[x] = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2}} 2\sqrt{\frac{m\omega}{\hbar}} (x-x_0) \text{Exp}\left[-\frac{m\omega}{2\hbar}(x-x_0)^2\right].$$

We have

$$a_n = \int_{-\infty}^{+\infty} dx\, \psi_0[x]\, \varphi_n[x],$$

and an elementary integral gives

$$P_0 = |a_0|^2 = e^{-\frac{m x_0^2 \omega}{2\hbar}} \;; \; P_1 = |a_1|^2 = \frac{m\omega}{2\hbar} x_0^2\, e^{-\frac{m x_0^2 \omega}{2\hbar}} .$$

It is interesting to perform the computation using creation and annihilation operators.

Let a, $a^\dagger$ the standard operators for H_0 and b, $b^\dagger$ the corresponding operators for H_1

$$a = \sqrt{\frac{m\omega}{2\hbar}}\, x + i \frac{1}{\sqrt{2 m\omega\hbar}}\, p; \quad a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\, x - i \frac{1}{\sqrt{2 m\omega\hbar}}\, p;$$

$$b = \sqrt{\frac{m\omega}{2\hbar}}\, (x-x_0) + i \frac{1}{\sqrt{2 m\omega\hbar}}\, p; \quad b^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\, (x-x_0) - i \frac{1}{\sqrt{2 m\omega\hbar}}\, p .$$

Clearly

$$b = a - C; \quad b^\dagger = a^\dagger - C; \quad C = \sqrt{\frac{m\omega}{2\hbar}}\, x_0 .$$

The expansion (2) reads

$$|0\rangle_0 = \sum_n a_n |n\rangle_1; \quad \text{where } b|n\rangle_1 = \sqrt{n}\, |n-1\rangle_1 .$$

The condition for $|0\rangle_0$ to be the ground state of H_0 is

$$a|0\rangle_0 = 0 \Rightarrow (b + C) \sum_n a_n |n\rangle_1 = 0 .$$

Using the definition of b we get the recurrence relation

$$a_n = \frac{(-C)}{\sqrt{n}} a_{n-1} = \ldots \frac{(-C)^n}{\sqrt{n!}} a_0 .$$

From the normalization condition

$$1 = \sum_n |a_n|^2 = a_0^2 \sum_n \frac{C^{2^n}}{n!} = a_0^2\, \text{Exp}\left[C^2\right] \quad \Rightarrow \quad a_0^2 = \text{Exp}\left[-C^2\right].$$

The probability to find the n - th state has a Poisson distribution

$$P_n = |a_n|^2 = \frac{C^{2^n}}{n!} \text{Exp}\left[-C^2\right] .$$

The result for n = 0, 1 coincides with the previous one.

■ 3

As the function do not change at t=0 the operators x and p do not have discontinuities. The same is true for $\dot{x}$= p/m . The derivative of p instead is discontinuous as

$$\frac{dp}{dt} = -\frac{\partial}{\partial x} V[x]$$

and V changes discontinuously.

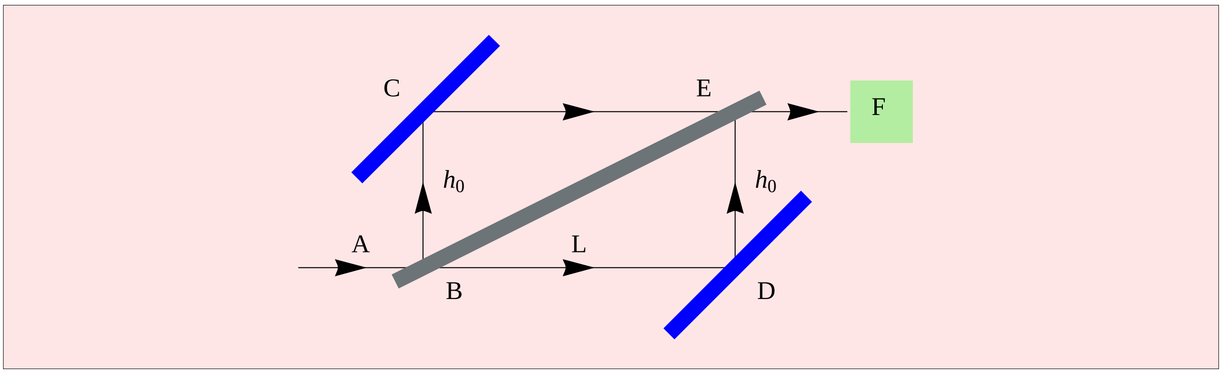
$$t < 0: \frac{\partial}{\partial x} V[x] = m\,\omega^2 x \quad ; \quad t > 0: \frac{\partial}{\partial x} V[x] = m\,\omega^2 (x - x_0) \quad ;$$

and the discontinuity is

$$\Delta \left\langle \frac{dp}{dt} \right\rangle = m\,\omega^2\, x_0.$$

Problem 14

A schematic neutron interferometer is shown in the figure



The Hamiltonian describing neutron motion is

$$H = \frac{p_x^2 + p_z^2}{2M} + M g z$$

M ≃ 1.7 10^{-24}gr is the neutron mass, g≃980 cm / sec^2. z is the vertical direction, x the horizontal direction, the incident direction of the beam.

Suppose that at $t = 0$ the plane BCED is vertical, while ABD is on the x axis.

1. Write the wavelength of the de Broglie wave for the incident neutron, as a function of the momentum p.

2. Assuming $p^2 / 2M \gg M g z$, compute the phase difference ΔΦ in the wave function between the two paths ABCEF and ABDEF, in terms of the parameters shown in the figure. Find the condition for maximal interference

3. The plane containing the apparatus BCED is now rotated by an angle φ around x axis, ABD. As φ varies the intensity I of the neutrons detected at the counter F varies, showing maxima and minima. Find how many maxima there are as φ varies from 0 and π/2.

Take v=3 10^5cm/sec for the velocity of the incident neutron. Neglect the small distortion in the paths due to gravitational field.

Solution

■ 1

$$\lambda = \frac{h}{p} = \frac{2\pi\hbar}{p}. \qquad (14.1)$$

■ 2

The wave function acquires a different phase in the two paths ABCEF and ABDEF not for the lengths of the paths(equal) but for the variation of the wavelength. In effect due to gravitational potential V[z] = M g z, in the horizontal paths CE and BD the wave number k = 2π/λ is given by

$$k_{CE} = \frac{\sqrt{2M(E - M g h_0)}}{\hbar} \; ; \quad k_{BD} = \frac{\sqrt{2ME}}{\hbar} = k. \qquad (14.2)$$

Their difference is

$$\Delta k = k - k \left(1 - \frac{M g h_0}{E}\right)^{1/2} \simeq \frac{M^2 g h_0}{k \hbar^2}. \tag{14.3}$$

The induced phase shift is

$$\Delta\Phi = L \Delta k \simeq \frac{M^2 g h_0 L \lambda}{2 \pi \hbar^2}. \tag{14.4}$$

The maximal phase difference is for

$$\Delta\Phi = 2 \pi n = \frac{M^2 g h_0 L \lambda}{2 \pi \hbar^2}. \tag{14.5}$$

3

As φ varies the height of CE path changes form $h_0$to 0. Using the preceding formula the number of times the two paths have maximal interference (i.e. maximal intensity in F) is

$$\mathcal{N} = \frac{M^2 g h_0 L \lambda}{(2 \pi)^2 \hbar^2} = \frac{Mg (h_0 L)}{2 \pi \hbar v} \simeq 8.8. \tag{14.6}$$

The interference has been measured experimentally, see R. Colella, A. Overhauser, S. Werner: *Phys. Rev. Lett.* **34**, 1472 , (1975).

Problem 15

An excited nucleus (with excitation energy G) decays into the ground state by emitting a photon.

1. Compute the energy of the emitted photon in the approximation of infinite nuclear mass.

2. Suppose now that the free nucleus of finite mass M is initially at rest. Compute the energy of the photon, taking into account the recoil of the final nucleus. Assume $Mc^2 \gg G$.

3. Assume now, instead, that the nucleus is bounded in a harmonic potential (e.g., a crystal lattice):

$$H = \frac{\mathbf{p}^2}{2 M} + \frac{1}{2} M \omega^2 \mathbf{r}^2.$$

r is the position of the center of mass of the nucleus. Before decaying the nucleus is in the ground state ψ_0 of the above oscillator (the internal state is in the excited state as in the previous points). Which energies are allowed for the photon in this case?

4. Justify the approximation for which the amplitude for the process is given by the matrix element of Exp[i k z], where the momentum of the emitted photon is (0,0, $\hbar$ k). By using such an approximation, compute the probability $P[E_\gamma]$ for emitting a photon with energy E_γ, one of the energies found in point 3 above.

This analysis shows that a fraction of absorption and re - emission of a gamma ray from a nucleus occurs in a recoil - free manner, when the nucleus is bound in a crystal solid. This phenomenon is known as the Moessbauer effect.

Solution

1

Energy conservation gives

$$E_\gamma = G . \tag{15.1}$$

2

If the photon carries a momentum p = $\hbar$ k, momentum conservation imply for the nucleus a momentum -p = - $\hbar$ k. The kinetic energy of the nucleus is

$$E_K = \frac{\hbar^2 k^2}{2 M}.$$

Energy conservation requires (as E_γ= p c):

$$k \hbar c + \frac{\hbar^2 k^2}{2 M} = G \; ; \quad \text{or} \quad E_\gamma + \frac{E_\gamma^2}{2 M c^2} = G. \tag{15.2}$$

Solving iteratively for E_γ we find

$$E_\gamma \simeq G - \frac{G^2}{2 M c^2} + ... \tag{15.3}$$

3

By hypothesis the oscillator center does not acquire momentum in the emission process (has an infinite mass). The energy of the photon can be affected by an excitation of the oscillator level, $0 \to N$, however. Energy conservation gives

$$G = E_\gamma + E_f - E_i ; \quad E_\gamma = G - \hbar\omega N, \quad N = 0, 1, 2 \ldots \tag{15.4}$$

N is related to the quantum numbers of the oscillator after the photon emission by

$$E_{osc} = \hbar\omega\left(\frac{3}{2} + n_1 + n_2 + n_3\right) = \hbar\omega\left(\frac{3}{2} + N\right). \tag{15.5}$$

4

The operator Exp[i k z] shifts the momentum of the nucleus by (0,0, $-\hbar$ k), this is the effect of the momentum conservation law. Then

$$P_N = |\langle \psi_f | e^{i k z} | \psi_0 \rangle|^2 = |\langle N | e^{i k z} | 0 \rangle|^2 ; \quad k \hbar c = G - \hbar\omega N . \tag{15.6}$$

The reader can easily convince himself that the normalization is consistent, i.e. the sum of probabilities is unity, due to completeness of harmonic oscillator eigenstates. The second expression in (6) refers to states of the one dimensional oscillator

$$H = \frac{p^2}{2M} + \frac{M\omega^2}{2} z^2 ,$$

the only one entering in the matrix elements of an operator which depends only on z. To compute the matrix element let us write

$$\langle N | = \frac{1}{\sqrt{N!}} \langle 0 | a^N ; \quad \text{Exp}[i k z] = \text{Exp}\left[i C \left(a + a^\dagger\right)\right]; \quad C = k\sqrt{\frac{\hbar}{2M\omega}} .$$

Using the identities

$$a\, e^{i C (a + a^\dagger)} |0\rangle = \left[a, e^{i C (a+a^\dagger)}\right] |0\rangle = i C e^{i C (a+a^\dagger)} |0\rangle ;$$
$$a^2 e^{i C (a+a^\dagger)} |0\rangle = \left[a^2, e^{i C (a+a^\dagger)}\right] |0\rangle = (i C)^2 e^{i C (a+a^\dagger)} |0\rangle ; \ldots .$$

one has

$$\langle N | e^{i k z} |0\rangle = \frac{(i C)^N}{\sqrt{N!}} \langle 0 | e^{i C (a+a^\dagger)} |0\rangle = \frac{(i C)^N}{\sqrt{N!}} e^{-\frac{C^2}{2}} .$$

In the last result we used Baker-Haussdorff formula, valid for [X,Y] a c-number :

$$e^X e^Y = e^{X+Y+\frac{1}{2}[X,Y]} .$$

The requested probability is thus a Poisson distribution,

$$P_N = \frac{C^{2N}}{N!} \text{Exp}\left[-C^2\right]. \tag{15.7}$$

Problem 16

*An electron (mass m and charge -e) moves in the x-y plane, subject to a magnetic field directed along z: **B** = (0, 0, B). In the following the electron spin is neglected. The Hamiltonian is*

$$H = \frac{\left(\mathbf{p} + \frac{e}{c}\mathbf{A}\right)^2}{2m} ;$$

***A** is the electromagnetic vector potential.*

1. Compute the spectrum of the Hamiltonian in the gauge **A** = (-B y, 0, 0) for the vector potential.

2. Do the same computation in the gauge **A** = (-B $\frac{y}{2}$, B $\frac{x}{2}$, 0).

(Laudau level: see Section 14.3 of the text for further discussions).

Solution

1

The Hamiltonian is

$$H = \frac{\left(\mathbf{p} + \frac{e}{c}\mathbf{A}\right)^2}{2m} = \frac{\left(p_x - \frac{eB}{c} y\right)^2}{2m} + \frac{p_y^2}{2m}.$$

H commutes with p_x, then we can look for H eigenstates in the form

$$\Psi[x, y] = e^{i p x/\hbar}\, \psi[y].$$

Substitution into the Schrödinger equation gives for ψ[y]

$$\left(\frac{p_y^2}{2m} + \frac{\left(\frac{eB}{c} y - p\right)^2}{2m}\right) \psi[y] = E\, \psi[y]. \tag{16.1}$$

This is the Schrödinger equation for a linear harmonic oscillator with center in

$$y_0 = \frac{p c}{e B},$$

and frequency equal to the Larmor frequency

$$\omega = \frac{e B}{m c}.$$

The eigenvalues are

$$E_n = \hbar \omega \left(n + \frac{1}{2}\right);\ n = 0, 1, \ldots \tag{16.2}$$

and are *independent* on p. This means that each level has an infinite degeneracy, labeled by p, with $-\infty < p < \infty$.

These energy levels are known as *Landau levels*. The present discussion in fact overlaps with the one in Section 14.3 of the main Text.

We remember from classical physics that the Larmor frequency corresponds to the angular frequency of an electron moving around a circle in a uniform magnetic field, with (the charge is - e):

$$\frac{d\mathbf{p}}{dt} = -\frac{e}{c} \mathbf{v} \wedge \mathbf{B}.$$

■ 2

In this gauge the Hamiltonian takes the form

$$H = \frac{\left(\mathbf{p} + \frac{e}{c}\mathbf{A}\right)^2}{2m} = \frac{\left(p_x - \frac{eB}{2c} y\right)^2}{2m} + \frac{\left(p_y + \frac{eB}{2c} x\right)^2}{2m}.$$

To simplify the formulas we will put in the following $\hbar = m = eB/c = 1$. In these new units

$$H = \frac{1}{2}\left(-i\,\partial_x - \frac{y}{2}\right)^2 + \frac{1}{2}\left(-i\,\partial_y + \frac{x}{2}\right)^2.$$

An elegant method of solution is to introduce complex variables

$$z = x + i y;\quad \bar{z} = x - i y.$$

The commutation relations are

$$[z, \partial_z] = [\bar{z}, \partial_{\bar{z}}] = -1;\ [\bar{z}, \partial_z] = [z, \partial_{\bar{z}}] = 0.$$

We can also define the analogous of annihilation and creation operators

$$a = \sqrt{2}\left(\frac{\partial}{\partial z} + \frac{\bar{z}}{4}\right);\quad a^\dagger = \sqrt{2}\left(-\frac{\partial}{\partial \bar{z}} + \frac{z}{4}\right);\quad \text{with}\quad \left[a, a^\dagger\right] = 1.$$

An elementary computation gives for H

$$H = -2\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} + \frac{1}{8}\bar{z} z + \bar{z}\frac{\partial}{\partial z} - z\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(a^\dagger a + a a^\dagger\right) = a^\dagger a + \frac{1}{2}. \tag{16.3}$$

The spectrum is again the Landau levels (2).

To compute the degeneracy let us note that the Schrödinger equation for the ground state

$$a\,\psi_0 = 0 \;\Rightarrow\; \left(\frac{\partial}{\partial z} + \frac{\bar{z}}{4}\right)\psi_0 = 0\,;$$

has infinite solutions

$$\psi_{0,m} = C_m\,\bar{z}^m\,e^{-\frac{1}{4}\bar{z}\,z} = C_m\,(x - i\,y)^m\,\mathrm{Exp}\left[-\frac{1}{4}\left(x^2 + y^2\right)\right];\quad m = 0, 1, 2\,... \qquad (16.4)$$

All solutions have $E_0 = \hbar\,\omega / 2$, an are labeled by the eigenvalues of angular momentum (L_3):

$$L = -i\left(x\,\partial_y - y\,\partial_x\right). \qquad (16.5)$$

L commutes with H and $L\,\psi_{0,m} = m\,\psi_{0,m}$.

Excited state can be constructed by acting with $a^\dagger$ on each function $\psi_{0,m}$.

$$\psi_{N,m} = \frac{1}{\sqrt{N!}}\left(a^\dagger\right)^N \psi_{0,m}\,. \qquad (16.6)$$

Remarks

a. In the symmetric gauge the electron appears localized in x and y, while in the first gauge it is localized only in y.

b. By choosing the gauge **A** = (-B $\frac{1}{2}\,(y - y_0)$, B $\frac{1}{2}\,(x - x_0)$, 0) we can construct localized eigenstates around an arbitrary point (x_0, y_0).

c. The two solutions are related by a unitary transformation, i.e. if ψ[x,y] satisfies

$$\left(\frac{\left(p_x - \frac{e\,B}{c}\,y\right)^2}{2\,m} + \frac{p_y^2}{2\,m}\right)\psi[x, y] = E\,\psi[x, y]\,;$$

then it can be easily checked that

$$\varphi[x, y] = \mathrm{Exp}\left[-\frac{\omega}{2\,\hbar}\,x\,y\right]\psi[x, y],$$

satisfies

$$\left(\frac{\left(p_x - \frac{e\,B}{2\,c}\,y\right)^2}{2\,m} + \frac{\left(p_y + \frac{e\,B}{2\,c}\,x\right)^2}{2\,m}\right)\varphi[x, y] = E\,\varphi[x, y]\,.$$

The spectrum of the Hamiltonian is, correctly, left invariant by this unitary transformation. The different aspect of the degeneracy must not be a trouble, each subspace at fixed n is an Hilbert space, the two description are the analogous of taking the plane waves basis or the harmonic oscillator basis for $\mathbb{L}^2$.