# Problems Chapter 4

**Quantum Mechanics**
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## Problem 1

*Two particle of spin 1/2 interact with a Hamiltonian* $H_0 = 1/2\ \mathbf{s_1} \cdot \mathbf{s_2}$. *List all the conserved quantities and write the energy eigenvalues. An external field* ***B*** *is applied to the system. Let us assume that this field interacts only with the first particle, with* $V = \mathbf{B} \cdot \mathbf{s_1}$. *Find the conserved quantities and determine the spectrum of* $H_0 + V$.

### Solution

#### Analytical solution

$H_0$ commutes with the 3 components of total spin $\mathbf{S} = \mathbf{s}_1 + \mathbf{s}_2$ . The total spin can be 0,1. Writing

$$\mathbf{s}_1 \cdot \mathbf{s}_2 = \frac{1}{2}\left(\mathbf{S}^2 - \mathbf{s}_1^2 - \mathbf{s}_2^2\right) = \frac{1}{2}\left(S(S+1) - \frac{3}{2}\right)$$

we find the eigenvalues E = (1/4,-3/4). The degeneracy is (2S+1) i.e. 3 and 1 respectively. A complete set of commuting variables is $S^2$ and $S_z$where z is an arbitrary axis. The eigenstates of $H_0$denoted with $|S, S_z\rangle$, are given by

$$|1, 1\rangle = u\otimes u; \quad |1, 0\rangle = (u\otimes d + d\otimes u)\big/\sqrt{2}; \quad |1, 1\rangle = d\otimes d; \quad |0, 0\rangle = (u\otimes d - d\otimes u)\big/\sqrt{2};$$

u and d represent up and down states for single particles.

We can choose the direction z along the external field B, in this way the interaction is written as $V = B\ s_{1z}$. V no longer commutes with all 3 components of total spin, as $s_{1z}$ is not rotationally invariant. The surviving symmetry is rotation around the z axis, and correspondingly $S_z$is still a good quantum number.
The action of $s_{1z}$ on the above basis is:

$$s_{1z}|1, 1\rangle = 1/2\,|1, 1\rangle; \qquad S_z|1, -1\rangle = -1/2\,|1, -1\rangle;$$
$$s_{1z}|1, 0\rangle = 1/2\,|0, 0\rangle; \qquad s_{1z}|0, 0\rangle = 1/2\,|1, 0\rangle;$$

The reader can check that the selection rules for the third component of a vector operator are satisfied, i.e.

$$\triangle S = 0, \pm 1; \quad 0 \to 0 \text{ forbidden}; \quad \triangle S_z = 0.$$

The resulting Hamiltonian is

$$H_1 = \begin{pmatrix} \frac{1}{4} + \frac{B}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & \frac{B}{2} \\ 0 & 0 & \frac{1}{4} - \frac{B}{2} & 0 \\ 0 & \frac{B}{2} & 0 & -\frac{3}{4} \end{pmatrix}.$$

The states 1 and 3 continue to be eigenstates, with eigenvalues 1/4 ± B/2. The submatrix involving states 2 and 4 must be diagonalized giving rise to eigenvalue equation

$$-\frac{3}{16} - \frac{B^2}{4} + \frac{\lambda}{2} + \lambda^2 == 0\,.$$

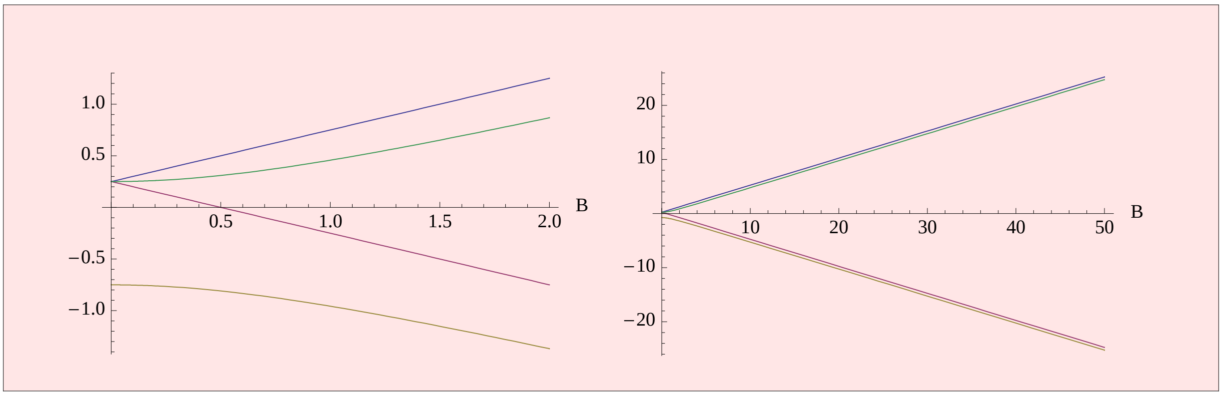
Solving this equation and adding the two trivial eigenvalues of $H_1$ (the elements (1,1) and (3,3) of the matrix), the eigenvalues are

$$\lambda_1 = \frac{1}{4} + \frac{B}{2}; \quad \lambda_2 = \frac{1}{4} - \frac{B}{2}; \quad \lambda_3 = \frac{1}{4}\left(-1 - 2\sqrt{1 + B^2}\right); \quad \lambda_4 = \frac{1}{4}\left(-1 + 2\sqrt{1 + B^2}\right).$$

For large B

$$\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\} \to \left\{\frac{B}{2}, -\frac{B}{2}, -\frac{B}{2}, \frac{B}{2}\right\};$$

i.e. two double degenerate levels. In this limit the eigenstates are eigenstates of V, while $H_0$ acts as a small perturbation. Here is a sketch of the behavior of the eigenvalues as a function of B.



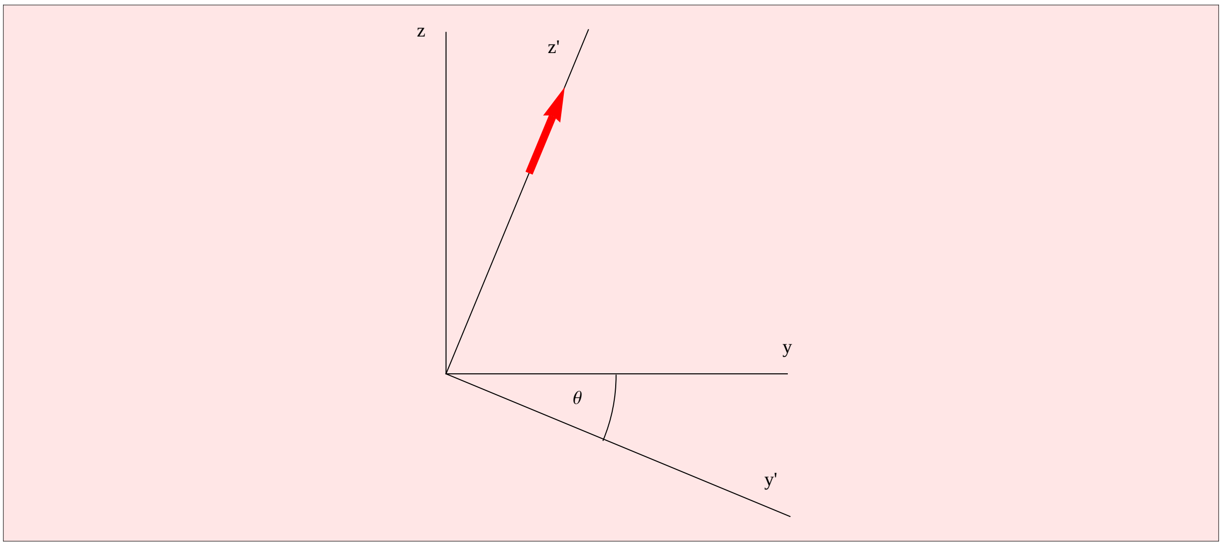

## Problem 2

1. *Write the components of a spinor (s = 1/2) with projection of spin 1/2 along an axis lying in the y, z plane and making an angle $\theta$ from the z axis.*
2. *Generalize the computation to an arbitrary axis $\zeta$ defined through polar angles ($\theta$,$\varphi$).*

### Solution

#### 1

The geometry of the problem is shown in figure below:




Consider for simplicity the double of the spin operator. We are looking for the eigenstate with eigenvalue +1 of

$$\Sigma = \text{Cos}[\theta]\,\sigma_z + \text{Sin}[\theta]\,\sigma_y = \begin{pmatrix} \text{Cos}[\theta] & -i\,\text{Sin}[\theta] \\ i\,\text{Sin}[\theta] & -\text{Cos}[\theta] \end{pmatrix}.$$

We can find eigenvalues and eigenvectors by diagonalizing the $\Sigma$, but it is instructive to proceed performing a rotation of reference frame. The rotated frame is obtained by a clockwise rotation of $\theta$ around x axis, i.e. $R_x[-\theta]$. In the *new* frame the state we are looking for is simply $\psi' = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Let us note that it is an eigenstate of $\sigma_z$ in the rotated frame. The state in the first frame is

$$\psi = R_x^{-1}[-\theta]\,\psi' = R_x[\theta]\,\psi' = \begin{pmatrix} \text{Cos}[\theta/2] & i\,\text{Sin}[\theta/2] \\ i\,\text{Sin}[\theta/2] & -\text{Cos}[\theta/2] \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \text{Cos}[\theta/2] \\ i\,\text{Sin}[\theta/2] \end{pmatrix}.$$

In fact

$$\Sigma\cdot\psi = \begin{pmatrix} \mathrm{Cos}[\theta] & -i\,\mathrm{Sin}[\theta] \\ i\,\mathrm{Sin}[\theta] & -\mathrm{Cos}[\theta] \end{pmatrix} \begin{pmatrix} \mathrm{Cos}[\theta/2] \\ i\,\mathrm{Sin}[\theta/2] \end{pmatrix} = \begin{pmatrix} \mathrm{Cos}[\theta/2] \\ i\,\mathrm{Sin}[\theta/2] \end{pmatrix} = \psi$$

It is perhaps useful to clarify a point: we used the relation $\psi' = R_x[-\theta]\psi$ which is the correct one for the components of the wave function.

In terms of kets the transformation law reads (primed kets refer to rotated frame)

$$U[R]\,|\,\sigma\rangle = R_{\sigma'\sigma}\,|\,\sigma\rangle' \;; \quad |\,\sigma\rangle' = R^*_{\sigma'\sigma}\,U[R]\,|\,\sigma\rangle\;.$$

The state we are looking for is $|+\rangle'$ (+ and - refers to indexes 1 and 2). This state, in the original non rotated frame is $U^{-1}[R]\,|+\rangle'$ i.e.

$$U^{-1}[R]\;|+\rangle' = R^*_{1\sigma}\,|\,\sigma\rangle\,.$$

The rotation matrix is

$$R_x[-\theta] = \begin{pmatrix} \mathrm{Cos}[\theta/2] & -i\,\mathrm{Sin}[\theta/2] \\ -i\,\mathrm{Sin}[\theta/2] & -\mathrm{Cos}[\theta/2] \end{pmatrix},$$

and it follows that

$$U^{-1}[R]\,\left|+\right\rangle' = R^*_{11}\,\left|+\right\rangle + R^*_{12}\,\left|-\right\rangle = \mathrm{Cos}[\theta/2]\,|+\rangle + i\,\mathrm{Sin}[\theta/2]\,|-\rangle$$

which coincides with the previous result.

■ **2**

The components, as shown in the previous answer, are the conjugate of first row of the rotation matrix which brings the old reference frame to a new one in which z axis is along $\zeta$. The axes directions transforms as a basis then the i-th axis will have as components, in general, the i-th row of the 3×3 rotation R. In 3 dimensions rotation matrices are real, and we have, indicating with $\mathbf{e}_i$ the old basis and with $\mathbf{f}_i$ the new basis:

$$\mathbf{f}_i = R_{ij}\,\mathbf{e}_j$$

The third row of the rotation matrix must be:

$$\zeta = \{\mathrm{Sin}[\theta]\,\mathrm{Cos}[\varphi],\ \mathrm{Sin}[\theta]\,\mathrm{Sin}[\varphi],\ \mathrm{Cos}[\theta]\};$$

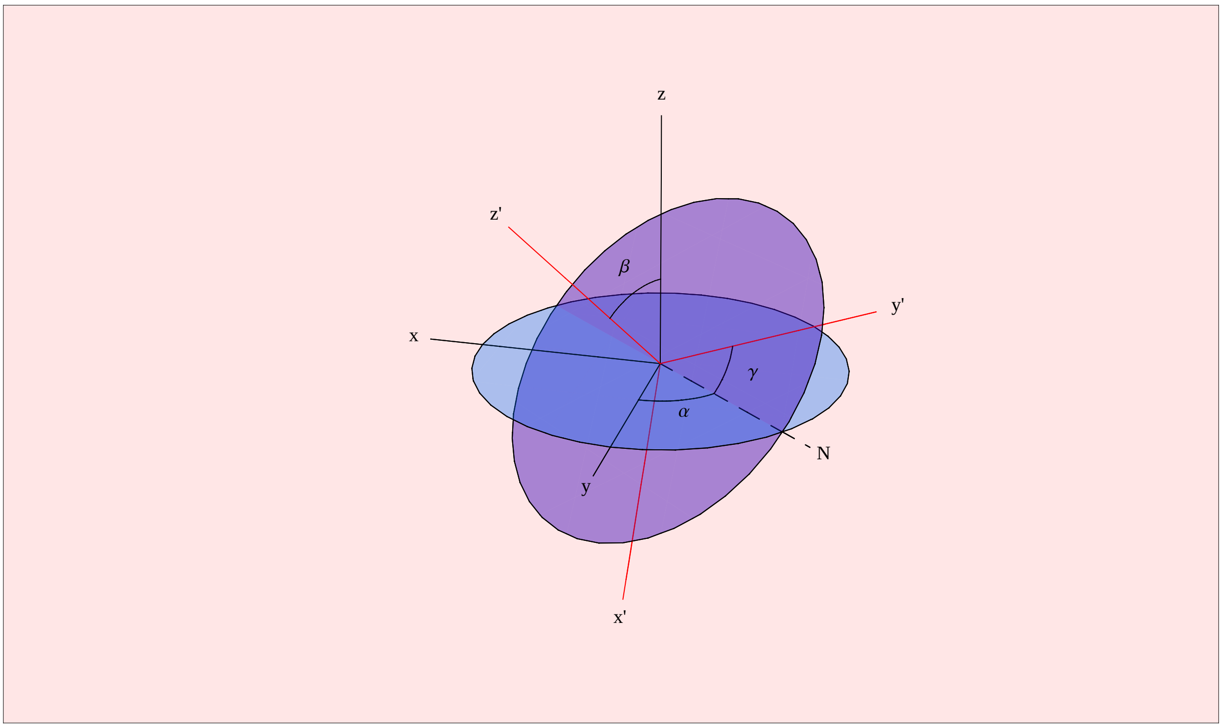
A generic rotation matrix is given through Euler angles, $\alpha, \beta, \gamma$ by

$$R(\alpha,\beta,\gamma) = Rz[\gamma]\,Ry[\beta]\,Rz[\alpha],$$

with the convention used in this course. With

$$Rx[\alpha] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathrm{Cos}[\alpha] & \mathrm{Sin}[\alpha] \\ 0 & -\mathrm{Sin}[\alpha] & \mathrm{Cos}[\alpha] \end{pmatrix};\ Ry[\alpha] = \begin{pmatrix} \mathrm{Cos}[\alpha] & 0 & -\mathrm{Sin}[\alpha] \\ 0 & 1 & 0 \\ \mathrm{Sin}[\alpha] & 0 & \mathrm{Cos}[\alpha] \end{pmatrix};\ Rz[\alpha] = \begin{pmatrix} \mathrm{Cos}[\alpha] & \mathrm{Sin}[\alpha] & 0 \\ -\mathrm{Sin}[\alpha] & \mathrm{Cos}[\alpha] & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The definitions are shown in the figure below:




It is apparent that the choice: $\alpha=\varphi$, $\beta=\theta$, $\gamma=0$ brings the third axis along the $\zeta$ direction, as the reader can check with

$$\mathcal{R} = Rz[0]\ Ry[\theta]\ Rz[\varphi] = \begin{pmatrix} \cos[\theta]\cos[\varphi] & \cos[\theta]\sin[\varphi] & -\sin[\theta] \\ -\sin[\varphi] & \cos[\varphi] & 0 \\ \cos[\varphi]\sin[\theta] & \sin[\theta]\sin[\varphi] & \cos[\theta] \end{pmatrix}. \quad (2.1)$$

In the spinor space the rotation matrices around the principal axes are constructed with the Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix};$$

$$R_k[\alpha] = \mathrm{Exp}\left[i\,\frac{\alpha}{2}\,\sigma_k\right] = \cos\left[\frac{\alpha}{2}\right] + i\,\sin\left[\frac{\alpha}{2}\right]\sigma_k.$$

The matrix $\mathcal{R}$, (1) is given in spinor space by

$$R = R_2[\theta]\,R_3[\varphi] = \begin{pmatrix} \cos\left[\frac{\theta}{2}\right] e^{i\varphi/2} & \sin\left[\frac{\theta}{2}\right] e^{-i\varphi/2} \\ -\sin\left[\frac{\theta}{2}\right] e^{i\varphi/2} & \cos\left[\frac{\theta}{2}\right] e^{-i\varphi/2} \end{pmatrix}. \quad (2.2)$$

The spinor we look for is then given by the complex conjugate of the first row of R,

$$\psi_1 = \begin{pmatrix} \cos\left[\frac{\theta}{2}\right] e^{-i\varphi/2} \\ \sin\left[\frac{\theta}{2}\right] e^{+i\varphi/2} \end{pmatrix}. \quad (2.3)$$

We can check the result by considering the projection of the spin along $\zeta$ and verifying that $\boldsymbol{\zeta}.\boldsymbol{\sigma}\ \psi_1 = \psi_1$:

$$\boldsymbol{\zeta}.\boldsymbol{\sigma} = \begin{pmatrix} \cos[\theta] & \sin[\theta]\,e^{-i\varphi} \\ \sin[\theta]\,e^{i\varphi} & -\cos[\theta] \end{pmatrix}; \qquad \boldsymbol{\zeta}.\boldsymbol{\sigma}\psi_1 = \psi_1.$$

The second row of R gives the opposite polarized spinor

$$\psi_2 = \begin{pmatrix} -\sin\left[\frac{\theta}{2}\right] e^{-i\varphi/2} \\ \cos\left[\frac{\theta}{2}\right] e^{+i\varphi/2} \end{pmatrix}.$$

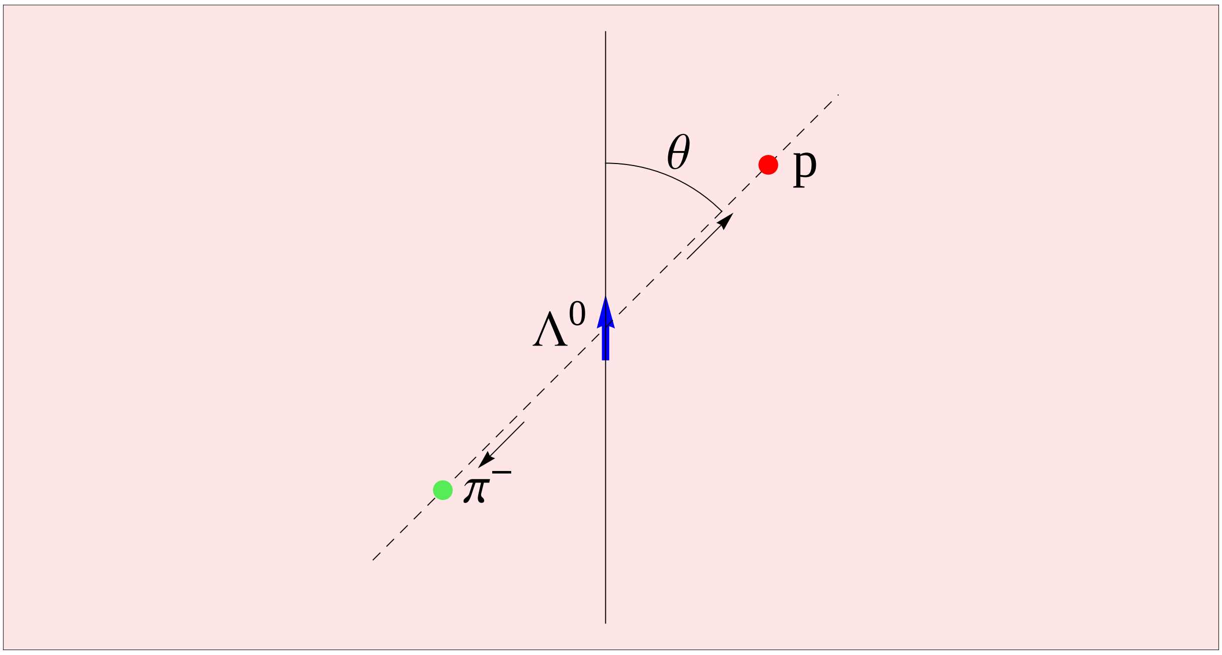
The use of $\gamma$ angle would give an additional arbitrary phase to the spinor, as expected.

## Problem 3

*The particle* $\Lambda^0$ *,with spin 1/2, decays into* p + $\pi$ in the rest system. *Assume that the* $\Lambda^0$ *is polarized along z axis. Write the most general amplitude for the process and the angular distribution for the secondary particles. What are the constraints if parity is conserved? Write the expression for the mean value of the longitudinal polarization of the proton (projection of spin along the direction of motion).*

### Solution

The kinematic is the following:




The problem is invariant under rotations around the z axis (polarization of the Λ particle), so we can put $\varphi$=0 for the azimuth angle. Let |Λ> the initial state and S the operator which causes the transition to p, $\pi$ states, denoted by |n>. We can observe the direction of the proton (surely opposite to that of the $\pi$) |$\theta$> and, possibly the spin state (eg. along z) of the proton, $\sigma$. The probability amplitude for the event is

$$A[\theta, \sigma] = \langle \theta, \sigma | S | \Lambda \rangle = \sum_n \langle \theta, \sigma | n \rangle \langle n | S | \Lambda \rangle .$$

One is interested only in the angular momentum variables here. The initial state have J=1/2, $J_z$= 1/2, the same must be for |n> states. Quantum number of intermediate states in the sum can be selected as L and $\sigma$, L is the angular momentum. As L ⊕ 1/2 must give 1/2 by angular momentum conservation, the only possible states have L=0,1. The two angular momenta must be combined via ClebschGordan coefficients, note that $L_z$= 1/2 - $\sigma$ as $J_z = 1/2$. The amplitude in $\theta$ are clearly $\delta_{\sigma\, s_z}\, Y_{Lm}[\theta, 0]$. If we call a and b the two matrix elements of S in these two states, the two possible terms in the sum are:

$$f1[\theta, \sigma] = a\, C\left[1, \frac{1}{2} - \sigma; \frac{1}{2}, \sigma, \frac{1}{2}, \frac{1}{2}\right] Y_{1, \frac{1}{2}-\sigma}[\theta, 0];$$

$$f2[\theta, \sigma] = b\, C\left[0, 0; \frac{1}{2}, \sigma, \frac{1}{2}, \frac{1}{2}\right] Y_{0,0}[\theta, 0];$$

To the second term only sz=1/2 gives a contribution. The total amplitude is

$$A[\theta, \sigma] = f0[\theta, \sigma] + f1[\theta, \sigma];$$

The (relative) probability distribution for the decay (i.e. we do not measure proton spin) is:

$$P[\theta] = \sum_{\sigma=-1/2}^{+1/2} |A[\theta, \sigma]|^2 = \frac{|a|^2 + |b|^2 - 2\,\mathrm{Re}[a\, b^*]\, \mathrm{Cos}[\theta]}{4\pi} \qquad (3.1)$$

#### Parity

The two states above have opposite parity, as orbital parity is $(-)^L$. This means that if both a and b are different from 0 parity is *violated* (as it is in Nature for this decay). If we look at the angular distribution (1)

$$P[\theta] = \frac{|a|^2 + |b|^2 - 2\,\mathrm{Re}[a\,b^*]\,\mathrm{Cos}[\theta]}{4\pi};$$

this is clear. If we consider a reflection in the (x,y) plane the spin of Λ remain unchanged (spin is an axial vector), while momentum of the p changes and $\theta \to \pi - \theta$. But $P[\pi - \theta] \neq P[\theta]$ if ab≠0 so parity must be violated.

### ■ Longitudinal polarization of the proton

In the previous problem it has been shown that the spinor with positive and negative spin projection along an axis with polar angles ( $\theta$, $\varphi$=0) are respectively:

$$\psi_+ = \begin{pmatrix} \mathrm{Cos}\left[\frac{\theta}{2}\right] e^{-i\varphi/2} \\ \mathrm{Sin}\left[\frac{\theta}{2}\right] e^{+i\varphi/2} \end{pmatrix}; \quad \psi_- = \begin{pmatrix} -\mathrm{Sin}\left[\frac{\theta}{2}\right] e^{-i\varphi/2} \\ \mathrm{Cos}\left[\frac{\theta}{2}\right] e^{+i\varphi/2} \end{pmatrix}. \qquad (3.2)$$

The corresponding amplitudes will be:

$$A_\pm = \sum A[\theta, \sigma]\,\psi_\pm[\sigma];$$

i.e.

$$A_+ = \frac{(-a + b)\,\mathrm{Cos}\left[\frac{\theta}{2}\right]}{2\sqrt{\pi}}; \quad A_- = -\frac{(a + b)\,\mathrm{Sin}\left[\frac{\theta}{2}\right]}{2\sqrt{\pi}}.$$

We see that for $\theta = 0$ $A_- = 0$, while for $\theta=\pi$, $A_+ = 0$. This is due to the conservation of angular momentum: for motions along z axis the orbital momentum $L_z$vanishes, and the spin of the proton must be in positive z direction, i.e. with positive longitudinal polarization for $\theta$=0 and negative longitudinal polarization for $\theta = \pi$. the sum over the final polarizations gives correctly the same result as before:

$$P[\theta] = |A_+|^2 + |A_-|^2 = \frac{|a|^2 + |b|^2 - 2\,\mathrm{Re}[a\,b^*]\,\mathrm{Cos}[\theta]}{4\pi}.$$

The mean value of the polarization will be given by the difference of counting with polarization + and - over the total events:

$$N_+ = \int |A_+|^2\,\mathrm{Sin}[\theta]\,d\theta; \quad N_- = \int |A_-|^2\,\mathrm{Sin}[\theta]\,d\theta;$$

$$\text{mean Polarization} = \frac{N_+ - N_-}{N_+ + N_-}.$$

Again this measurement checks parity conservation: mean longitudinal polarization is the mean value of a pseudoscalar, < **s·p** >, so must be zero if parity is conserved.

# Problem 4

*A particle of spin S interacts with an external uniform magnetic field B,*

$$H = -\mu \cdot B = -\beta\,S_z.$$

*At time t = 0 the spin is oriented along* **n** = (Sin[$\theta$]Cos[$\varphi$],Sin[$\theta$]Sin[$\varphi$], Cos[$\theta$]) , i.e. **S·n** |0⟩ = S |0⟩ .

1. Compute the mean values $\langle S_x \rangle$, $\langle S_y \rangle$ and $\langle S_z \rangle$ at time $t = 0$. (Suggestion: consider an equivalent system composed by N = 2 S particles of spin 1/2).
2. Compute the probabilities to find the different $S_z$values with a measure of $S_z$ at t=0.
3. Consider the Heisenberg operators $S_x^H[t]$, $S_y^H[t]$, $S_z^H[t]$. Write and solve the Heisenberg equations of motion. From the computation extract the values of $\langle t | S_x | t \rangle$, $\langle t | S_y | t \rangle$, $\langle t | S_z | t \rangle$ for the Schrödinger operators at time t. Find the minimum value of $t$ such that $\langle t | S_x | t \rangle$ has the initial value.

## ● Solution

### ■ 1

Let us first consider the eigenstate of **S·n** for a spin 1/2 particle:

$$|\mathbf{n}\rangle = \begin{pmatrix} e^{-i\varphi/2}\,\mathrm{Cos}\left[\frac{\theta}{2}\right] \\ e^{i\varphi/2}\,\mathrm{Sin}\left[\frac{\theta}{2}\right] \end{pmatrix}. \qquad (4.1)$$

To verify that | **n**⟩ is an eigenvector the reader can easily check that **n·σ** | **n**⟩ = | **n**⟩ .

For the general case with spin S in the direction **n,** we consider as suggested the equivalent system of N = 2S spin 1/2 particles, with parallel spin:

$$| 0\rangle = \begin{pmatrix} e^{-i\varphi/2}\,\mathrm{Cos}\left[\frac{\theta}{2}\right] \\ e^{i\varphi/2}\,\mathrm{Sin}\left[\frac{\theta}{2}\right] \end{pmatrix}_1 \begin{pmatrix} e^{-i\varphi/2}\,\mathrm{Cos}\left[\frac{\theta}{2}\right] \\ e^{i\varphi/2}\,\mathrm{Sin}\left[\frac{\theta}{2}\right] \end{pmatrix}_2 \ldots \begin{pmatrix} e^{-i\varphi/2}\,\mathrm{Cos}\left[\frac{\theta}{2}\right] \\ e^{i\varphi/2}\,\mathrm{Sin}\left[\frac{\theta}{2}\right] \end{pmatrix}_N \tag{4.2}$$

The spin operator is

$$\mathbf{S} = \sum \mathbf{s}_i\,. \tag{4.3}$$

The mean value of $S_x$ is

$$\langle S_x\rangle = N\,\langle \mathbf{n}\,|\,s_{1\,x}\,|\,\mathbf{n}\rangle = \frac{N}{2}\left(e^{i\varphi}\,\mathrm{Cos}\left[\frac{\theta}{2}\right]\mathrm{Sin}\left[\frac{\theta}{2}\right] + e^{-i\varphi}\,\mathrm{Cos}\left[\frac{\theta}{2}\right]\mathrm{Sin}\left[\frac{\theta}{2}\right]\right) = S\,\mathrm{Cos}[\varphi]\,\mathrm{Sin}[\theta]\,. \tag{4.4}$$

Likewise

$$\langle S_y\rangle = S\,\mathrm{Cos}[\varphi]\,\mathrm{Sin}[\theta]\,;\quad \langle S_z\rangle = S\,\mathrm{Cos}[\theta]\,. \tag{4.5}$$

■ **2**

We can write the state (2) as

$$| 0\rangle = \left(e^{-i\varphi/2}\,\mathrm{Cos}\left[\frac{\theta}{2}\right]\alpha_1 + e^{i\varphi/2}\,\mathrm{Sin}\left[\frac{\theta}{2}\right]\beta_1\right)\ \ldots\ \left(e^{-i\varphi/2}\,\mathrm{Cos}\left[\frac{\theta}{2}\right]\alpha_N + e^{i\varphi/2}\,\mathrm{Sin}\left[\frac{\theta}{2}\right]\beta_N\right) = \sum_{k=-N}^{N} c_k\,|\,S,\ S_z = \frac{k}{2}\rangle\,. \tag{4.6}$$

where $\alpha, \beta$ are the up and down spinors. The vector $|\,S,\ S_z = \frac{k}{2}\rangle$ is the symmetric combination

$$|\,S,\ S_z = \frac{k}{2}\rangle = \binom{N}{\ell}^{-1/2} \sum \alpha_1\,\alpha_2 \ldots \beta_{\ell+1} \ldots \beta_N \tag{4.7}$$

of terms with $\ell$ spins up and N-$\ell$ spins down, We must have

$$\frac{\ell}{2} - \frac{N-\ell}{2} = \frac{2\ell - N}{2} = \frac{k}{2} \quad\Rightarrow\quad k = 2\ell - N;\quad \ell = \frac{N+k}{2}\,.$$

The coefficients $c_k$are

$$c_k = \binom{N}{\ell}^{1/2}\left(e^{-i\varphi/2}\,\mathrm{Cos}\left[\frac{\theta}{2}\right]\right)^{\ell}\left(e^{i\varphi/2}\,\mathrm{Sin}\left[\frac{\theta}{2}\right]\right)^{N-\ell}. \tag{4.8}$$

The requested probability is then

$$P_k = \binom{N}{\ell}\,\mathrm{Cos}\left[\frac{\theta}{2}\right]^{2\ell}\,\mathrm{Sin}\left[\frac{\theta}{2}\right]^{2\,(N-\ell)};\quad \ell = \frac{N+k}{2}. \tag{4.9}$$

The reader can easily check that the probabilities sum up to 1.

■ **3**

The time evolution of the operators is given by

$$S_x[t] = e^{i\,H\,t/\hbar}\,S_x\,e^{-i\,H\,t/\hbar} = e^{-i\,\beta\,S_z\,t/\hbar}\,S_x\,e^{i\,\beta S_z\,t/\hbar}\,;\ \text{etc}. \tag{4.10}$$

In particular $S_z[t] = S_z[0] = S_z$. The time evolution equations are

$$\begin{aligned} i\,\frac{d}{dt}S_x[t] &= \frac{1}{\hbar}[S_x,\,H] = -\frac{\beta}{\hbar}[S_x,\,S_z] = i\,\beta\,S_y; \\ i\,\frac{d}{dt}S_y[t] &= \frac{1}{\hbar}[S_y,\,H] = -\frac{\beta}{\hbar}[S_y,\,S_z] = -i\,\beta\,S_x; \\ i\,\frac{d}{dt}S_z[t] &= 0. \end{aligned} \tag{4.11}$$

From the first two equations

$$\ddot{S}_x = -\beta^2\,S_x\,;\ \ddot{S}_y = -\beta^2\,S_y\,. \tag{4.12}$$

Solving and imposing the initial condition one easily get

$$S_x[t] = S_x \mathrm{Cos}[\beta t] + S_y \mathrm{Sin}[\beta t]; \quad S_y[t] = S_y \mathrm{Cos}[\beta t] - S_x \mathrm{Sin}[\beta t]. \tag{4.13}$$

For the average values at time t

$$\langle t \mid S_x \mid t \rangle = \langle 0 \mid S_x[t] \mid 0 \rangle = S\,(\mathrm{Cos}[\varphi]\,\mathrm{Sin}[\theta]\,\mathrm{Cos}[\beta t] + \mathrm{Sin}[\varphi]\,\mathrm{Sin}[\theta]\,\mathrm{Sin}[\beta t]) = S\,\mathrm{Cos}[\varphi - \beta t]\,\mathrm{Sin}[\theta];$$

$$\langle t \mid S_y \mid t \rangle = \langle 0 \mid S_y[t] \mid 0 \rangle = S\,(\mathrm{Sin}[\varphi]\,\mathrm{Sin}[\theta]\,\mathrm{Cos}[\beta t] - \mathrm{Cos}[\varphi]\,\mathrm{Sin}[\theta]\,\mathrm{Sin}[\beta t]) = S\,\mathrm{Sin}[\varphi - \beta t]\,\mathrm{Sin}[\theta].$$

# Problem 5

*A particle of mass m, spin 1/2 and charge e is bound in a three dimensional harmonic potential. A weak uniform magnetic field is applied to the system. The Hamiltonian is*

$$H = \frac{1}{2m}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)^2 + \frac{1}{2} m\,\omega^2 \mathbf{r}^2 - \boldsymbol{\mu}\cdot\mathbf{B}; \quad \boldsymbol{\mu} = g\,\mu_B\,S = \frac{g}{2}\mu_B\,\sigma. \qquad \mu_B = \frac{e\hbar}{2\,mc}.$$

1. With $\mathbf{B} = 0$, compute the first few energy levels of the system, their orbital angular momentum and the total angular momentum.
2. Consider now a weak magnetic field, with a vector potential $\mathbf{A} = (-B\frac{y}{2}, B\frac{x}{2}, 0)$. List the conserved operators, neglecting order $B^2$effects.
3. Using previous results compute, in the weak field approximation, the energy for the first two levels of the system.

## Solution

### 1

The Hamiltonian for B = 0 is the sum of three one dimensional oscillators, so the energy levels are

$$E_N = \hbar\,\omega\left(N + \frac{3}{2}\right); \quad N = 0, 1, 2 \ldots \tag{5.1}$$

We have an orbital degeneracy equal to the number of ways to get N as the sum of three numbers ( $n_1$, $n_2$, $n_3$), corresponding to the quantum numbers of the three independent one dimensional oscillators:
N = 0, deg. = 1; N = 1, deg. = 2, N = 3, deg. = 6, N = 4, deg. = 12 ... In general deg[N] = (N+1)(N+2). The eigenfunctions have the form

$$\psi_N[x, y, z] = C\,H_{n_1}\left[\sqrt{\frac{m\,\omega}{\hbar}}\,x\right] H_{n_2}\left[\sqrt{\frac{m\,\omega}{\hbar}}\,y\right] H_{n_3}\left[\sqrt{\frac{m\,\omega}{\hbar}}\,z\right] \mathrm{Exp}\left[-\frac{m\,\omega}{2\hbar}r^2\right],$$

Where $H_i$are Hermite polynomials. The parity is $(-1)^N$, product of the parity of single oscillators. Each eigenfunction is a product of a polynomial of degree N times a rotation invariant exponential, then the possible angular momenta are

$$L = N, N-2, \ldots .0\,(1).$$

Even / Odd sequence of L for even/odd parity levels.

Each level has an additional doubling in the degeneracy due to spin.

The first levels are

1. N = 0, L = 0, J = 1/2 ;
2. N=1, L = 1, J=3/2, 1/2 ;
3. N=2, L=2, 0, J = 5/2, 3/2, 1/2.

### 2

In weak field approximation

$$H = \frac{1}{2\,m}\mathbf{p}^2 + \frac{1}{2} m\,\omega^2\,\mathbf{r}^2 - \frac{e\,B\,\hbar}{2\,m\,c} L_z - \frac{g\,B\,\mu_B}{2}\sigma_z = \frac{1}{2\,m}\mathbf{p}^2 + \frac{1}{2} m\,\omega^2\,\mathbf{r}^2 - B\,\mu_B\left(L_z + \frac{g}{2}\sigma_z\right).$$

Operators $L^2$, $L_z$, $S^2$, $S_z$ commute with H.

### 3

Eigenstates of H can be classified by N, $L_z$, $S_z$. The eigenvalues are

$$E_{N, L_z, S_z} = \hbar\,\omega\left(N + \frac{3}{2}\right) - \mu_B\,B\,(L_z + g\,S_z).$$

## Problem 6

*A constant electric field directed along z axis acts on a particle of mass m and charge q.*

1. Write and solve the Heisenberg equations for the position **r**[t] and the momentum **p**[t].
2. Find the operator **L**[t] using the results of point 1. Verify that the result is consistent with equations of motion for **L**[t].
3. Compute the mean value of **L**[t] knowing that

$$\langle \mathbf{r}[0] \rangle = 0; \quad \langle \mathbf{p}[0] \rangle = p_0\, \hat{\mathbf{x}}; \quad \langle \mathbf{L}[t] \rangle = 0.$$

### Solution

#### 1

The Hamiltonian is

$$H = \frac{\mathbf{p}^2}{2\,m} - q\,E_0\,z \tag{6.1}$$

The equations of motion are

$$\dot{\mathbf{p}} = \frac{i}{\hbar}[H, \mathbf{p}] = q\,E_0\,\hat{\mathbf{z}}; \quad \dot{\mathbf{r}} = \frac{i}{\hbar}[H, \mathbf{r}] = \frac{\mathbf{p}}{m}; \quad \Rightarrow \ddot{\mathbf{r}} = \frac{q\,E_0}{m}\hat{\mathbf{z}} \quad (\text{constant acceleration}). \tag{6.2}$$

Their solution is identical to the classical solution

$$\mathbf{r}[t] = \mathbf{r}[0] + \frac{t}{m}\mathbf{p}[0] + \frac{1}{2}\frac{q\,E_0}{m}t^2\,\hat{\mathbf{z}}; \qquad \mathbf{p}[t] = \mathbf{p}[0] + q\,E_0\,t\,\hat{\mathbf{z}}. \tag{6.3}$$

#### 2

By definition

$$\mathbf{L}[t] = \mathbf{r}[t] \wedge \mathbf{p}[t]. \tag{6.4}$$

From the solutions (3) it follows that

$$\begin{aligned} L_z[t] &= L_z[0]; \\ L_x[t] &= L_x[0] + q\,E_0\,t\,y[0] + \frac{1}{2}\frac{q\,E_0}{m}t^2\,p_y[0]; \\ L_y[t] &= L_y[0] - q\,E_0\,t\,x[0] - \frac{1}{2}\frac{q\,E_0}{m}t^2\,p_x[0]. \end{aligned} \tag{6.5}$$

The equations of motion for **L** are

$$\dot{\mathbf{L}} = \frac{i}{\hbar}[H, \mathbf{L}]. \tag{6.6}$$

or

$$\dot{L}_z[t] = 0; \quad \dot{L}_x[t] = q\,E_0\,y[t]; \quad \dot{L}_y[t] = -q\,E_0\,x[t]. \tag{6.7}$$

$L_z$ is a constant of motion, in agreement with (5). The equations for $L_x$ and $L_y$ can be integrated using (3) and reproduce (5):

$$\begin{aligned} L_x[t] &= L_x[0] + q\,E_0\int_0^t dt\,y[t] = L_x[0] + q\,E_0\,t\,y[0] + \frac{1}{2}\frac{q\,E_0}{m}t^2\,p_y[0]; \\ L_y[t] &= L_y[0] - q\,E_0\int_0^t dt\,x[t] = L_y[0] - q\,E_0\,t\,x[0] - \frac{1}{2}\frac{q\,E_0}{m}t^2\,p_x[0]. \end{aligned} \tag{6.8}$$

#### 3

With the initial conditions given in the text the mean value of (5) give immediately:

$$\langle L_x[t] \rangle = 0; \quad \langle L_z[t] \rangle = 0; \quad \langle L_y[t] \rangle = -\frac{1}{2}\frac{q\,E_0}{m}t^2\,p_0.$$

## Problem 7

*A particle of mass m and spin 1/2 is described by a one-dimensional Hamiltonian*

$$H = \frac{p^2}{2m} + V[x]; \quad V[x] = \frac{\sigma_3 + 1}{4} m\,\omega^2 x^2 .$$

**1.** Compute the spectrum of H and its eigenfunctions.

**2.** At time $t = 0$ the particle is in the state

$$\psi[0, x] = \psi_0[x] \frac{|+\rangle - |-\rangle}{\sqrt{2}};$$

where

$$|+\rangle, |-\rangle \text{ up, down eigenstates of } s_z \,; \quad \psi_0[x] = \left(\frac{m\,\omega}{\pi\,\hbar}\right)^{1/4} \text{Exp}\left[-\frac{m\,\omega}{2\,\hbar} x^2\right] = \text{ground} - \text{state wave function for the oscillator (freq. } \omega\text{)}.$$

Determine if this state is an eigenstate of H and compute the mean value of H on it.

**3.** Compute the wave function at time $t$. At time $t$ a measurement of $s_z$ is performed with a Stern-Gerlach experiment. Compute the probability of finding $s_z = -1/2$.

### Solution

#### 1

H commutes with $s_z$ so we can classify the states with the eigenvalues ±1/2 of this operator. For *up* states (eigenvalues +1/2, or +1 for $\sigma_3$) H is the Hamiltonian $H_\omega$ of an harmonic oscillator, with eigenvalues and eigenfunctions

$$E_n = \hbar\,\omega \left(n + \frac{1}{2}\right), \quad n = 0, 1, \ldots; \quad \psi_n[x] = C_n H_n\left[\sqrt{\frac{m\,\omega}{\hbar}} x\right] \text{Exp}\left[-\frac{m\,\omega}{2\,\hbar} x^2\right]. \tag{7.1}$$

For *down* states (eigenvalues -1/2, or -1 for $\sigma_3$) , the system describes a free particle of an Hamiltonian $H^{(\text{free})}$ with eigenvalues and eigenfunctions

$$E_p = \frac{p^2}{2m}, \quad -\infty < p < \infty; \quad \psi_p[x] = \frac{1}{\sqrt{2\,\pi\,\hbar}} \text{Exp}[i\,p\,x]. \tag{7.2}$$

#### 2

The state is *not* an eigenfunction of H as it is not an eigenfunction of $s_z$. The mean value of H on this state is

$$\langle\psi\,|\,H\,|\,\psi\rangle = \frac{1}{2}\langle\psi_0\,|\,H_\omega\,|\,\psi_0\rangle + \frac{1}{2}\left\langle\psi_0\,\middle|\,H^{(\text{free})}\,\middle|\,\psi_0\right\rangle = \frac{1}{4}\hbar\,\omega + \frac{1}{2}\left\langle\psi_0\,\middle|\,\frac{p^2}{2m}\,\middle|\,\psi_0\right\rangle = \frac{1}{4}\hbar\,\omega + \frac{1}{2}\frac{\hbar\,\omega}{4} = \frac{3}{8}\hbar\,\omega. \tag{7.3}$$

#### 3

The spin up part of the state evolve as an eigenstate of H:

$$\psi_0[x]\,|+\rangle \to e^{-i\,E_0\,t/\hbar}\,\psi_0[x]\,|+\rangle\,; \quad E_0 = \frac{\hbar\,\omega}{2}.$$

For the down part of the state we can write

$$\psi_0[x]\,|-\rangle = \int_{-\infty}^{+\infty} dk\,a[k]\,e^{i\,k\,x}\,|-\rangle \to \int_{-\infty}^{+\infty} dk\,a[k]\,e^{-i\frac{k^2\hbar\,t}{2m}}\,e^{i\,k\,x}\,|-\rangle .$$

Using the general formula

$$\int_{-\infty}^{+\infty} dz\,\text{Exp}\left[-A\,z^2 + B\,z\right] = \int_{-\infty}^{+\infty} dz\,\text{Exp}\left[-A\left(z - \frac{B}{2A}\right)^2 + \frac{B^2}{4A}\right] = \sqrt{\frac{\pi}{A}}\,\text{Exp}\left[\frac{B^2}{4A}\right]; \quad \text{Re}[A] > 0 . \tag{7.4}$$

for

$$\psi_0[x] = \left(\frac{m\,\omega}{\pi\,\hbar}\right)^{1/4} \text{Exp}\left[-\frac{m\,\omega}{2\,\hbar} x^2\right] \equiv N\,\text{Exp}\left[-A\,x^2\right]$$

one easily gets

$$a[k] = \frac{N}{2\pi}\sqrt{\frac{\pi}{A}}\ \mathrm{Exp}\left[-\frac{k^2}{2A}\right]. \tag{7.5}$$

The free part of the wave function at x = 0 and time t is then

$$\int_{-\infty}^{+\infty} dk\, a[k]\, e^{-i\frac{k^2\hbar t}{2m}} = \frac{N}{2\pi}\sqrt{\frac{\pi}{A}}\int_{-\infty}^{+\infty} dk\, e^{-\frac{k^2}{2A}}\, e^{-i\frac{k^2\hbar t}{2m}} = \frac{N}{\sqrt{1+i\,\omega\,t}}.$$

The state is

$$\psi[x=0,\,t] = \frac{N}{\sqrt{2}}\left(e^{-i\,\omega\,t/2}\,|+\rangle - \frac{1}{\sqrt{1+i\,\omega\,t}}\,|-\rangle\right). \tag{7.6}$$

To compute the probabilities for measuring up and down components it is convenient to normalize (6) in spin space:

$$|\psi\rangle = \sqrt{\frac{\sqrt{1+\omega^2 t^2}}{1+\sqrt{1+\omega^2 t^2}}}\left(e^{-i\,\omega\,t/2}\,|+\rangle - \frac{1}{\sqrt{1+i\,\omega\,t}}\,|-\rangle\right).$$

The requested probability is then

$$P_- = |\langle -|\psi\rangle|^2 = \frac{1}{1+\sqrt{1+\omega^2 t^2}}.$$

## Problem 8

*Let us consider the Hamiltonian of Hydrogen atom, together with the spin-orbit interaction term for the electron and the spin-spin interaction between electron and proton (hyperfine interaction):*

$$H_{so} = A\,\mathbf{L}\cdot\mathbf{s}_e\,; \qquad H_{ss} = B\,\mathbf{s}_p\cdot\mathbf{s}_e\,.$$

*A and B are constants.*

1. Compute the degeneracy for the levels n=1, n=2 of the hydrogen atom taking into account electron and proton spin but neglecting the above interactions.
2. Classify the same levels by neglecting only $H_{ss}$. Use the operators

$$\mathbf{L}^2,\ \mathbf{J}^2,\ \mathbf{F}^2,\ F_z;\quad \mathbf{J} = \mathbf{L}+\mathbf{s}_e\,;\ \mathbf{F} = \mathbf{J}+\mathbf{s}_p\,.$$

3. Some of the states classified above are also eigenstates of the total Hamiltonian: find which and compute their eigenvalues of H.

### Solution

#### 1

The energy levels are

$$E_n = -\frac{1}{2n^2}\frac{m e^4}{\hbar^2}. \tag{8.1}$$

Taking into account the spin variables the total degeneracy is 4 $n^2$, a factor $n^2$ is the Coulomb degeneracy, a factor 2 is due to electron spin and another factor of 2 is due to nuclear spin.

#### 2

Within the degenerate subspace of fixed n, L we can write

$$H_{so} = A\,\mathbf{L}\cdot\mathbf{s}_e = \frac{A}{2}\left[\mathbf{J}^2 - \mathbf{L}^2 - \mathbf{s}_e^2\right] = \frac{A}{2}\left[J\,(J+1) - L\,(L+1) - \frac{3}{4}\right]. \tag{8.2}$$

Taking into account the rules for angular momentum sum we can compute energies. The results for energies and degeneracies are summarized in the following table.

| n | L | J | F | E | deg. |
|---|---|---|---|---|---|
| 1 | 0 | $\frac{1}{2}$ | 0,1 | $E_1^{(0)}$ | 4 |
| 2 | 0 | $\frac{1}{2}$ | 0,1 | $E_2^{(0)}$ | 4 |
| 2 | 1 | $\frac{1}{2}$ | 0,1 | $E_2^{(0)} - A$ | 4 |
| 2 | 1 | $\frac{3}{2}$ | 1,2 | $E_2^{(0)} + \frac{1}{2} A$ | 8 |

■ **3**

In a similar way we can write

$$H = H_0 + A\,\mathbf{L}\cdot\mathbf{s}_e + B\,\mathbf{s}_p\cdot\mathbf{s}_e = H_0 + \frac{A}{2}\left[\mathbf{J}^2 - \mathbf{L}^2 - \mathbf{s}_e^2\right] + \frac{B}{2}\left[\mathbf{S}^2 - \frac{3}{4} - \frac{3}{4}\right]. \tag{8.3}$$

The states $|L, J, F, F_z\rangle$ of the previous table are not, in general, eigenstates of H, as $\mathbf{J}^2$ do not commutes with $\mathbf{S}^2$, this happens only when the values of L and F determine uniquely the S value. We are using the notation

$$S = s_e + s_p\,;\; F = L + S.$$

One gets easily the following table

| n | L | J | F | S | E | deg. |
|---|---|---|---|---|---|---|
| 1 | 0 | $\frac{1}{2}$ | 0 | 0 | $E_1^{(0)} - \frac{3}{4} B$ | 1 |
| 1 | 0 | $\frac{1}{2}$ | 1 | 1 | $E_1^{(0)} + \frac{1}{4} B$ | 3 |
| 2 | 0 | $\frac{1}{2}$ | 0 | 0 | $E_2^{(0)} - \frac{3}{4} B$ | 1 |
| 2 | 0 | $\frac{1}{2}$ | 1 | 1 | $E_2^{(0)} + \frac{1}{4} B$ | 3 |
| 2 | 1 | $\frac{1}{2}$ | 0 | 1 | $E_2^{(0)} - A + \frac{1}{4} B$ | 1 |
| 2 | 1 | $\frac{3}{2}$ | 2 | 1 | $E_2^{(0)} + \frac{1}{2} A + \frac{1}{4} B$ | 5 |